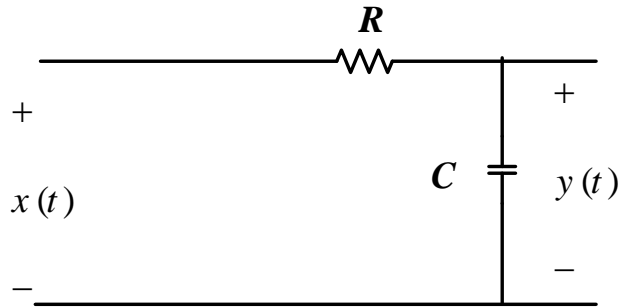


Chapter 7 The Laplace Transform

Consider the following RC circuit (**System**)



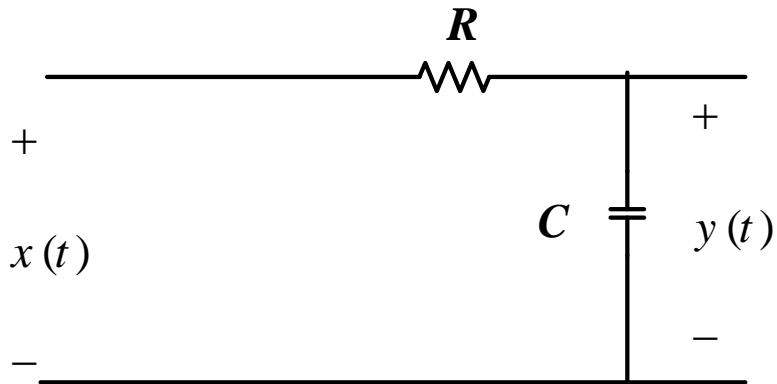
System analysis in the time domain involves (finding $y(t)$):

Solving the differential equation $RC \frac{dy(t)}{dt} + y(t) = x(t)$

OR

Using the convolution integral $y(t) = x(t) * h(t)$

Both Techniques can results in tedious (**ممل**) mathematical operation



$$RC \frac{dy(t)}{dt} + y(t) = x(t)$$

Fourier Transform provided an alternative approach

Differential Equation

$$RC \frac{dy(t)}{dt} + y(t) = x(t)$$

Algebraic Equation

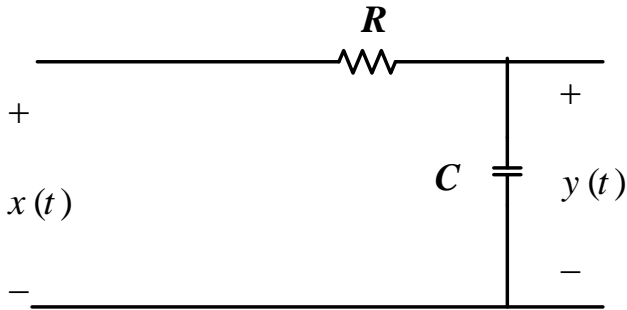
$$RC(j\omega) Y(\omega) + Y(\omega) = X(\omega)$$

solve for $Y(\omega)$

$$Y(\omega) = \frac{X(\omega)}{[(j\omega RC) + 1]}$$

Inverse Back

$y(t)$



$$RC \frac{dy(t)}{dt} + y(t) = x(t)$$

Fourier Transform both side

$$RC(j\omega) Y(\omega) + Y(\omega) = X(\omega)$$

Solve for $Y(\omega)$

$$y(t) \longleftarrow \text{Inverse Back} \quad Y(\omega) = \frac{X(\omega)}{[(j\omega RC) + 1]}$$

Unfortunately, there are many signals of interest that arise in system analysis for which the **Fourier Transform does not exist**
(not absolutely integrable)

$$\int_{-\infty}^{\infty} |x(t)| dt \rightarrow \infty \quad \text{Example } x(t) = t, \quad x(t) = t^2, \quad x(t) = e^t$$

In fact signals such as

$$x(t) = 1 \Leftrightarrow 2\pi\delta(\omega)$$

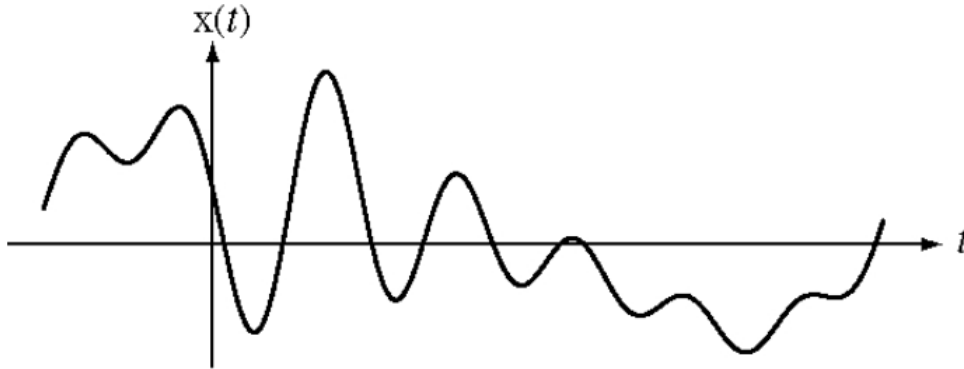
$$x(t) = u(t) \Leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

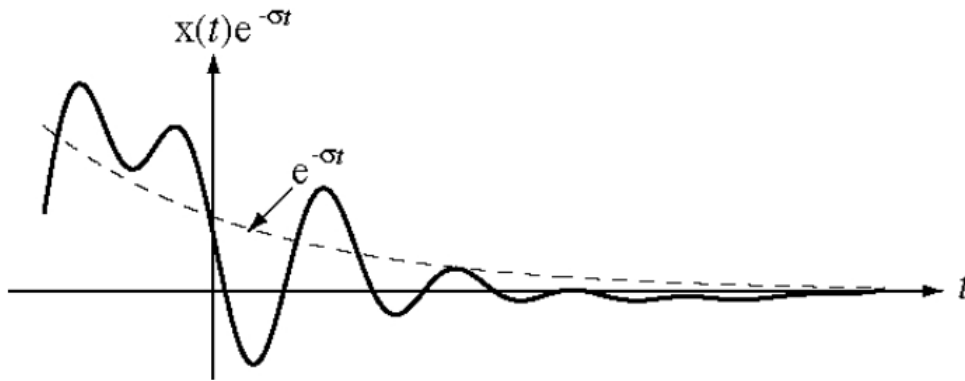
are not strictly integrable and their Fourier transforms all contain some non – conventional function such as $\delta(\omega)$.

A more general transform is needed

Suppose we have a function or signal $x(t)$ that is not absolutely integrable as shown



Now multiply $x(t)$ by $e^{-\sigma t}$



The signal $x(t)e^{-\sigma t}$ is absolutely integrable



The signal $x(t)e^{-\sigma t}$ has Fourier Transform

Fourier Transform pairs was defined as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \Leftrightarrow X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Now multiply $x(t)$ by $e^{-\sigma t}$ and takes the Fourier Transform

$$FT \left[e^{-\sigma t} x(t) \right] = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt = X(\sigma + j\omega)$$

}
 Fourier Transform of $e^{-\sigma t} x(t)$
 The domain is not ω but $\sigma + j\omega$

Let us defined this $(\sigma + j\omega)$ as $s \square \sigma + j\omega$ Complex Frequency

$$\Rightarrow FT \left[e^{-\sigma t} x(t) \right] = \int_0^{\infty} x(t) e^{-st} dt = X(s) \quad \square \quad L_b \left[x(t) \right]$$

were $L_b \left[\quad \right]$ Denotes the operation of obtaining the **Bilateral Laplace Transform**

7.1 DEFINITIONS OF LAPLACE TRANSFORMS

The Laplace transform of a time function is given by the integral

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt$$

where $\mathcal{L}_b[\cdot]$ indicates the Laplace transform

This definition is called the **bilateral**, or **two-sided**, Laplace transform—hence, the subscript b

Notice that the bilateral Laplace transform integral becomes the Fourier transform integral if s is replaced by $j\omega$ (i.e., $\sigma \rightarrow 0$)

The Laplace transform variable is complex $s = \sigma + j\omega$

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t} dt = \int_{-\infty}^{\infty} (f(t)e^{-\sigma t})e^{-j\omega t} dt$$

This shows that the **bilateral Laplace transform** of a signal can be interpreted as the **Fourier transform** of that signal multiplied by an exponential function $e^{-\sigma t}$

The inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds \quad \text{where } j = \sqrt{-1}$$

where $\mathcal{L}^{-1}[\cdot]$ indicates the inverse Laplace transform

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st} dt = \int_{-\infty}^0 f(t)e^{-st} dt + \int_0^{\infty} f(t)e^{-st} dt$$

For most application , $f(t)$ is zero for $t < 0$

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

where $\mathcal{L}[\cdot]$ denotes the unilateral Laplace transform

This transform is usually called, simply, the **Laplace transform** , the subscript b dropped

$$f(t) \xleftrightarrow{\mathcal{L}} F(s) \quad \text{Laplace transform pairs}$$

Unilateral (single sided) Laplace transform pairs

$$f(t) \xleftrightarrow{\mathcal{L}} F(s)$$

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

Forward transform

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds$$

Inverse transform

$$s = \sigma + j\omega$$

Complex domain

Because of the difficulty in evaluating the complex inversion integral

Simpler procedure to find the inverse of **Laplace Transform** (i.e $f(t)$) will be presented later



Using Properties and **table** of known transform (similar to Fourier)

Laplace Transform for an impulse $\delta(t)$

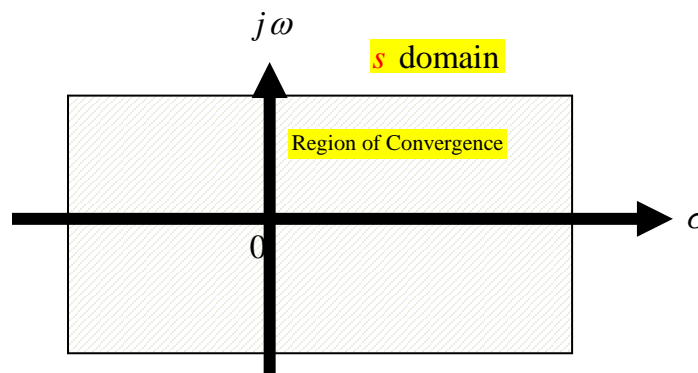
$$\mathcal{L}[\delta(t)] = \int_0^{\infty} \delta(t) e^{-st} dt$$

Assuming the lower limit is 0^-

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-st} \Big|_{t=0} = 1$$

The value of s make no difference

Region of Convergence



$$\mathcal{L}[\delta(t - t_0)] = \int_0^{\infty} \delta(t - t_0) e^{-st} dt = e^{-st} \Big|_{t=t_0} = e^{-t_0 s}$$

$$\delta(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-t_0 s}$$

We also can drive it from the shifting in time property that will be discussed later

