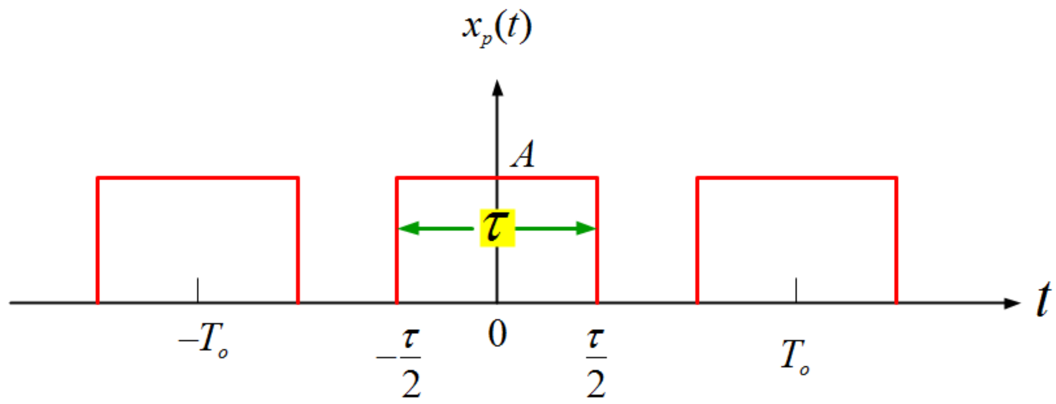
EE 207 Dr. Adil Balghonaim

Chapter 4 The Fourier Transform

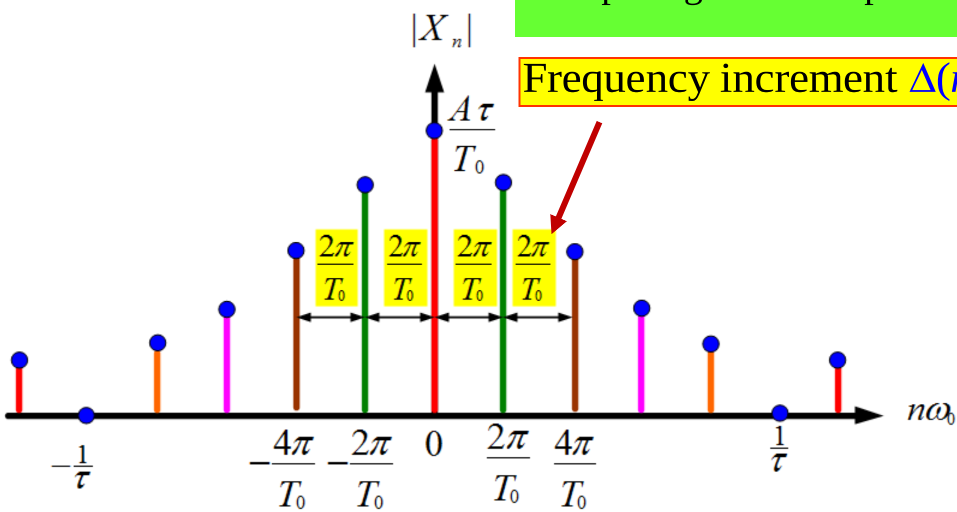


$$X_n = \frac{A\tau}{T_0}\,\mathrm{sinc}\left(\frac{\tau n\omega_0}{2}\right)$$

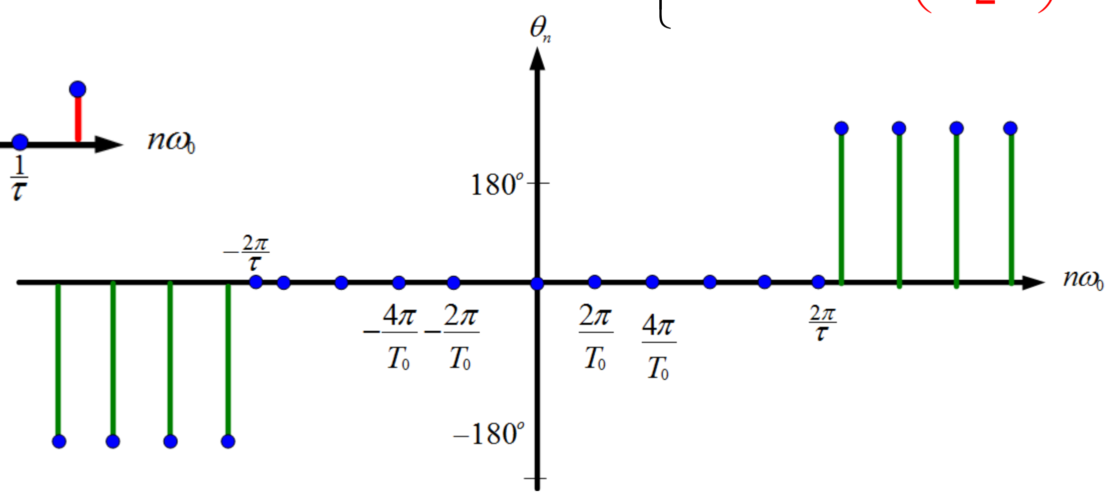
$$|X_n| = \frac{A\tau}{T_0}\left|\,\mathrm{sinc}\left(\frac{\tau n\omega_0}{2}\right)\right|$$

The spacing between spectrum lines is $\frac{2\pi}{T_0} = \omega_0$

Frequency increment $\Delta(n\omega_0) = \omega_0$



$$\theta_n = \begin{cases} 0 & \mathrm{sinc}\left(\frac{n\omega_0\tau}{2}\right) > 0 \\ 180^o & \mathrm{sinc}\left(\frac{n\omega_0\tau}{2}\right) < 0 \end{cases}$$