EXAMPLE 7.1

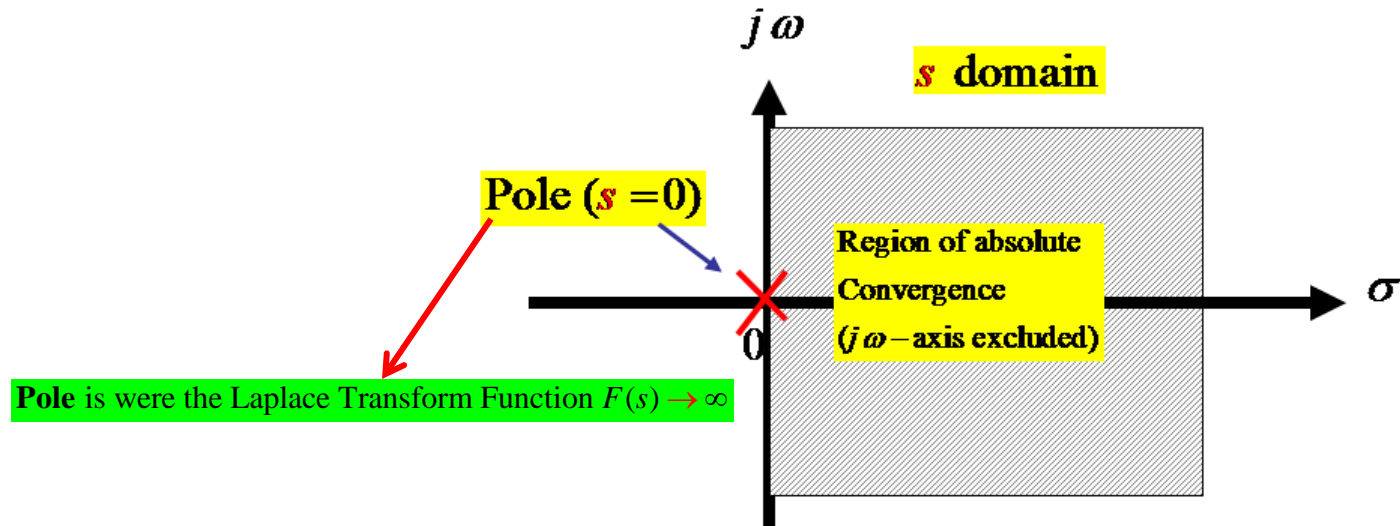
Laplace transform of a unit step function

$$\mathcal{L}[u(t)] = \int_0^{\infty} u(t)e^{-st} dt = \int_0^{\infty} e^{-st} dt = \frac{e^{-st}}{-s} \Big|_0^{\infty} = \frac{-1}{s} \left[\lim_{t \rightarrow \infty} e^{-st} - 1 \right]$$

$$= \frac{-1}{s} \left[\lim_{t \rightarrow \infty} e^{-(\sigma+j\omega)t} - 1 \right] = \frac{-1}{s} \left[\lim_{t \rightarrow \infty} e^{-\sigma t} e^{-j\omega t} - 1 \right]$$

For this to converge then $\lim_{t \rightarrow \infty} e^{-\sigma t} \rightarrow 0 \Rightarrow \sigma > 0 \Rightarrow \underbrace{\text{Re}(s) > 0}_{\text{region of convergence (ROC)}}$

$$\mathcal{L}[u(t)] = \frac{1}{s}, \quad \text{Re}(s) > 0$$



EXAMPLE 7.2

Laplace transform of an exponential function

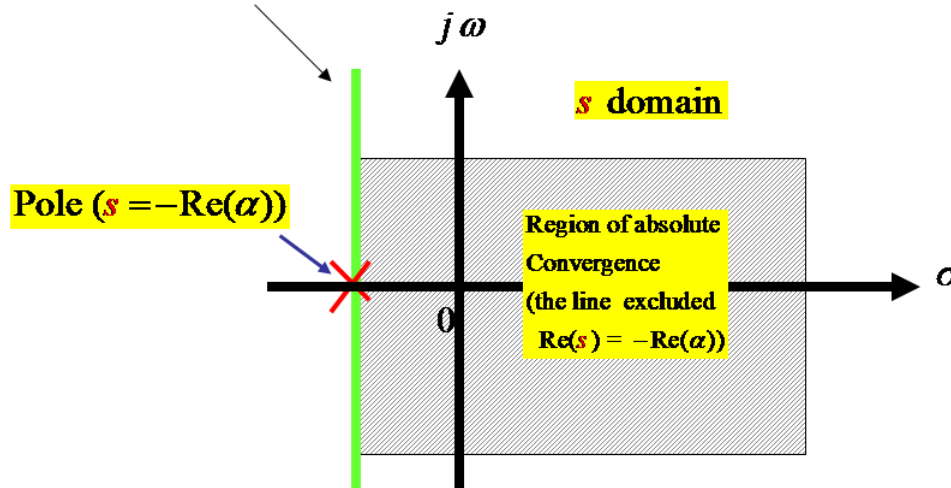
We derive the Laplace transform of the exponential function $f(t) = e^{-at}$

$$F(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt = \frac{e^{-(s+a)t}}{-(s+a)} \Big|_0^{\infty} = \frac{-1}{s+a} \left[\lim_{t \rightarrow \infty} e^{-(s+a)t} - 1 \right]$$

This transform exists only if $\text{Re}(s + a)$ is positive

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}, \quad \text{Re}(s+a) > 0$$

$$\text{Re}(s) = -\text{Re}(\alpha)$$



$$e^{-at} \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}$$

EXAMPLE 7.3**Laplace transform of a unit ramp function**

We now find the Laplace transform of the unit ramp function $f(t) = t$.

$$\mathcal{L}[t] = \int_0^{\infty} te^{-st} dt$$

From the table of integrals, Appendix A $\int ue^u du = e^u(u - 1) + C$

Then, letting $u = -st$, we get

$$\int_0^{\infty} te^{-st} dt = \frac{1}{(-s)^2} \int_0^{\infty} (-st)e^{(-st)} d(-st) = \frac{1}{s^2} e^{-st}(-st - 1) \Big|_0^{\infty}$$

by L'Hôpital's rule $\lim_{t \rightarrow \infty} \frac{t}{e^{st}} = \lim_{t \rightarrow \infty} \frac{1}{se^{st}} = 0$

$$\rightarrow \int_0^{\infty} te^{-st} dt = \frac{1}{s^2} [0 - (-1)] = \frac{1}{s^2} \quad \text{Re}(s) > 0$$

$$\rightarrow t \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}$$

A short table of Laplace transforms is constructed from Examples 7.1 and 7.2 and is given as Table 7.2

TABLE 7.2 Laplace Transforms

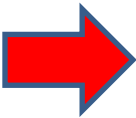
f(t), t ≥ 0	F(s)	ROC
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
3. t	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
4. t^n	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
5. e^{-at}	$\frac{1}{s + a}$	$\text{Re}(s) > -a$

We know derive the Laplace Transform (LT) for a cosine function

$$\cos bt = \frac{e^{jbt} + e^{-jbt}}{2}$$

$$\mathcal{L}[\cos bt] = \frac{1}{2}[\mathcal{L}[e^{jbt}] + \mathcal{L}[e^{-jbt}]]$$

Since $e^{-at} \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}$


$$\mathcal{L}[\cos bt] = \frac{1}{2} \left[\frac{1}{s-jb} + \frac{1}{s+jb} \right] = \frac{s+jb + s-jb}{2(s-jb)(s+jb)} = \frac{s}{s^2 + b^2}$$

By the same procedure, because $\sin bt = (e^{jbt} - e^{-jbt})/2j$.

$$\begin{aligned} \mathcal{L}[\sin bt] &= \frac{1}{2j}[\mathcal{L}[e^{jbt}] - \mathcal{L}[e^{-jbt}]] = \frac{1}{2j} \left[\frac{1}{s-jb} - \frac{1}{s+jb} \right] \\ &= \frac{s+jb - s+jb}{2j(s-jb)(s+jb)} = \frac{b}{s^2 + b^2} \end{aligned}$$

7.4 LAPLACE TRANSFORM PROPERTIES

linearity $\mathcal{L}[a_1f_1(t) + a_2f_2(t)] = a_1F_1(s) + a_2F_2(s)$

$$\begin{aligned}\mathcal{L}[f(t)] &= \mathcal{L}[f_1(t) + f_2(t)] = \int_0^{\infty} [f_1(t) + f_2(t)]e^{-st} dt \\ &= \int_0^{\infty} f_1(t)e^{-st} dt + \int_0^{\infty} f_2(t)e^{-st} dt \\ &= \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)] = F_1(s) + F_2(s)\end{aligned}$$

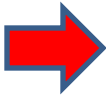
$$\mathcal{L}[af_5(t)] = \int_0^{\infty} af_5(t)e^{-st} dt = a \int_0^{\infty} f_5(t)e^{-st} dt = a\mathcal{L}[f_5(t)] = aF_5(s)$$

$$\mathcal{L}[a_1f_1(t) + a_2f_2(t)] = a_1F_1(s) + a_2F_2(s),$$

complex shifting $\mathcal{L}[e^{-at}f(t)] = F(s) \Big|_{s \leftarrow s+a} = F(s+a)$

Proof

$$L[e^{-at}f(t)] = \int_0^{\infty} [e^{-at}f(t)]e^{-st} dt = \int_0^{\infty} f(t)e^{-(s+a)t} dt$$

Let $s^* = s+a$  $L[e^{-at}f(t)] = \int_0^{\infty} f(t)e^{-s^*t} dt = F(s^*) = F(s+a)$

$$L[\cos \omega_0 t] = \frac{s}{s^2 + \omega_0^2} \quad \img alt="red arrow" data-bbox="415 475 470 540"/> \quad \mathcal{L}[e^{-\alpha t} \cos \omega_0 t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}$$

$$L[\sin \omega_0 t] = \frac{\omega_0}{s^2 + \omega_0^2} \quad \img alt="red arrow" data-bbox="405 665 460 730"/> \quad \mathcal{L}[e^{-\alpha t} \sin \omega_0 t] = \frac{\omega_0}{(s + \alpha)^2 + \omega_0^2}$$

Example

$$\mathbf{L}[\cos \omega_0 t] = \mathbf{L}\left[\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right] = \frac{1}{2}\mathbf{L}[e^{j\omega_0 t}] + \frac{1}{2}\mathbf{L}[e^{-j\omega_0 t}]$$

Since $\mathbf{L}[e^{-\alpha t}u(t)] = \frac{1}{(\alpha + s)}$

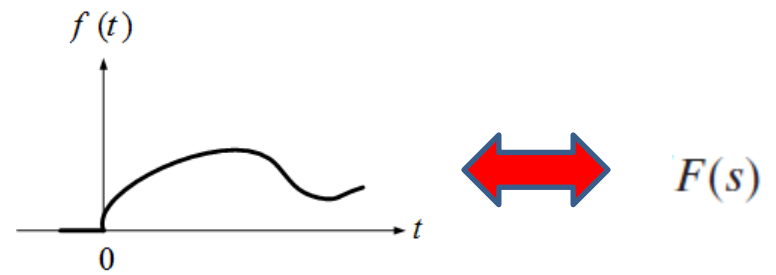
Then $\mathbf{L}[e^{j\omega_0 t}] = \frac{1}{(j\omega_0 + s)}$ $\mathbf{L}[e^{-j\omega_0 t}] = \frac{1}{(-j\omega_0 + s)}$

$$\mathbf{L}[\cos \omega_0 t] = \frac{1}{2} \frac{1}{(s + j\omega_0)} + \frac{1}{2} \frac{1}{(s - j\omega_0)} = \frac{s}{s^2 + \omega_0^2}$$

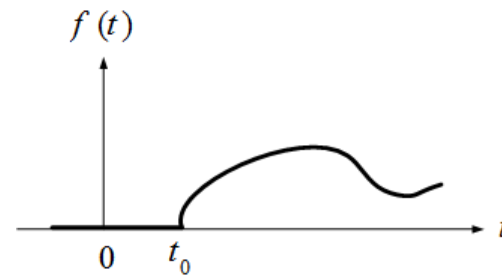
Similarly $\mathbf{L}[\sin \omega_0 t] = \frac{\omega_0}{s^2 + \omega_0^2}$

Real Shifting

$$f(t)u(t) = \begin{cases} f(t), & t > 0 \\ 0, & t < 0 \end{cases}$$

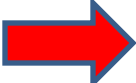


Let $f(t - t_0)u(t - t_0) = \begin{cases} f(t - t_0), & t > t_0 \\ 0, & t < t_0 \end{cases}$
where $t_0 \cong 0$



The Laplace transform of the function

$$\mathcal{L}[f(t - t_0)u(t - t_0)] = \int_0^{\infty} f(t - t_0)u(t - t_0)e^{-st} dt = \int_{t_0}^{\infty} f(t - t_0)e^{-st} dt$$

We make the change of variable $(t - t_0) = \tau$  $t = (\tau + t_0)$ $dt = d\tau$

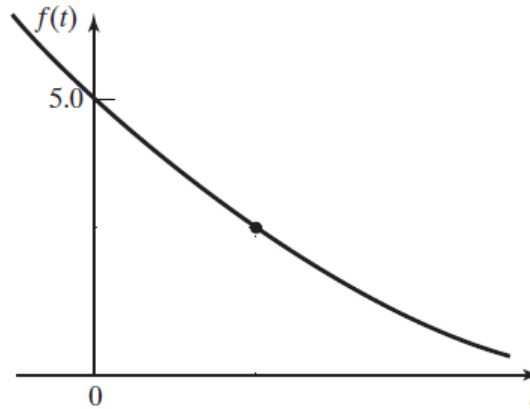
$$\mathcal{L}[f(t - t_0)u(t - t_0)] = \int_0^{\infty} f(\tau)e^{-s(\tau+t_0)} d\tau = e^{-t_0s} \underbrace{\int_0^{\infty} f(\tau)e^{-s\tau} d\tau}_{F(s)}$$

$$\mathcal{L}[f(t - t_0)u(t - t_0)] = e^{-t_0s}F(s)$$

EXAMPLE 7.4**Laplace transform of a delayed exponential function**

Consider the exponential function

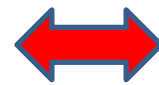
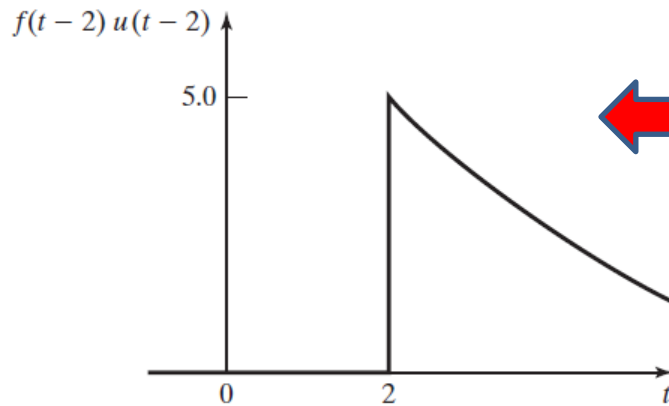
$$f(t) = 5e^{-0.3t}$$



$$e^{-at} \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}$$

$$\longleftrightarrow F(s) = \frac{5}{s+0.3}$$

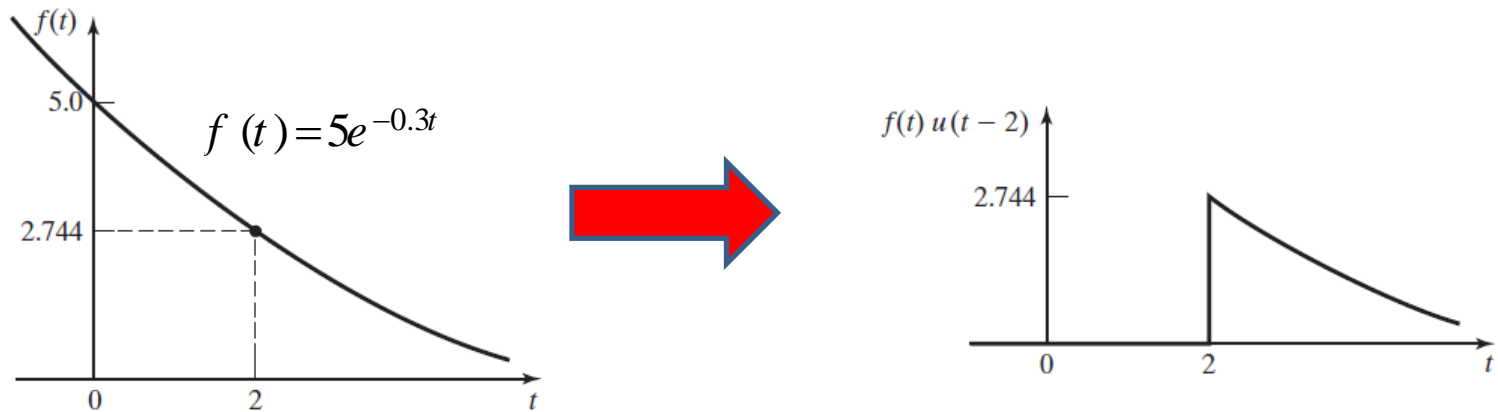
$$f_1(t) = 5e^{-0.3(t-2)}u(t-2)$$



$$\mathcal{L}[f_1(t)] = F_1(s) = e^{-2s}F(s) = \frac{5e^{-2s}}{s+0.3}$$

EXAMPLE 7.5**Laplace transform of a more complex delayed function**

Consider now the function $f_2(t) = 5e^{-0.3t} u(t - 2)$



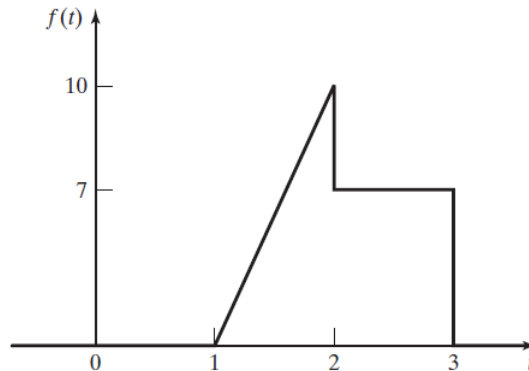
we need to put the function $f_2(t)$ in a delay form $f(t - t_0)u(t - t_0)$ in order to use the Laplace Transform for time shift $\mathcal{L}[f(t - t_0)u(t - t_0)] = e^{-t_0s} F(s)$

$$\begin{aligned} f_2(t) &= 5e^{-0.3t} u(t - 2) = 5e^{-0.3t} u(t - 2)[e^{0.3(2)}e^{-0.3(2)}] = (5e^{-0.6})e^{-0.3(t-2)}u(t - 2) \\ &= 2.744e^{-0.3(t-2)}u(t - 2) \end{aligned}$$

$$\Rightarrow F_2(s) = \mathcal{L}[f_2(t)] = \frac{2.744e^{-2s}}{s + 0.3}$$

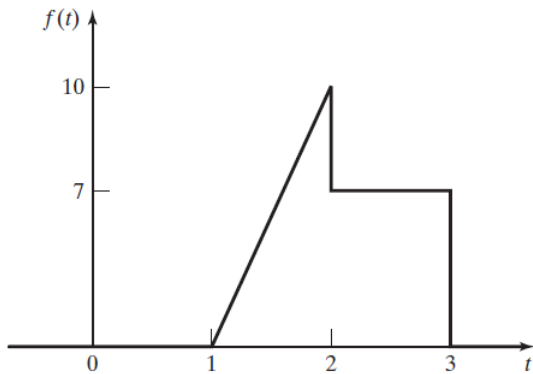
EXAMPLE 7.6**Laplace transform of a straight-line-segments function**

It is sometimes necessary to construct complex waveforms from simpler waveforms



1. The slope of the function changes from 0 to 10 at $t = 1$: $f_1(t) = 10(t - 1)u(t - 1)$
2. The slope of the function changes from 10 to 0 at $t = 2$: $f_2(t) = f_1(t) - 10(t - 2)u(t - 2)$
3. The function steps by -3 at $t = 2$: $f_3(t) = f_2(t) - 3u(t - 2)$
4. The function steps by -7 at $t = 3$:

$$\begin{aligned} f(t) &= f_3(t) - 7u(t - 3) \\ &= 10(t - 1)u(t - 1) - 10(t - 2)u(t - 2) - 3u(t - 2) - 7u(t - 3) \end{aligned}$$

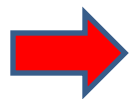


$$f(t) = 10(t - 1)u(t - 1) - 10(t - 2)u(t - 2) - 3u(t - 2) - 7u(t - 3)$$

Laplace transform of the unit ramp function $f(t) = t$

$$\mathcal{L}[t] = \int_0^{\infty} te^{-st} dt = \frac{1}{s^2} \quad (\text{See example 7.3})$$

Since $\mathcal{L}[f(t - t_0)u(t - t_0)] = e^{-t_0s}F(s)$ (Shift in time property)



$$F(s) = \frac{10e^{-s}}{s^2} - \frac{10e^{-2s}}{s^2} - \frac{3e^{-2s}}{s} - \frac{7e^{-3s}}{s} = \frac{10e^{-s} - 10e^{-2s} - 3se^{-2s} - 7se^{-3s}}{s^2}$$