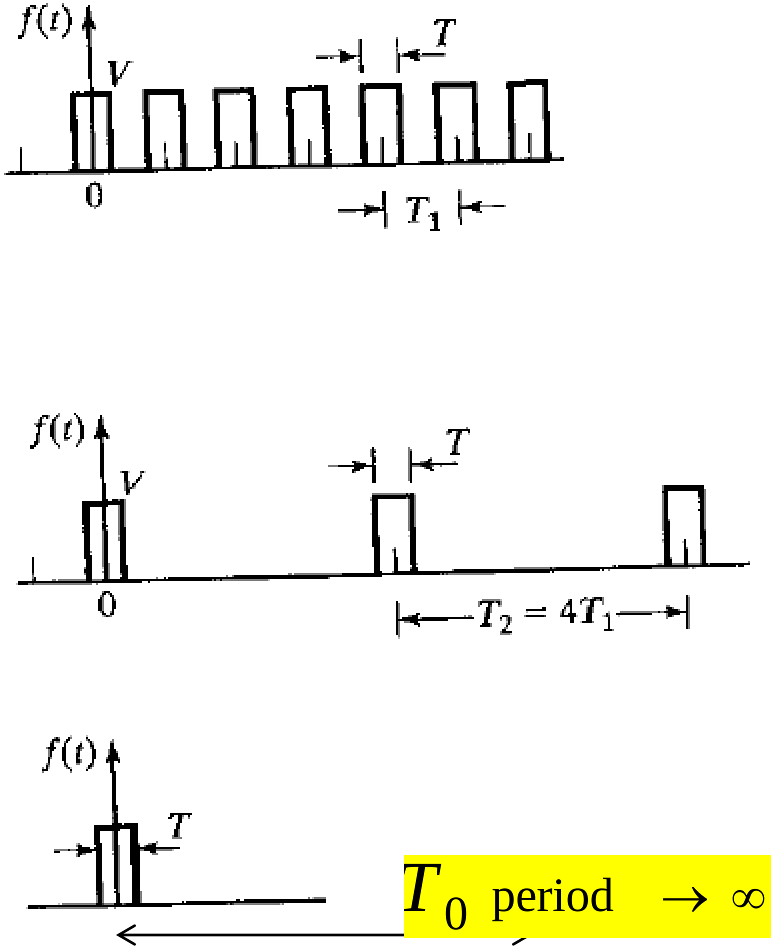
Let $x_p(t)$ be a periodical wave, then expanding the periodical function

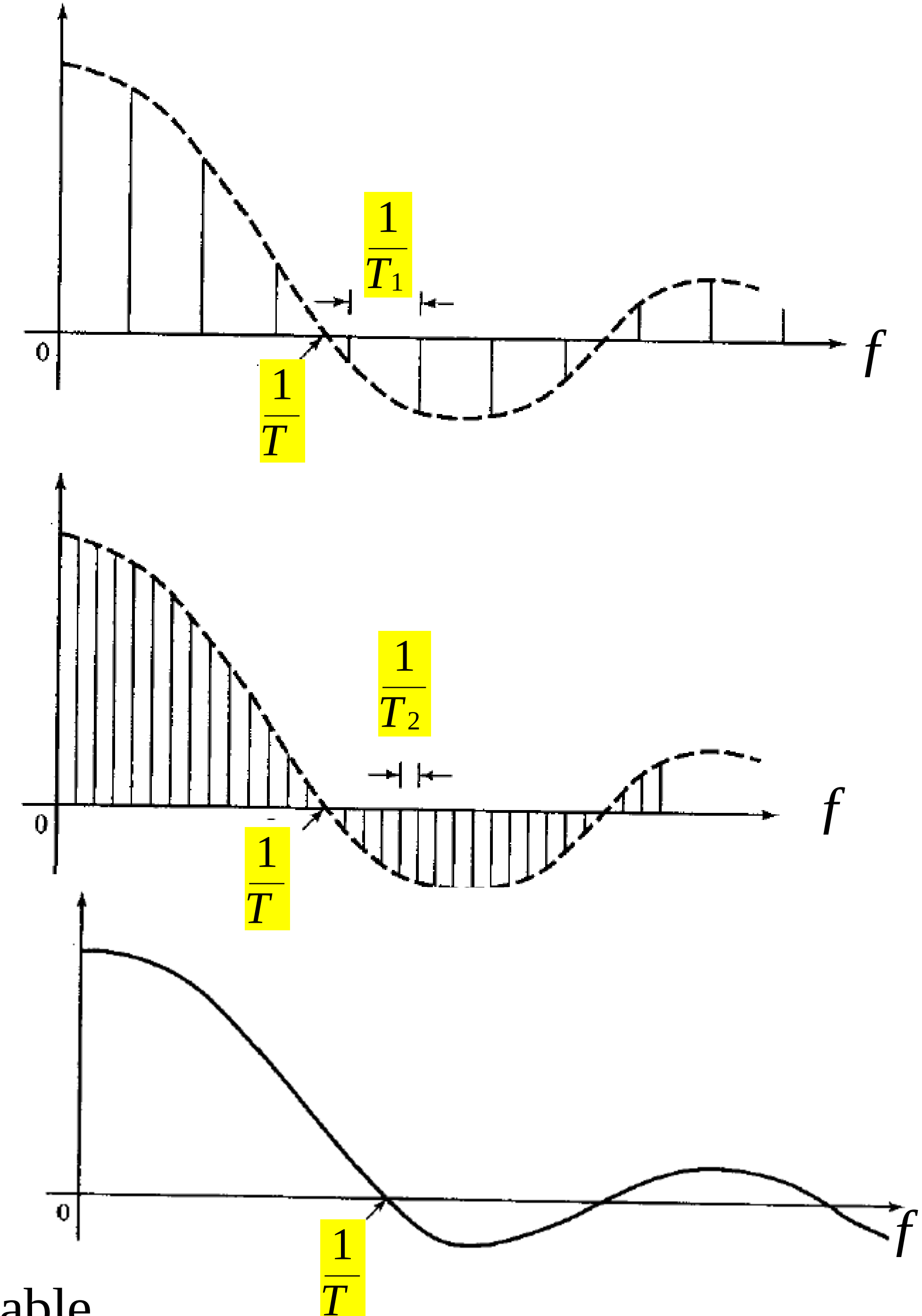
$$x_p(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt = \frac{\omega_o}{2\pi}\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x(t) e^{-jn\omega_0 t} dt$$

Rewriting $x_p(t)$ and X_n

$$x_p(t) = \sum_{n=-\infty}^{\infty} \frac{X_n}{\omega_0} e^{jn\omega_0 t} \Delta(n\omega_0) = \sum_{n\omega_0=-\infty}^{\infty} \frac{X_n}{\omega_0} e^{jn\omega_0 t} \Delta(n\omega_0)$$

$$\tilde{X}(nf_0) \square \frac{X_n}{\omega_0} = \frac{1}{2\pi}\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x(t) e^{-jn\omega_0 t} dt$$