Transform of Derivatives

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

We evaluate this integral by parts $\int u dv = uv - \int v du$

$$\mathcal{L}[f(t)] = \int_0^{\infty} \boxed{f(t)} \boxed{e^{-st} dt}$$

$$u = f(t), \quad dv = e^{-st} dt$$

$$\Rightarrow du = \frac{df(t)}{dt} dt \quad v = \frac{e^{-st}}{-s}$$

$$\mathcal{L}[f(t)] = F(s) = -\frac{1}{s} f(t) e^{-st} \Big|_0^{\infty} + \int_0^{\infty} \frac{df(t)}{dt} \frac{e^{-st}}{s} dt$$

$$F(s) = \frac{1}{s} [-0 + f(0)] + \frac{1}{s} \int_0^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

$$\Rightarrow \mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) \quad \underbrace{\int_0^{\infty} \frac{df(t)}{dt} e^{-st} dt}_{\mathcal{L}\left[\frac{df(t)}{dt}\right]}$$

$$\mathbf{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^-)$$

Proof

$$\mathbf{L}\left[\frac{df(t)}{dt}\right] = \int_0^{\infty} \left[\frac{df(t)}{dt}\right] e^{-st} dt$$

Integrating by parts, $u = e^{-st}$ $dv(t) = df(t)$

$$\Rightarrow du = -se^{-st} \quad v(t) = f(t)$$

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

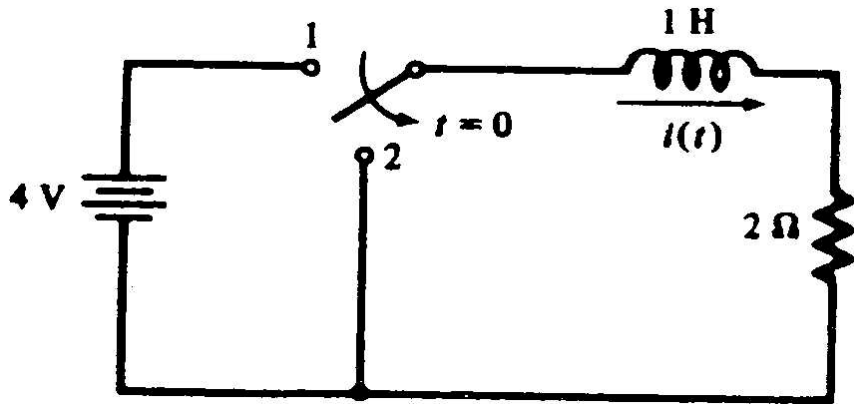
$$u = e^{-st} \quad dv(t) = df(t) \quad du = -s e^{-st} \quad v(t) = f(t)$$

$$\mathcal{L} \left[\frac{df(t)}{dt} \right] = \int_0^{\infty} \left[\frac{df(t)}{dt} \right] e^{-st} dt = e^{-st} f(t) \Big|_{0^-}^{\infty} + s \int_{0^-}^{\infty} f(t) e^{-st} dt$$

$$= \left[e^{-s(\infty)} f(\infty) - e^{-s(0)} f(0^-) \right] + sF(s) = sF(s) - f(0^-)$$

$$\Rightarrow \frac{df(t)}{dt} \Leftrightarrow sF(s) - f(0^-)$$

Consider the circuit shown



$$\frac{di(t)}{dt} + 2i(t) = \begin{cases} 4, & t \leq 0 \\ 0, & t > 0 \end{cases}$$

Taking the Laplace transform of both sides starting at $t = 0^-$

$$sI(s) - i(0^-) + 2I(s) = 0$$

Assuming that the circuit was in steady state for $t < 0$. $\rightarrow i(0^-) = \frac{4}{2} = 2$

$$\rightarrow I(s)(s + 2) - 2 = 0 \quad \rightarrow I(s) = \frac{2}{s + 2} \quad \rightarrow i(t) = 2e^{-2t}u(t)$$

$$\rightarrow i(t) = \begin{cases} 2e^{-2t}, & t > 0 \\ 2, & t \leq 0 \end{cases}$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s[sF(s) - f(0^+)] - f'(0^+) = s^2F(s) - sf(0^+) - f'(0^+)$$

where $f'(0^+)$ is the value of $df(t)/dt$ as $t \rightarrow 0^+$

$$\mathcal{L}\left[\frac{d^3f(t)}{dt^3}\right] = s^3F(s) - s^2f(0^+) - sf'(0^+) - f''(0^+)$$

where $f''(0^+)$ is the value of $d^2f(t)/dt^2$ as $t \rightarrow 0^+$

$$\mathcal{L}\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - \dots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$$

where $f^{(i)}(0^+)$ is the value of $d^if(t)/dt^i$ as $t \rightarrow 0^+$

$$\frac{df(t)}{dt} \Leftrightarrow sF(s) - f(0^-)$$

$$\frac{d^2f(t)}{dt^2} \Leftrightarrow s^2F(s) - sf(t)\Big|_{t=0} - \frac{df(t)}{dt}\Big|_{t=0}$$

$$\frac{d^3f(t)}{dt^3} \Leftrightarrow s^3F(s) - s^2f(t)\Big|_{t=0} - s\frac{df(t)}{dt}\Big|_{t=0} - \frac{d^2f(t)}{dt^2}\Big|_{t=0}$$

$$\frac{d^nf(t)}{dt^n} \Leftrightarrow s^nF(s) - s^{n-1}f(t)\Big|_{t=0} - s^{n-2}\frac{df(t)}{dt}\Big|_{t=0} - s^{n-3}\frac{d^2f(t)}{dt^2}\Big|_{t=0} - \dots - \frac{d^{n-1}f(t)}{dt^{n-1}}\Big|_{t=0}$$

Integration

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

Proof

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \int_0^\infty \left[\int_0^t f(\tau)d\tau\right] e^{-st} dt$$

We integrate this expression by parts

$$u = \int_0^t f(\tau)d\tau, \quad dv = e^{-st}dt \quad \Rightarrow \quad du = f(t)dt, \quad v = \frac{e^{-st}}{-s}$$

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{e^{-st}}{-s} \int_0^t f(\tau)d\tau \Big|_{t=0}^{\infty} + \underbrace{\frac{1}{s} \int_0^\infty f(t)e^{-st}dt}_{F(s)}$$

$$= -[0 - 0] + \frac{1}{s}F(s)$$

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

EXAMPLE 7.8**Illustration of the integration property**

Consider the following relationship, for $t > 0$

$$\int_0^t u(\tau) d\tau = \tau \Big|_0^t = t$$

The Laplace transform of the unit step function is $1/s$


$$\mathcal{L}[u(t)] = \frac{1}{s}, \quad \text{Re}(s) > 0$$


$$\mathcal{L}[t] = \mathcal{L}\left[\int_0^t u(\tau) d\tau\right] = \frac{1}{s} \mathcal{L}[u(t)] = \frac{1}{s} \frac{1}{s} = \frac{1}{s^2}$$

which is the Laplace transform of $f(t) = t$

find the Laplace transforms of the following integrals:

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

$$\mathcal{L}\left\{\int_0^t \cos at dt\right\} = \frac{s}{s^2 + a^2}$$


$$\mathcal{L}\left\{\int_0^t e^{at} \cos btdt\right\} = \frac{s-a}{(s-a)^2 + b^2} = \frac{s-a}{s((s-a)^2 + b^2)}$$


$$\mathcal{L}\left\{\int_0^t te^{-3t} dt\right\} = \frac{1}{s} \times \frac{1}{(s+3)^2} = \frac{1}{s(s+3)^2}$$

Multiplication by t

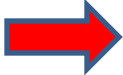
$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$$

Proof (See the book)

EXAMPLE 7.9 Illustration of the multiplication-by- t property

We now derive the transform of $f(t) = t \cos t$

Since $\mathcal{L}[\cos bt] = \frac{s}{s^2 + b^2}$

 $\mathcal{L}[t \cos bt] = -\frac{d}{ds} \left[\frac{s}{s^2 + b^2} \right] = -\frac{(s^2 + b^2)(1) - s(2s)}{(s^2 + b^2)^2} = \frac{s^2 - b^2}{(s^2 + b^2)^2}$

Two more examples

$$\mathcal{L}[t^2] = -\frac{d}{ds} \mathcal{L}[t] = -\frac{d}{ds} \left[\frac{1}{s^2} \right] = \frac{2}{s^3} = \frac{2!}{s^3}$$

$$\mathcal{L}[t^3] = -\frac{d}{ds} \left[\frac{2}{s^3} \right] = \frac{2 \cdot 3}{s^4} = \frac{3!}{s^4}$$

TABLE 7.2 Laplace Transforms

f(t), t ≥ 0	F(s)	ROC
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
3. t	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
4. t^n	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
5. e^{-at}	$\frac{1}{s+a}$	$\text{Re}(s) > -a$
6. te^{-at}	$\frac{1}{(s+a)^2}$	$\text{Re}(s) > -a$
7. $t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}(s) > -a$
8. $\sin bt$	$\frac{b}{s^2 + b^2}$	$\text{Re}(s) > 0$
9. $\cos bt$	$\frac{s}{s^2 + b^2}$	$\text{Re}(s) > 0$
10. $e^{-at} \sin bt$	$\frac{b}{(s+a)^2 + b^2}$	$\text{Re}(s) > -a$
11. $e^{-at} \cos bt$	$\frac{s+a}{(s+a)^2 + b^2}$	$\text{Re}(s) > -a$
12. $t \sin bt$	$\frac{2bs}{(s^2 + b^2)^2}$	$\text{Re}(s) > 0$
13. $t \cos bt$	$\frac{s^2 - b^2}{(s^2 + b^2)^2}$	$\text{Re}(s) > 0$

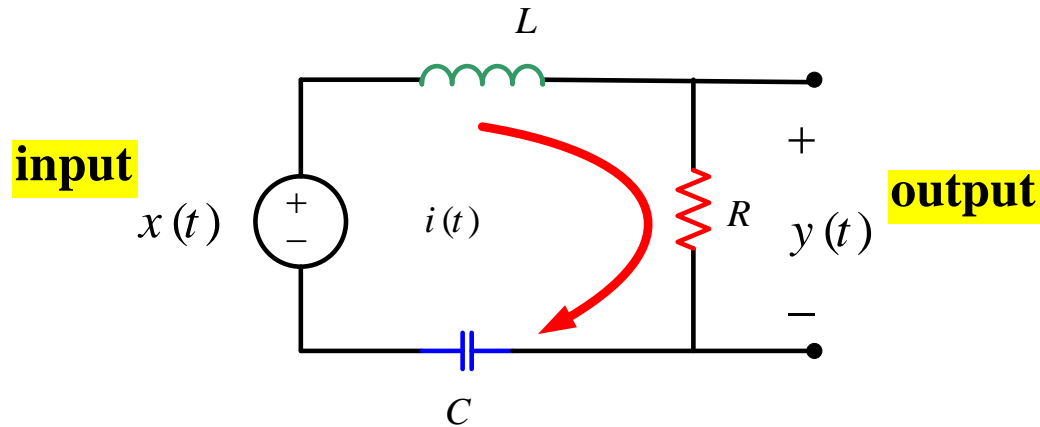
TABLE 7.3 Laplace Transform Properties

Name	Property
1. Linearity, (7.10)	$\mathcal{L}[a_1f_1(t) + a_2f_2(t)] = a_1F_1(s) + a_2F_2(s)$
2. Derivative, (7.15)	$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$
3. n th-order derivative, (7.29)	$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1}f(0^+) - \dots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$
4. Integral, (7.31)	$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$
5. Real shifting, (7.22)	$\mathcal{L}[f(t - t_0)u(t - t_0)] = e^{-t_0s}F(s)$
6. Complex shifting, (7.20)	$\mathcal{L}[e^{-at}f(t)] = F(s + a)$
7. Initial value, (7.36)	$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$
8. Final value, (7.39)	$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$
9. Multiplication by t , (7.34)	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
10. Time transformation, (7.42) ($a > 0; b \geq 0$)	$\mathcal{L}[f(at - b)u(at - b)] = \frac{e^{-sb/a}}{a} F\left(\frac{s}{a}\right)$
11. Convolution	$\begin{aligned} \mathcal{L}^{-1}[F_1(s)F_2(s)] &= \int_0^t f_1(t - \tau)f_2(\tau) d\tau \\ &= \int_0^t f_1(\tau)f_2(t - \tau)d\tau \end{aligned}$
12. Time periodicity	$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-sT}}F_1(s)$, where
$[f(t) = f(t + T)], t \geq 0$	$F_1(s) = \int_0^T f(t)e^{-st} dt$