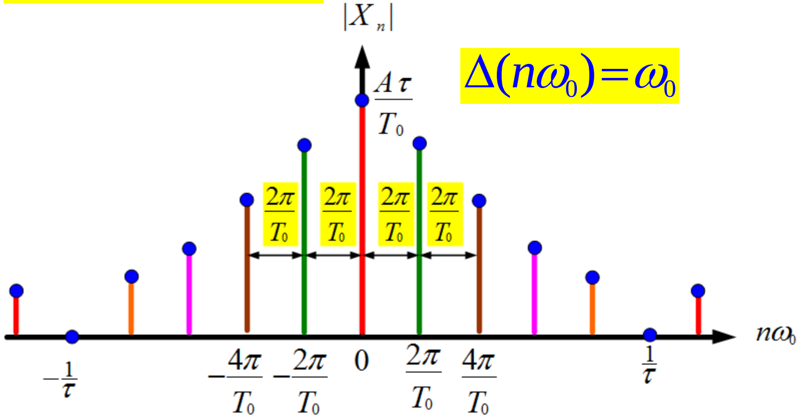


$$T_0 \rightarrow \infty \quad \Rightarrow \quad \omega_0 \rightarrow 0$$

$$n \rightarrow \infty$$

$$\Rightarrow \quad n\omega_0 \rightarrow \omega$$ Continuous Variable

$x_p(t)$ **Periodic Signal**



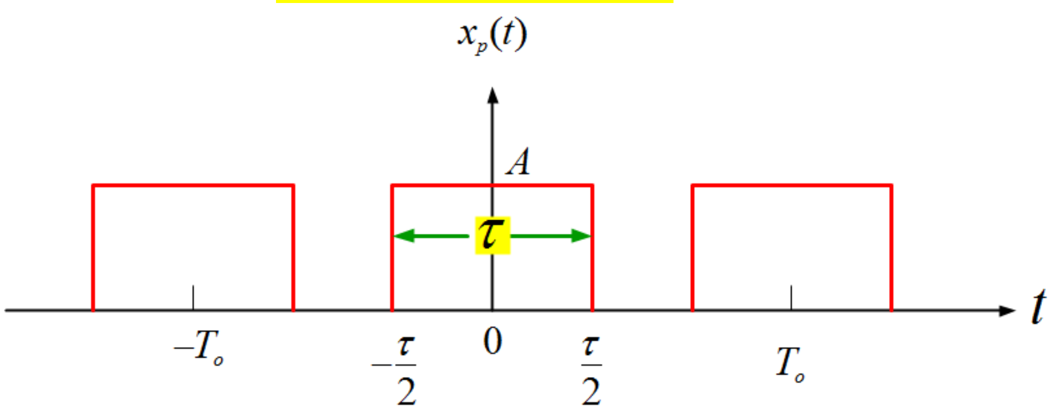
$\Delta(n\omega_0)=\omega_0$

$$x_p(t)=\sum_{n\omega_0=-\infty}^{\infty}\left(\frac{X_n}{\omega_0}\right)e^{jn\omega_0 t}\,\Delta(n\omega_0)$$

$$\tilde{X}(n\omega_0)\triangleq\left(\frac{X_n}{\omega_0}\right)=\frac{1}{2\pi}\int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}}x(t)e^{-jn\omega_0 t}\,dt$$

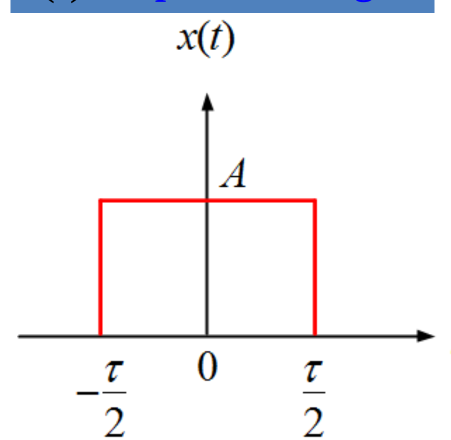
$$\Rightarrow\quad x_p(t)=\sum_{n\omega_0=-\infty}^{\infty}\tilde{X}(n\omega_0)\,e^{jn\omega_0 t}\,\Delta(n\omega_0)$$

$x_p(t)$ **Periodic Signal**

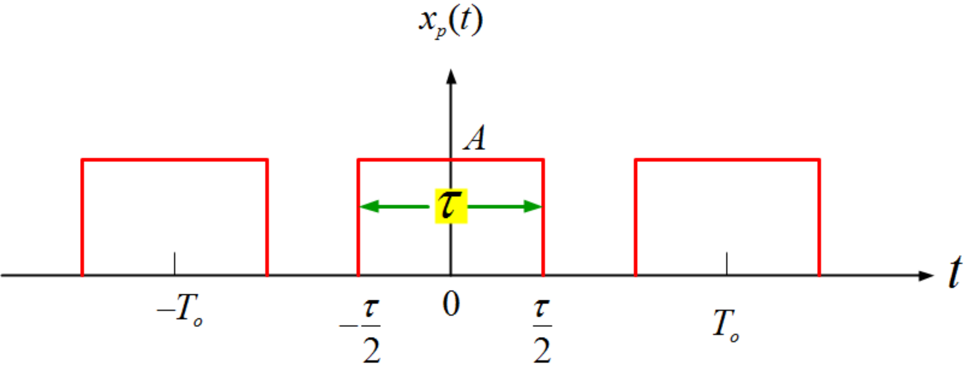


$T_0\rightarrow\infty$

$x(t)$ **Aperiodic Signal**

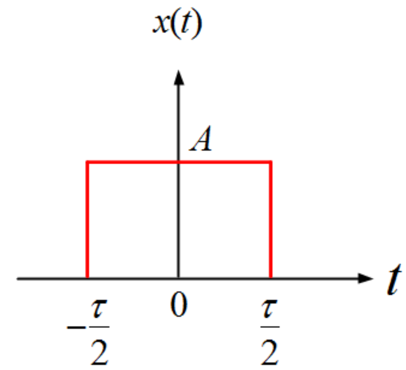


$x_p(t)$ **Periodic Power Signal**



$T_0 \to \infty$

$x(t)$ **Aperiodic Energy Signal**



$$x_p(t) = \sum_{n\omega_0=-\infty}^{\infty} \tilde{X}(n\omega_0)\, e^{jn\omega_0 t}\, \Delta(n\omega_0)$$

$$\tilde{X}(n\omega_0) \triangleq \left(\frac{X_n}{\omega_0}\right) = \frac{1}{2\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x(t) e^{-jn\omega_0 t} dt$$

$$\Rightarrow \quad \omega_0 \to 0 \qquad n\omega_0 \to \omega$$

$$\Delta(n\omega_0) \to d\omega \qquad \sum \to \int$$

$$x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform Pairs

Fourier Transform Pairs

$$x(t)=\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \qquad X(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Sufficient conditions for the existence of the Fourier transform are **(Dirichlet conditions)**