7.6 Response of LTI Systems

Transfer Functions

Consider the following circuit

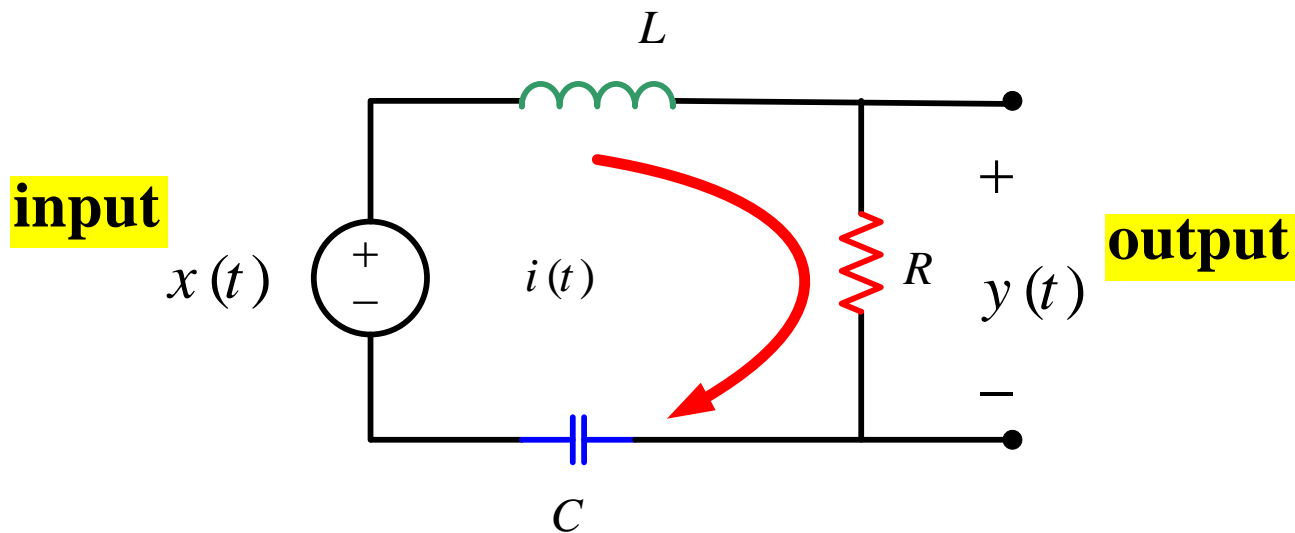


We want a relation (an equation) between the input $x(t)$ and output $y(t)$

KVL

$$x(t) = L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_{-\infty}^t i(t') dt'$$

$$\frac{dx(t)}{dt} = L \frac{d^2 i(t)}{dt^2} + R \frac{di(t)}{dt} + \frac{i(t)}{C}$$

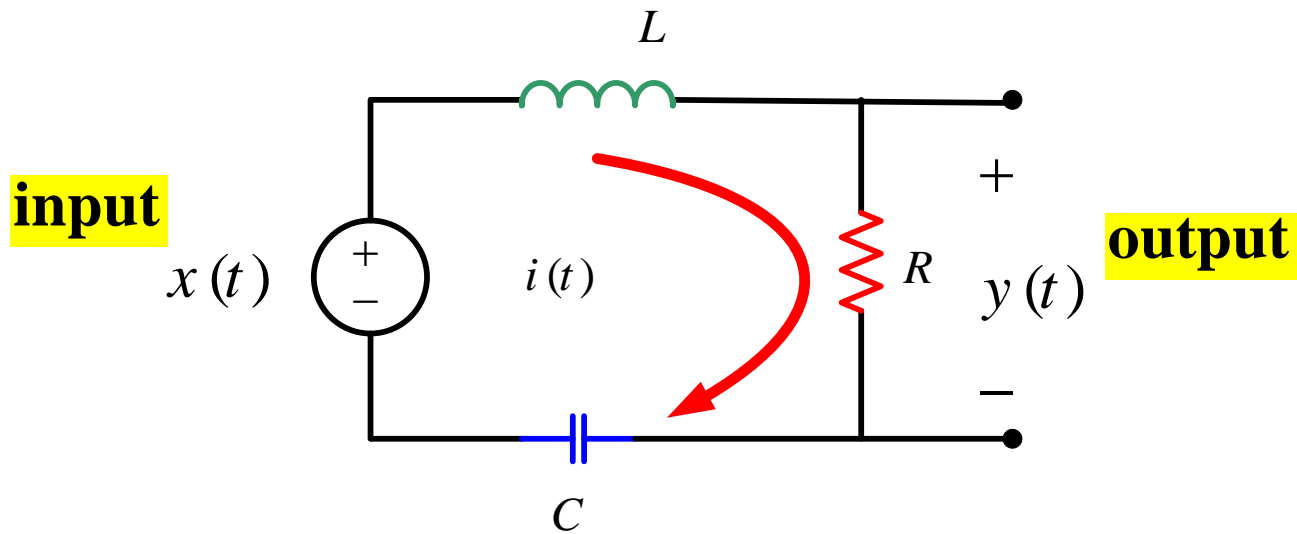


$$\frac{dx(t)}{dt} = L \frac{di^2(t)}{dt^2} + R \frac{di(t)}{dt} + \frac{i(t)}{C}$$

Since $i(t) = \frac{y(t)}{R} \Rightarrow \frac{d \cancel{x}(t)}{dt} = \frac{L}{R} \frac{dy^2(t)}{dt^2} + R \frac{d \cancel{y}(t)}{dt} + \frac{y(t)}{RC}$

Writing the differential equation as

$$RC \frac{dx(t)}{dt} = LC \frac{dy^2(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t)$$



$$\underbrace{\{RC\}}_{\text{RC}} \frac{dx(t)}{dt} = \underbrace{\{LC\}}_{\text{LC}} \frac{d^2 y(t)}{dt^2} + \underbrace{\{RC\}}_{\text{RC}} \frac{dy(t)}{dt} + \underbrace{\{1\}}_{\text{1}} y(t)$$

Real coefficients, non negative which results from system components R, L, C

In general,

$$a_n \frac{dy^n(t)}{dt^n} + a_{n-1} \frac{dy^{n-1}(t)}{dt^{n-1}} + \dots + a_0 y(t) = b_m \frac{dx^m(t)}{dt^m} + b_{m-1} \frac{dx^{m-1}(t)}{dt^{m-1}} + \dots + b_0 y(t)$$

were a_n 's , b_m 's are real, non negative which results from system components R, L, C

Now if we take the Laplace Transform of both side (Assuming Zero initial Conditions)

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \dots + a_0 Y(s) = b_m s^m X(s) + b_{m-1} s^{m-1} X(s) + \dots + b_0 X(s)$$

We now define the transfer function $H(s)$,

$$H(s) \square \left. \frac{Y(s)}{X(s)} \right| = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

all initial conditions are zero

$$a_n \frac{dy^n(t)}{dt^n} + a_{n-1} \frac{dy^{n-1}(t)}{dt^{n-1}} + \dots + a_0 y(t) = b_m \frac{dx^m(t)}{dt^m} + b_{m-1} \frac{dx^{m-1}(t)}{dt^{m-1}} + \dots + b_0 x(t)$$

$$H(s) \square \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} \square \frac{N(s)}{D(s)}$$

Since a_n 's , b_m 's are real, non negative

The roots of the polynomials $N(s)$, $D(s)$ are either real or occur in complex conjugate

The roots of $N(s)$ are referred to as the zero of $H(s)$ ($H(s) = 0$)

The roots of $D(s)$ are referred to as the pole of $H(s)$ ($H(s) = \pm \infty$)

$$H(s) \square \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} \square \frac{N(s)}{D(s)}$$

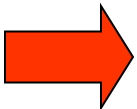
The Degree of $N(s)$ (which is related to input) must be less than or Equal of $D(s)$ (which is related to output) for the system to be Bounded-input, bounded-output (**BIBO**)

Example :
$$H(s) = \frac{4s^3 + 2s^2 + s + 1}{s^2 + 6s + 8}$$

Using polynomial division , we obtain
$$H(s) = 4s + 2 + \frac{-19s + 17}{s^2 + 6s + 8}$$

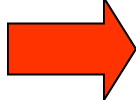
Now assume the input $x(t) = u(t)$ (**bounded input**) $\Rightarrow X(s) = \frac{1}{s}$

$$Y(s) = X(s)H(s) = 4 + \frac{2}{s} + \frac{1}{s} \left(\frac{-19s + 17}{s^2 + 6s + 8} \right)$$


$$y(t) = \underbrace{4\delta(t)}_{\text{unbounded } (\rightarrow \infty)} + 2 + L^{-1} \left(\frac{-19s + 17}{s(s^2 + 6s + 8)} \right)$$

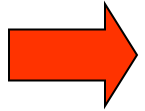
We see that for finite bounded Input (i.e $x(t) = u(t)$)

We get an infinite (unbounded) output

 **$m \leq n$ for BIBO**

$$H(s) \square \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} \square \frac{N(s)}{D(s)}$$

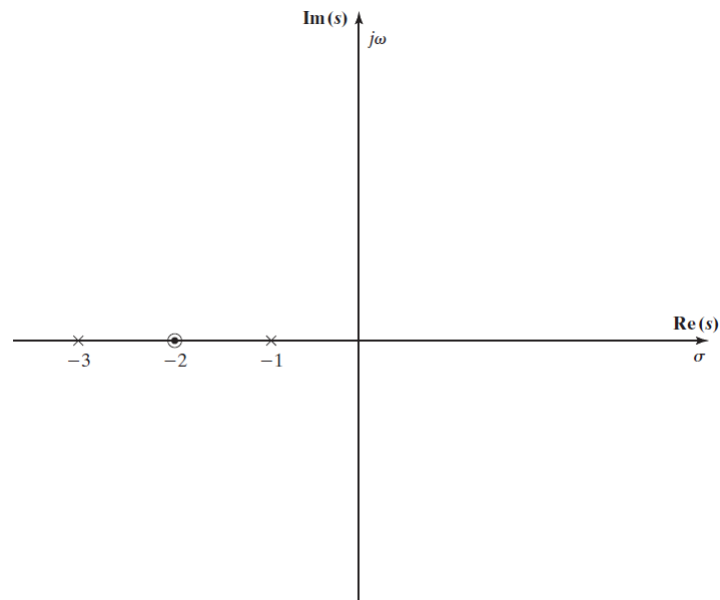
The poles of $H(s)$ must have real parts which are negative



The poles must lie in the left half of the s -plan

EXAMPLE 7.13**Poles and zeros of a transfer function**

$$H(s) = \frac{4s + 8}{2s^2 + 8s + 6} = \frac{2(s + 2)}{(s + 1)(s + 3)}$$



Convolution

$$h(t)*x(t) \Rightarrow H(s)X(s)$$

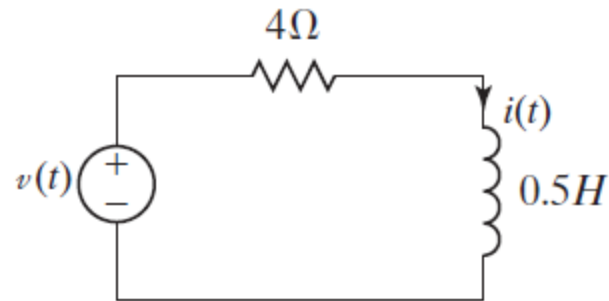
$$\mathcal{L}[x(t)*h(t)] = H(s)X(s)$$

Proof (See the book)

(It is very similar to the Fourier Transform Property)

EXAMPLE 7.12 LTI system response using Laplace transforms

Consider again the RL circuit



The loop equation (**KVL**) for this circuit is given by $0.5 \frac{di(t)}{dt} + 4i(t) = v(t)$

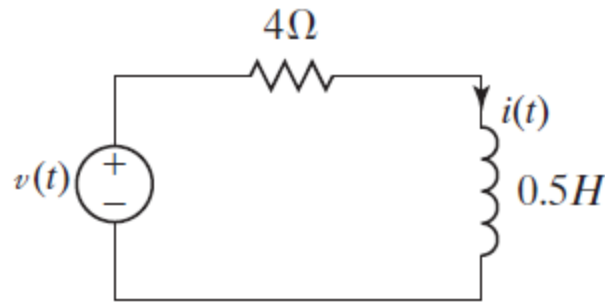
The Laplace transform of the loop equation (ignoring initial conditions) is given by

$$(0.5s + 4)I(s) = V(s)$$

We define the circuit input to be the voltage $v(t)$ and the output to be the current $i(t)$

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{0.5s + 4}$$

If $v(t) = 12u(t)$, find $i(t)$? $I(s) = H(s)V(s) = \frac{1}{0.5s + 4} \frac{12}{s} = \frac{24}{s(s + 8)}$



$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{0.5s + 4}$$

$$v(t) = 12u(t)$$

$$I(s) = \frac{24}{s(s + 8)}$$

Now using the Laplace Table 7.2 to find $i(t)$

The partial-fraction expansion of $I(s)$ is then

$$I(s) = \frac{24}{s(s + 8)} = \frac{k_1}{s} + \frac{k_2}{s + 8}$$

$$k_1 = s \left[\frac{24}{s(s + 8)} \right]_{s=0} = \frac{24}{s + 8} \Big|_{s=0} = 3$$

$$k_2 = (s + 8) \left[\frac{24}{s(s + 8)} \right]_{s=-8} = \frac{24}{s} \Big|_{s=-8} = -3$$

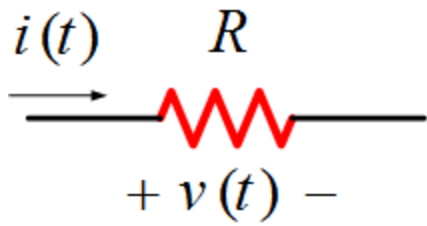
$$I(s) = \frac{24}{s(s + 8)} = \frac{3}{s} + \frac{-3}{s + 8}$$

The inverse transform, from **Table 7.2**

$$3 \leftarrow \frac{3}{s}$$

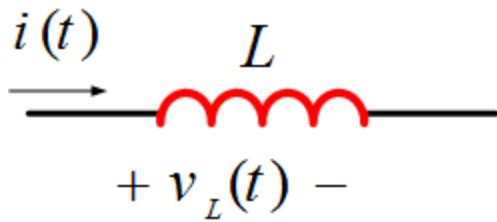
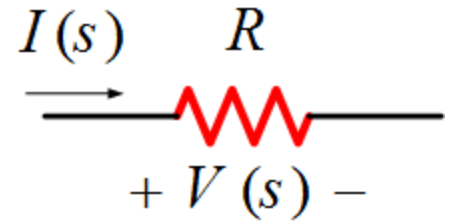
$$-3e^{-8t} \leftarrow \frac{-3}{s + 8}$$

$$\Rightarrow i(t) = 3[1 - e^{-8t}]$$



$$v(t) = Ri(t) \Leftrightarrow V(s) = RI(s)$$

$$Z_R = R \Omega$$

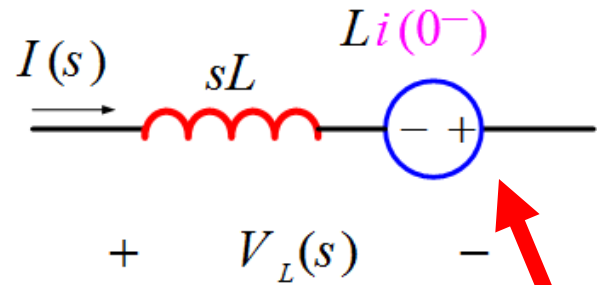


$$v_L(t) = L \frac{di(t)}{dt} \Leftrightarrow V_L(s) = L[sI(s) - i(0^-)]$$

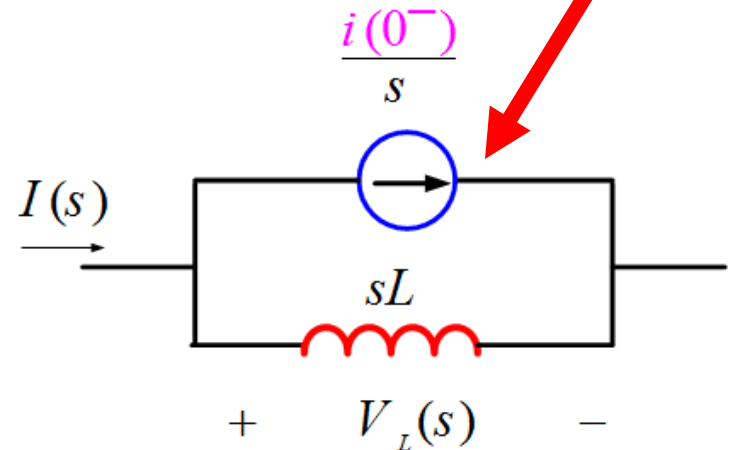
$$= sLI(s) - Li(0^-)$$

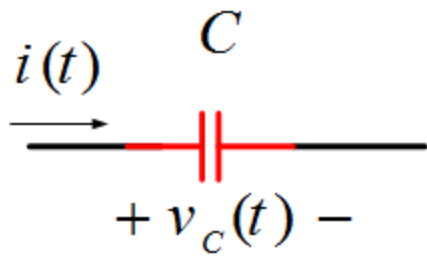
$$Z_L = sL \Omega$$

$$I(s) = \frac{1}{sL} V_L(s) + \frac{i(0^-)}{s}$$



Source Transformation



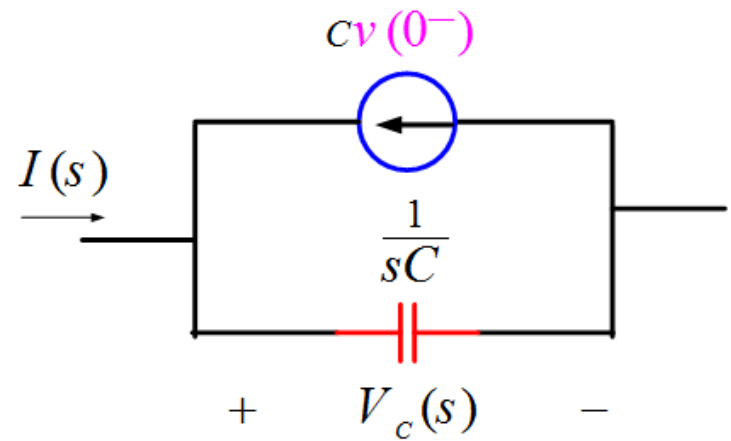


$$i(t) = C \frac{dv_c(t)}{dt} \Leftrightarrow I(s) = C[sV_c(s) - v(0^-)]$$

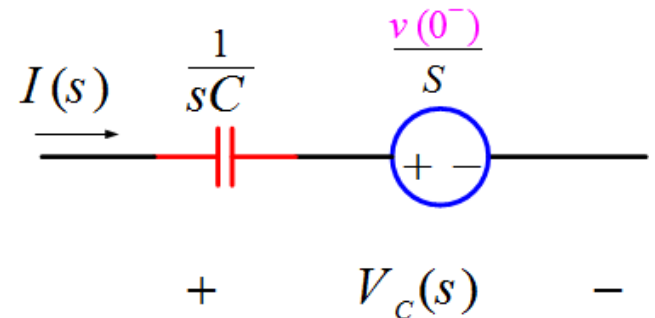
$$= sCV_c(s) - Cv(0^-)$$

$$Z_c = \frac{1}{sC} \Omega$$

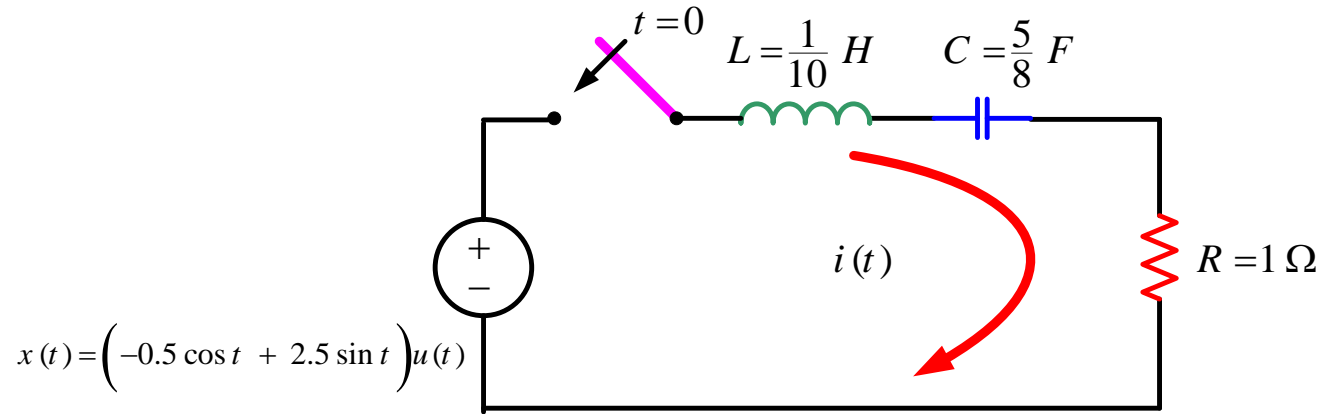
$$V_c(s) = \frac{1}{sC} I(s) - \frac{v(0^-)}{s}$$