1. On any finite interval,

a. $f(t)$ is bounded;

b. $f(t)$ has a finite number of maxima and minima; and

c. $f(t)$ has a finite number of discontinuities.

2. f (t) is absolutely integrable; that is, $\int_{-\infty}^{\infty} |f(t)|dt < \infty$

Note that these are **sufficient** conditions and not **necessary** conditions

Note you can have a function that is not absolutely integrable however it has Fourier Transform like cos(ωt) (will be shown later)

Fourier Transform Pairs

$$x(t)=\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad X(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

absolutely integrable; that is, $\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$

Examples of functions that is not absolutely integrable

e^{-t}, $\cos(\omega t)$, $\sin(\omega t)$, $u(t)$

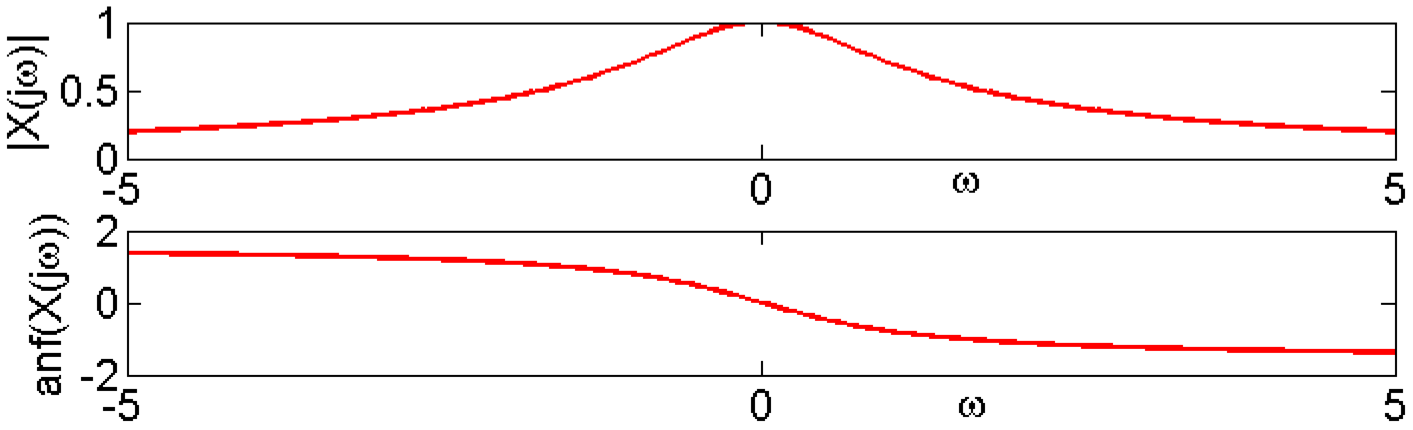
e^{-t} does not have Fourier Transform however $e^{-t}u(t)$ does have

$\cos(\omega t)$, $\sin(\omega t)$, $u(t)$ has Fourier Transform (will be shown later)

Finding the Fourier Transform

Exponential Pulse $x(t) = e^{-at}u(t) \qquad a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \frac{1}{-(a+j\omega)}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{(a+j\omega)}$$



$a = 1$

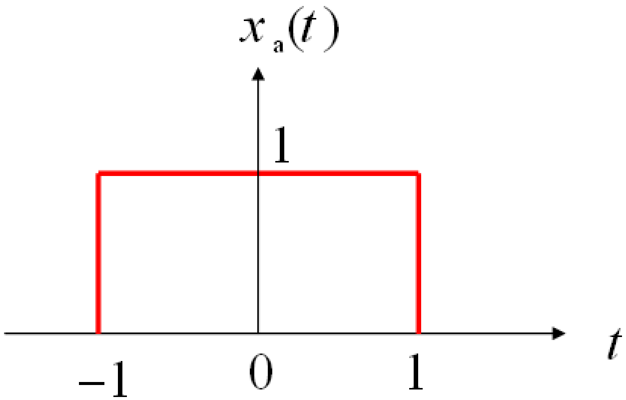
$x(t) = e^{-at}u(t) \qquad a > 0 \quad \Longleftrightarrow \quad X(j\omega) = \frac{1}{(a+j\omega)}$

$$x(t) = e^{-at}u(t) \qquad a > 0 \quad \Longleftrightarrow \quad X(j\omega) = \frac{1}{(a + j\omega)}$$

TABLE 5.2 Fourier Transform Pairs

Time Domain Signal	Fourier Transform
$\sin(\omega_0 t)u(t)$	$\frac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$
$\text{rect}(t/T)\cos(\omega_0 t)$	$\frac{T}{2}\left[\text{sinc}\left(\frac{(\omega - \omega_0)T}{2}\right) + \text{sinc}\left(\frac{(\omega + \omega_0)T}{2}\right)\right]$
$\frac{\beta}{\pi}\text{sinc}(\beta t)$	$\text{rect}(\omega/2\beta)$
$\text{tri}(t/T)$	$T\,\text{sinc}^2(T\omega/2)$
$\text{sinc}^2(Tt/2)$	$\frac{2\pi}{T}\text{tri}(\omega/T)$
$e^{-at}u(t)$, $\text{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$
$te^{-at}u(t)$, $\text{Re}\{a\} > 0$	$\left(\frac{1}{a + j\omega}\right)^2$
$t^{n-1}e^{-at}u(t)$, $\text{Re}\{a\} > 0$	$\frac{(n-1)!}{(a + j\omega)^n}$
$e^{-a\|t\|}$, $\text{Re}\{a\} > 0$	$\frac{2a}{a^2 + \omega^2}$
$\sum_{n=-\infty}^{\infty} g(t - nT_0)$	$\sum_{n=-\infty}^{\infty} \omega_0 G(n\omega_0)\delta(\omega - n\omega_0)$, $\omega_0 = \frac{2\pi}{T_0}$
$\sum_{n=-\infty}^{\infty} g(t - nT_0) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$	$2\pi \sum_{n=-\infty}^{\infty} C_n\delta(\omega - n\omega_0)$, $\omega_0 = \frac{2\pi}{T_0}$
$\delta_T(t)$	$\sum_{k=-\infty}^{\infty} \omega_0\delta(\omega - k\omega_0)$

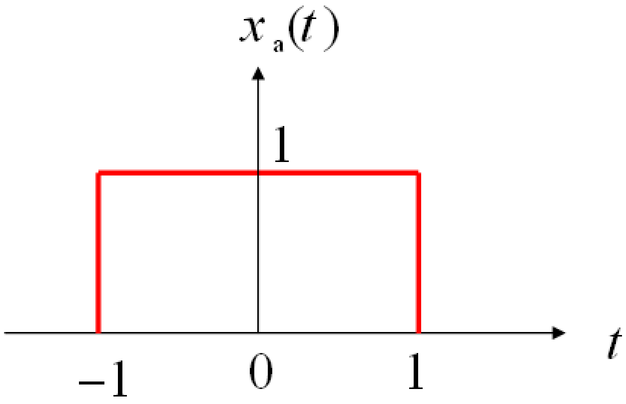
Example Find the Fourier Transform for the following function



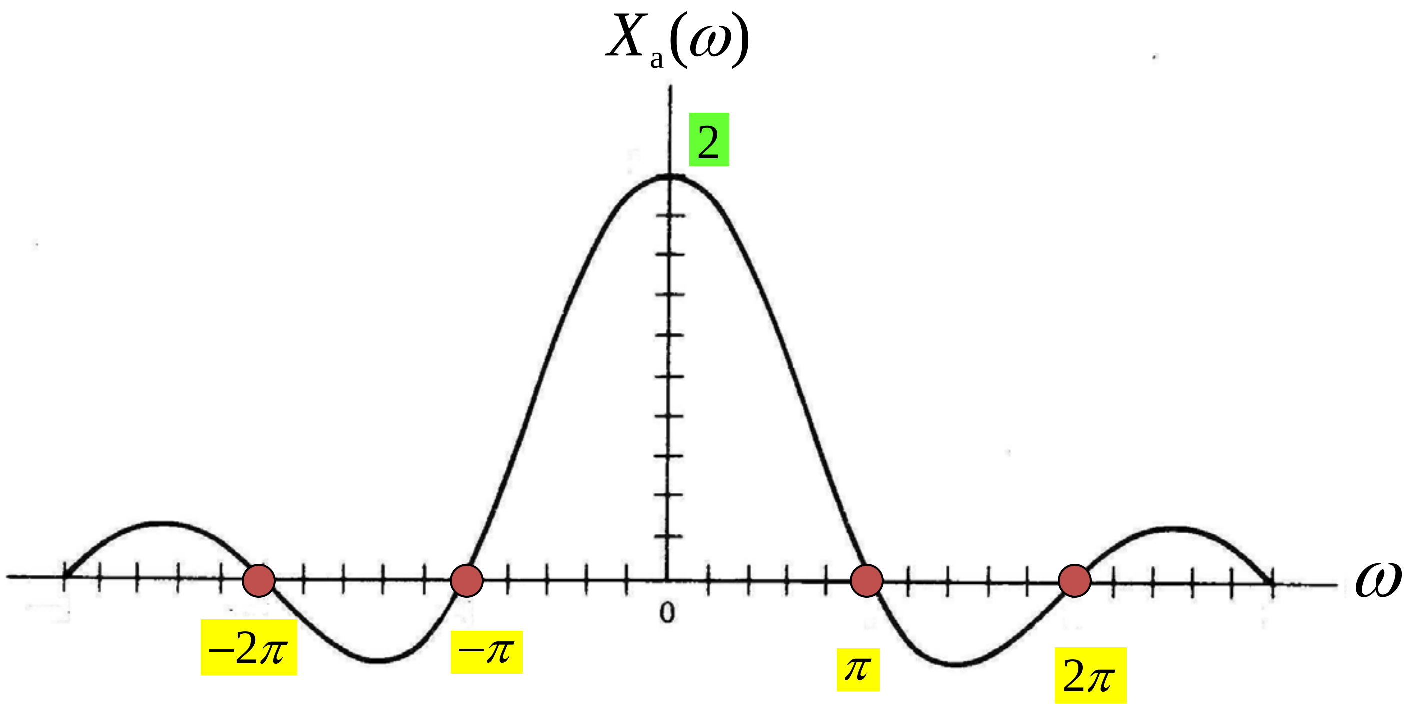
$$X_a(\omega)=\int_{-\infty}^{\infty} x_a(t)e^{-j\omega t}dt = \int_{-1}^{1} e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1}^{1} = \frac{e^{-j\omega(1)}-e^{-j\omega(-1)}}{-j\omega}$$

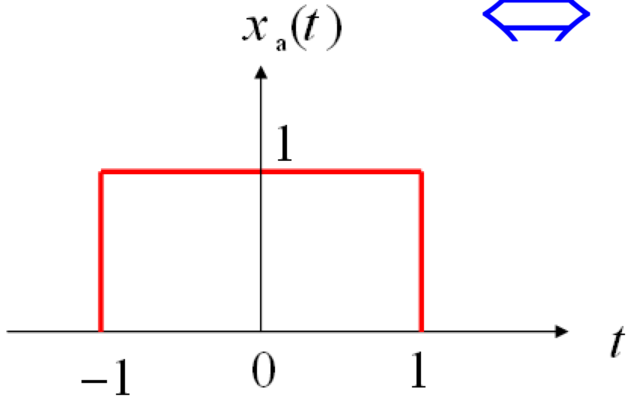
$$=\frac{e^{j\omega}-e^{-j\omega}}{j\omega}=\left(\frac{2}{\omega}\right)\frac{e^{j\omega}-e^{-j\omega}}{j2}=\left(\frac{2}{\omega}\right)\sin(\omega)=2\frac{\sin(\omega)}{\omega}$$