Source Transformation



Example Using Laplace Transform Method find $i(t)$



$$i_L(0^-) = 0 \text{ V}$$

$$v_C(0^-) = -2 \text{ V}$$

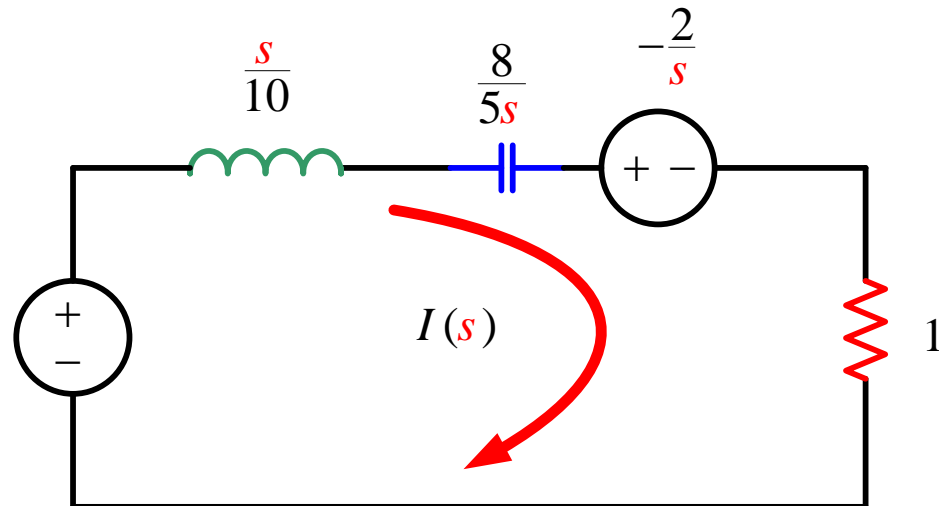
$$x(t) \Rightarrow X(s) = -\frac{0.5s}{s^2 + 1} + \frac{2.5s}{s^2 + 1}$$

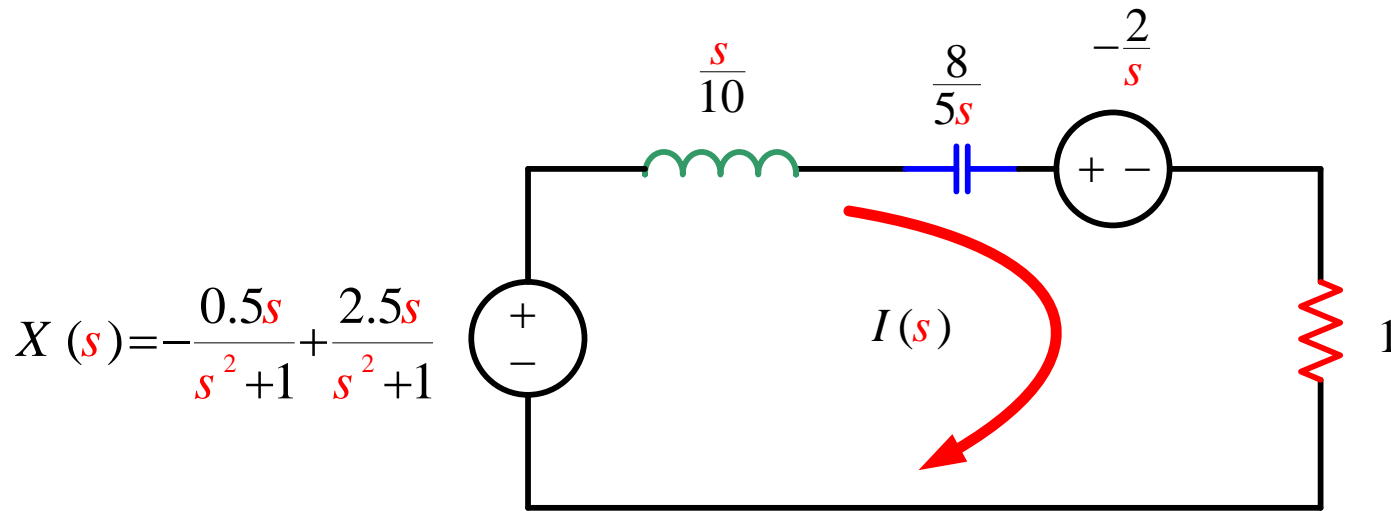
$$L = \frac{1}{10} \text{ H} \Rightarrow Z_L = \frac{s}{10} \Omega$$

$$C = \frac{5}{8} \text{ F} \Rightarrow Z_C = \frac{8}{5s} \Omega$$

$$R = 1 \Omega \Rightarrow Z_R = 1 \Omega$$

$$X(s) = -\frac{0.5s}{s^2 + 1} + \frac{2.5s}{s^2 + 1}$$

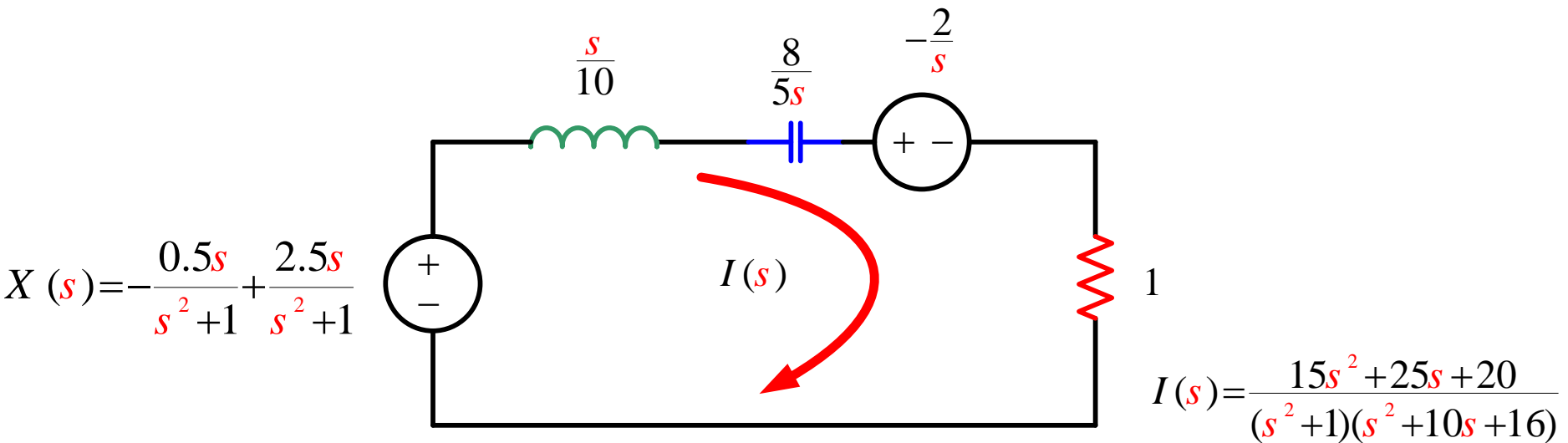




KVL $\Rightarrow -X(s) + \left(\frac{s}{10}\right)I(s) + \left(\frac{8}{5s}\right)I(s) - \frac{2}{s} + (1)I(s) = 0$

$\Rightarrow -\frac{0.5s}{s^2+1} + \frac{2.5s}{s^2+1} = \left(\frac{s}{10} + \frac{8}{5s} + 1\right)I(s) - \frac{2}{s}$

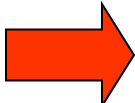
$\Rightarrow I(s) = \frac{15s^2 + 25s + 20}{(s^2+1)(s^2+10s+16)}$



$$I(s) = \frac{15s^2 + 25s + 20}{(s^2+1)(s^2+10s+16)} = \frac{(15s^2 + 25s + 20)}{(s+j)(s-j)(s+2)(s+8)}$$

$$= \frac{A_1}{(s+j)} + \frac{A_2}{(s-j)} + \frac{A_3}{(s+2)} + \frac{A_4}{(s+8)}$$

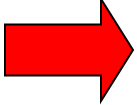
(Imaginary Roots)


 $i(t) = (\cos t + \sin t + e^{-2t} + e^{-8t})u(t)$

Inversion of Rational Function (Inverse Laplace Transform)

Let $Y(s)$ be Laplace Transform of some function $y(t)$.

We want to find $y(t)$ without using the inversion formula .