$$=2\operatorname{sinc}(\omega)$$

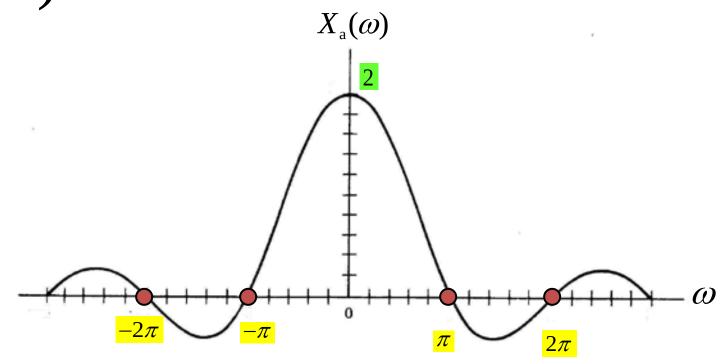


$$\Leftrightarrow \quad X_a(\omega) = 2\,\mathrm{sinc}(\omega)$$

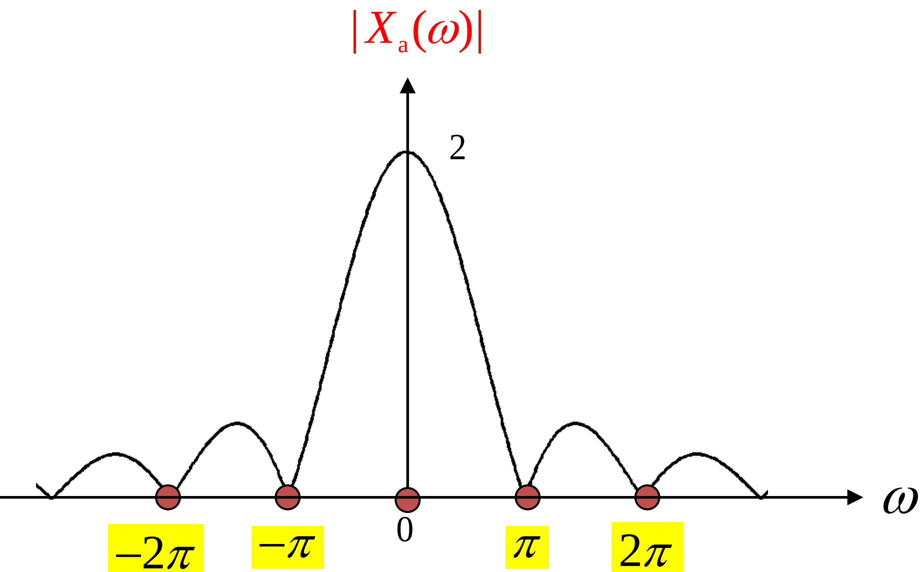


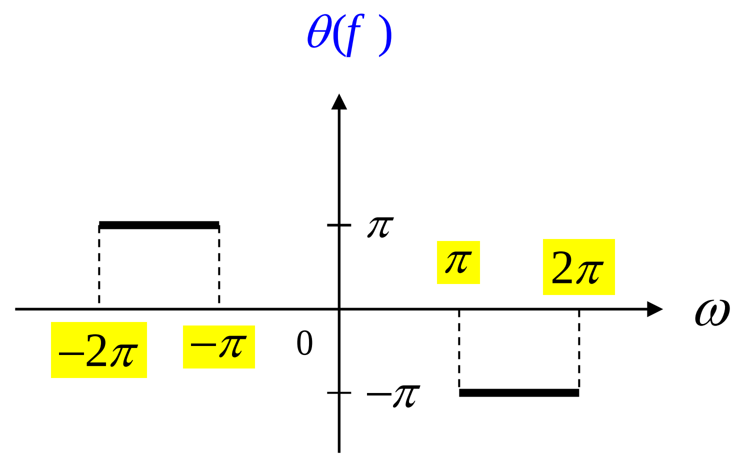


$$\Leftrightarrow \quad X_a(\omega) = 2\sin\ (\omega)$$

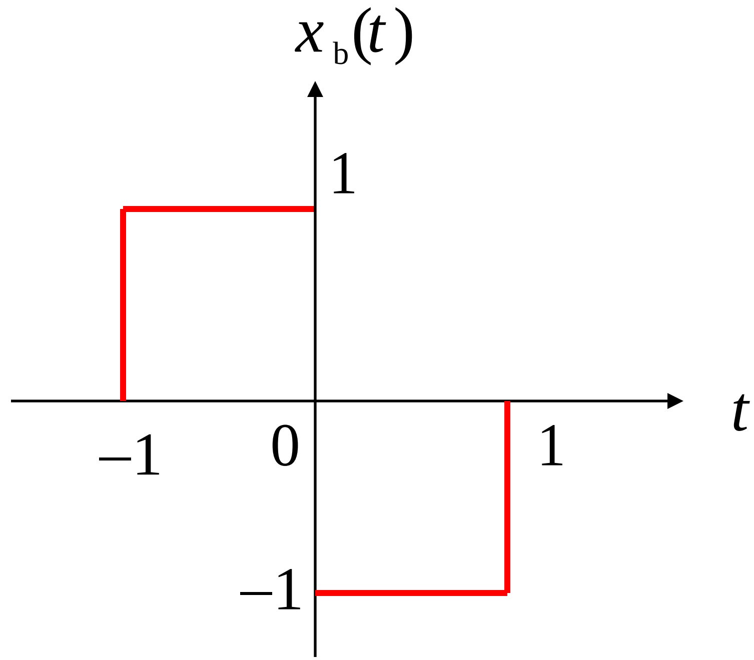


$$X_a(\omega) = |X_a(\omega)| e^{j\theta(\omega)}$$





Example

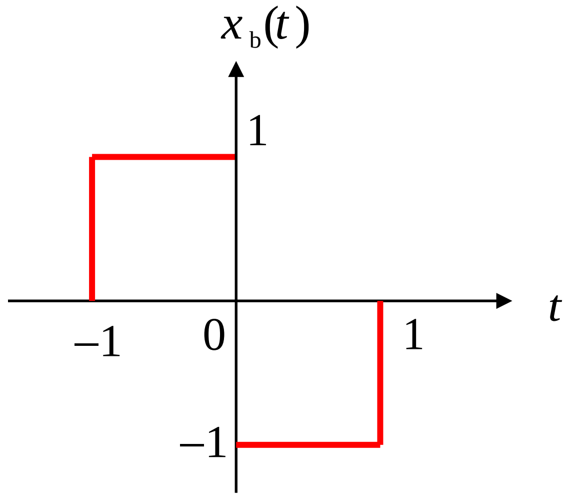


$$X_b(\omega) = \int_{-\infty}^{\infty} x_b(t) e^{-j\omega t} dt$$

$$= \int_{-1}^{0} (1)\, e^{-j\omega t} dt + \int_{0}^{1} (-1)\, e^{-\omega t} dt \quad = j\omega \operatorname{sinc}^2\left(\frac{\omega}{2}\right)$$