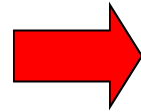
 We want to find $y(t)$ using the Laplace Transform known table and properties

Objective : Put $Y(s)$ in a form or a sum of forms that we know it is in the Laplace Transform Table

$Y(s)$ in general is a ratio of two polynomials  Rational Function

Example $Y(s) = \frac{s^2 - 2s}{2s^3 - 5s^2 + 3s + 2}$

When the degree of the numerator of rational function is less the Degree of the dominator



Proper Rational Function

Highest Degree is 2

Example $Y(s) = \frac{s^2 - 2s}{2s^3 - 5s^2 + 3s + 2}$

Highest Degree is 3

Examples of proper rational Functions

$$Y_1(s) = \frac{1}{s+1}$$

$$Y_2(s) = \frac{2s^2 + 6s + 6}{(s+2)(s^2 + 2s^2 + 2)}$$

Examples of not proper rational Functions

$$Y_3(s) = \frac{s+2}{s+1}$$

However we can obtain a proper rational Function through long division

$$Y_3(s) = \frac{s+2}{s+1} = 1 + \frac{1}{s+1}$$

We will discuss different techniques of factoring $Y(s)$ into simple known forms

Simple Factors

$$\text{Let } Y(s) = \frac{10}{(s^2 + 10s + 16)}$$

If we check the Table, we see there is no form similar to $Y(s)$

However if we expand $Y(s)$ in partial fractions:

$$\frac{10}{(s^2 + 10s + 16)} = \frac{A}{(s + 2)} + \frac{B}{(s + 8)}$$

$\frac{A}{(s + 2)}$ and $\frac{B}{(s + 8)}$ Are available on the Table

Next we develop Techniques of finding A and B

Heaviside's Expansion Theorem

$$Y(s) = \frac{10}{(s^2+10s+16)} = \frac{10}{(s+2)(s+8)} = \frac{A}{(s+2)} + \frac{B}{(s+8)}$$

Multiply (X) both side by (s+2) and set s = -2

$$\frac{10}{\cancel{(s+2)}(s+8)} \mathbf{X}(\cancel{s+2}) \Big|_{s=-2} = \frac{A}{\cancel{(s+2)}} \mathbf{X}(\cancel{s+2}) \Big|_{s=-2} + \frac{B}{(s+8)} \mathbf{X}(s+2) \Big|_{s=-2}$$

$$\rightarrow \frac{10}{(-2+8)} = A + \frac{B(-2+2)}{(-2+8)}$$

$$\rightarrow \frac{10}{(6)} = A + 0 \rightarrow A = \frac{5}{3}$$

Similarly Multiply both side by (s+8) and set s = -8

$$\frac{10}{(s+2)\cancel{(s+8)}} \mathbf{X}(\cancel{s+8}) \Big|_{s=-8} = \frac{A}{(s+2)} \mathbf{X}(s+8) \Big|_{s=-8} + \frac{B}{\cancel{(s+8)}} \mathbf{X}(\cancel{s+8}) \Big|_{s=-8}$$

$$\rightarrow B = -\frac{5}{3}$$

$$\rightarrow y(t) = \frac{5}{3}e^{-2t}u(t) - \frac{5}{3}e^{-8t}u(t) = \frac{5}{3}(e^{-2t} - e^{-8t})u(t)$$

Imaginary Roots

$$\begin{aligned}\text{Let } Y(s) &= \frac{(15s^2 + 25s + 20)}{(s^2 + 1)(s^2 + 10s + 16)} = \frac{(15s^2 + 25s + 20)}{(s + j)(s - j)(s + 2)(s + 8)} \\ &= \frac{A_1}{(s + j)} + \frac{A_2}{(s - j)} + \frac{A_3}{(s + 2)} + \frac{A_4}{(s + 8)}\end{aligned}$$

Using Heaviside's Expansions, by multiplying the left hand side and Right hand side by the factors

$$(s + j), (s - j), (s + 2), (s + 8)$$

and substitute $s = -j$, $s = j$, $s = -2$, $s = -8$ respectively

$$\text{We obtain } A_1 = \frac{1}{2}(1 + j), A_2 = \frac{1}{2}(1 - j), A_3 = 1, A_4 = -2$$

From Table

$$Y(s) = \frac{(1/2)(1+j)}{(s+j)} + \frac{(1/2)(1-j)}{(s-j)} + \frac{1}{(s+2)} - \frac{2}{(s+8)}$$

$e^{-2t}u(t)$ $e^{-8t}u(t)$

$$\frac{(1/2)(1+j)}{(s+j)} \quad \text{and} \quad \frac{(1/2)(1-j)}{(s-j)}$$

Can be inverted in two methods:

(a) $\frac{(1/2)(1+j)}{(s+j)} \Rightarrow (1/2)(1+j) e^{-jt}u(t)$

$$\frac{(1/2)(1-j)}{(s-j)} \Rightarrow (1/2)(1-j) e^{jt}u(t)$$

combine

$$\frac{(1/2)(1+j)}{(s+j)} \Rightarrow (1/2)(1+j) e^{-jt} u(t)$$

$$\frac{(1/2)(1-j)}{(s-j)} \Rightarrow (1/2)(1-j) e^{jt} u(t)$$

$$\frac{(1/2)(1+j)}{(s+j)} + \frac{(1/2)(1-j)}{(s-j)} \Rightarrow \frac{1}{2}(1+j) e^{-jt} u(t) + \frac{1}{2}(1-j) e^{jt} u(t)$$

$$= \frac{1}{2}(e^{jt} + e^{-jt}) u(t) + \frac{j}{2}(e^{-jt} - e^{jt}) u(t)$$

$$= \frac{1}{2}(e^{jt} + e^{-jt}) u(t) + \frac{1}{2j}(e^{jt} - e^{-jt}) u(t)$$

$$= \cos t u(t) + \sin t u(t)$$

(b) $\frac{(1/2)(1+j)}{(s+j)}$ and $\frac{(1/2)(1-j)}{(s-j)}$

Can be combined as

$$\frac{(1/2)(1+j)}{(s+j)} + \frac{(1/2)(1-j)}{(s-j)}$$

$$= \frac{(1/2)(1+j)(s-j) + (1/2)(1-j)(s+j)}{(s+j)(s-j)}$$

$$= \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1} \Rightarrow \cos t) u(t) + \sin t) u(t)$$

$$\begin{array}{c}
 \cos t \, u(t) + \sin t \, u(t) \quad e^{-2t} u(t) \quad e^{-8t} u(t) \\
 \underbrace{\hspace{15em}} \\
 Y(s) = \frac{(1/2)(1+j)}{(s+j)} + \frac{(1/2)(1-j)}{(s-j)} + \frac{1}{(s+2)} - \frac{2}{(s+8)}
 \end{array}$$

$$y(t) = \cos t \, u(t) + \sin t \, u(t) + e^{-2t} u(t) + e^{-8t} u(t)$$

$$y(t) = (\cos t + \sin t + e^{-2t} + e^{-8t}) u(t)$$

Repeated Linear Factor

$$\text{If } Y(s) = \frac{P(s)}{(s+\alpha)^n Q(s)} \quad \left\{ \text{example } Y(s) = \frac{10s}{(s+2)^2(s+8)} \right\}$$

Then its partial fraction

$$Y(s) = \frac{A_1}{(s+\alpha)} + \frac{A_2}{(s+\alpha)^2} + \frac{A_3}{(s+\alpha)^3} + \dots + \frac{A_n}{(s+\alpha)^n} + \frac{R(s)}{Q(s)}$$

Where

$$A_m = \frac{1}{(n-m)!} \frac{d^{(n-m)}}{ds^{(n-m)}} \left[(s+\alpha)^n Y(s) \right]_{s=-\alpha}$$

Repeated Linear Factor

Let $Y(s) = \frac{10s}{(s+2)^2(s+8)}$

Also A_2 can be found using **Heavisdie's** by multiplying both sides by $(s+2)^2$

B can be found using **Heavisdie's** by multiplying both sides by $(s+8)$

Then $Y(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{B}{(s+8)}$

$n=2, m=1$ (pointing to A_1)
 $n=2, m=2$ (pointing to A_2)

$$A_m = \frac{1}{(n-m)!} \frac{d^{(n-m)}}{ds^{(n-m)}} \left[(s+\alpha)^n Y(s) \right]_{s=-\alpha}$$

$$A_1 = \frac{1}{(2-1)!} \frac{d^{(2-1)}}{ds^{(2-1)}} \left[\cancel{(s+2)^2} \frac{10s}{\cancel{(s+2)^2} (s+8)} \right]_{s=-2} = \frac{d}{ds} \left[\frac{10s}{(s+8)} \right]_{s=-2}$$

$$= \left[\frac{10(s+8) - 10s}{(s+8)^2} \right]_{s=-2} = \left[\frac{10(-2+8) - 10(-2)}{(-2+8)^2} \right] = \left[\frac{80}{36} \right] = \frac{20}{9}$$

$$Y(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{B}{(s+8)}$$

$\frac{20}{9}$
 $-\frac{10}{3}$

$$A_2 = \frac{1}{(2-2)!} \frac{d^{(2-2)}}{ds^{(2-2)}} \left[\frac{10s}{(s+2)^2 (s+8)} \right]_{s=-2} = \left[\frac{10s}{(s+8)} \right]_{s=-2}$$

$$= \left[\frac{10(-2)}{(-2+8)} \right] = \left[\frac{-20}{6} \right] = -\frac{10}{3}$$

$$Y(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{B}{(s+8)}$$

$\frac{20}{9}$
 $-\frac{10}{3}$
 $-\frac{20}{9}$

To find B , we use Heaviside

$$\cancel{(s+8)} \frac{10s}{(s+2)^2 \cancel{(s+8)}} \Big|_{s=-8} = \frac{A_1}{(s+2)} \cancel{X(s+8)} \Big|_{s=-8} + \frac{A_2}{(s+2)^2} \cancel{X(s+8)} \Big|_{s=-8} + \frac{B}{\cancel{(s+8)}} \cancel{X(s+8)} \Big|_{s=-8}$$

$$\Rightarrow B = -\frac{20}{9} \quad \Rightarrow Y(s) = \frac{(20/9)}{(s+2)} - \frac{(10/3)}{(s+2)^2} - \frac{(20/9)}{(s+8)}$$

$$Y(s) = \frac{20}{9} \left[-\frac{1}{(s+8)} + \frac{1}{(s+2)} - \frac{(3/2)}{(s+2)^2} \right]$$

$$y(t) = \frac{20}{9} \left(-e^{-8t} + e^{-2t} - \frac{3}{2} t e^{-2t} \right) u(t)$$

Repeated Linear Factor

Let $Y(s) = \frac{10s}{(s+2)^3(s+8)}$

Can be found using Heaviside's expansion



Then $Y(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)^3} + \frac{B}{(s+8)}$

$$B = (s+8) Y(s) \Big|_{s=-8} = \frac{10}{27}$$

$$A_3 = (s+2)^3 Y(s) \Big|_{s=-2} = -\frac{10}{3}$$

$$A_2 = \frac{1}{(3-2)!} \frac{d^{(3-2)}}{ds^{(3-2)}} \left[(s+2)^3 Y(s) \right]_{s=-2} = \frac{d}{ds} \left[\frac{10s}{(s+8)} \right]_{s=-2} = \frac{20}{9}$$

$$Y(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)^3} + \frac{B}{(s+8)}$$

$\frac{20}{27}$
 $\frac{20}{9}$
 $\frac{10}{3}$
 $\frac{10}{27}$

A_1 Can be found using Heaviside differentiation techniques

$$A_1 = \frac{1}{(3-1)!} \frac{d^{(3-1)}}{ds^{(3-1)}} \left[(s+2)^3 Y(s) \right]_{s=-2} = \frac{1}{2} \frac{d^2}{ds^2} \left[\frac{10s}{(s+8)} \right]_{s=-2}$$

$$= \frac{1}{2} \left[\frac{-160}{(s+8)^3} \right]_{s=-2} = -\frac{20}{27}$$

$$Y(s) = \frac{10}{27} \left[\frac{1}{(s+8)} - \frac{1}{(s+2)} \right] + \frac{20}{9} \frac{1}{(s+2)^2} - \frac{10}{3} \frac{1}{(s+2)^3}$$

$$\Rightarrow y(t) = \left[\frac{10}{27} (e^{-8t} - e^{-2t}) - \frac{5}{3} t \left(t - \frac{4}{3} \right) e^{-2t} \right] u(t)$$