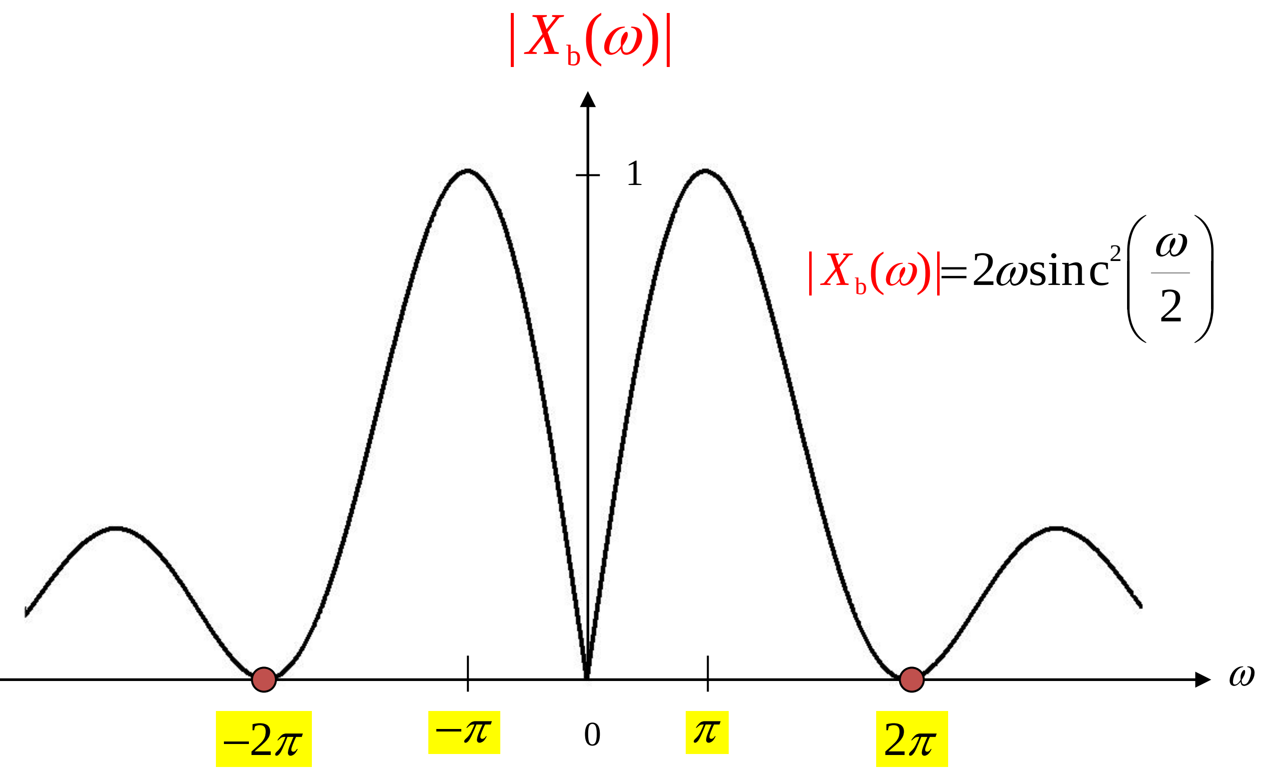
$$X_b(\omega) = |X_b(\omega)| e^{j\theta(\omega)}$$

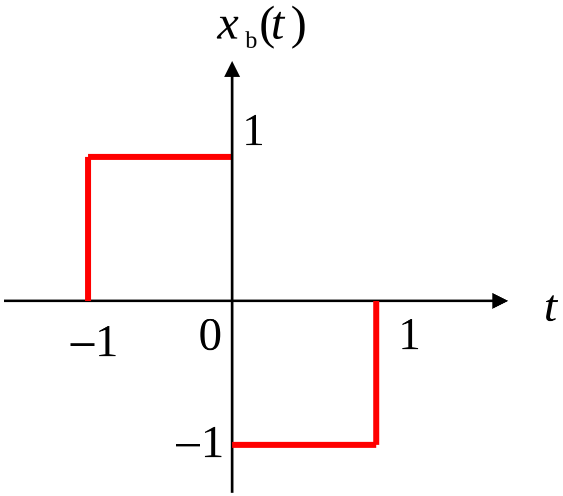
$$|X_b(\omega)| = \omega \operatorname{sinc}^2\left(\frac{\omega}{2}\right)$$



$$\Leftrightarrow \quad X_b(\omega) = j\omega \operatorname{sinc}^2\left(\frac{\omega}{2}\right)$$

$$X_b(\omega) = |X_b(\omega)| e^{j\theta(\omega)}$$





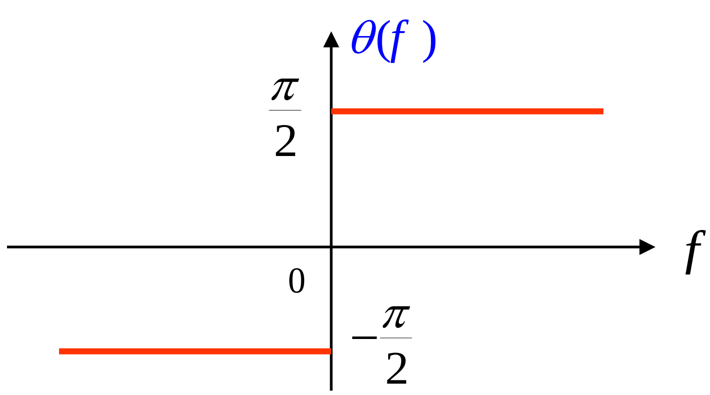
$$\Leftrightarrow \quad X_b(\omega) = j\omega \,\mathrm{sinc}^2\left(\frac{\omega}{2}\right)$$

$$X_b(\omega) = |X_b(\omega)|e^{j\theta(\omega)}$$

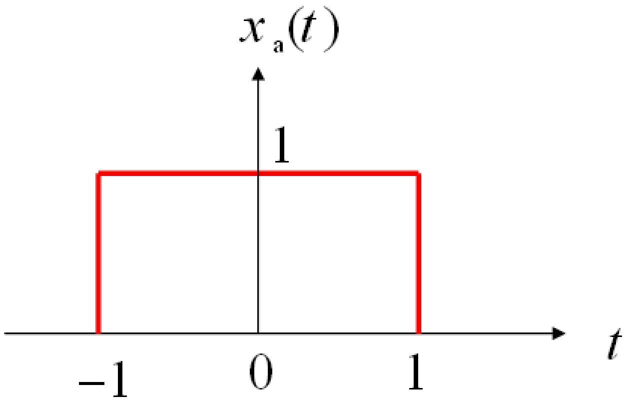
$\frac{\pi}{2}$

Always > 0 it add no angle (0^o)

$$X_b(f) = j\omega \,\mathrm{sinc}^2\left(\frac{\omega}{2}\right)$$



It was shown previously

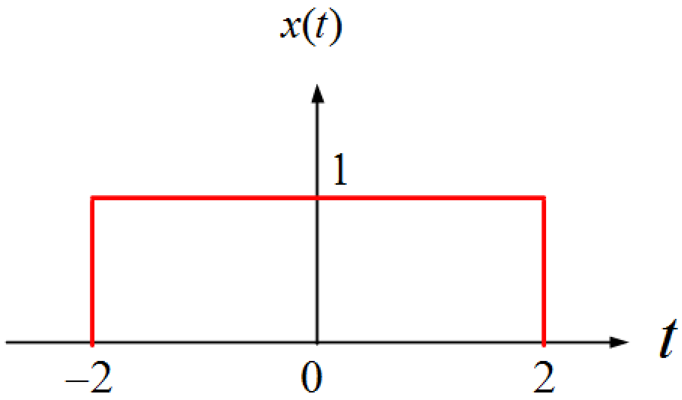


$$X_a(\omega)=\int_{-\infty}^{\infty} x_a(t)e^{-j\omega t}dt \;=\int_{-1}^{1} e^{-j\omega t}dt \;=\left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1}^{1} \;=\frac{e^{-j\omega(1)}-e^{-j\omega(-1)}}{-j\omega}$$

$$=\frac{e^{j\omega}-e^{-j\omega}}{j\omega}=\left(\frac{2}{\omega}\right)\frac{e^{j\omega}-e^{-j\omega}}{j2}=\left(\frac{2}{\omega}\right)\sin(\omega)=2\frac{\sin(\omega)}{\omega}$$

$$=2\text{sinc}(\omega)$$

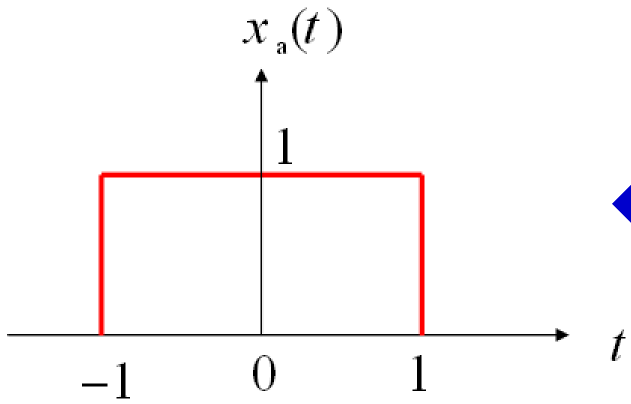
The Fourier Transform for the following function



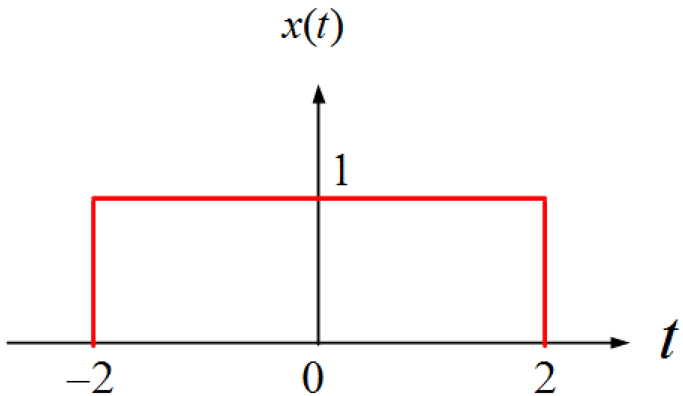
$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad =\int_{-2}^{2} e^{-j\omega t}dt \quad =\left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-2}^{2} =\frac{e^{-j\omega(2)}-e^{-j\omega(-2)}}{-j\omega}$$

$$=\frac{e^{j2\omega}-e^{-j2\omega}}{j\omega} \quad =\left(\frac{4}{2\omega}\right)\frac{e^{j2\omega}-e^{-j2\omega}}{j2} =\left(\frac{4}{2\omega}\right)\sin(2\omega) \quad =4\frac{\sin(2\omega)}{2\omega}$$

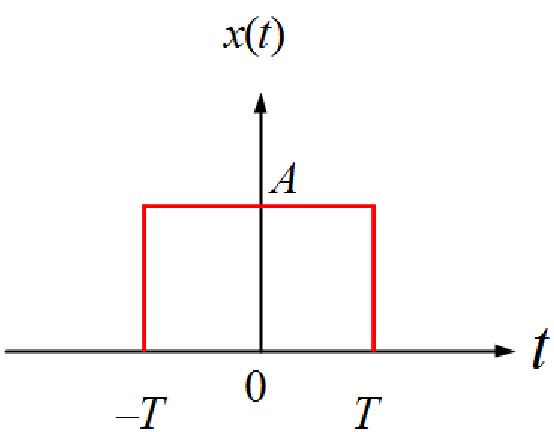
$$=4\operatorname{sinc}(2\omega)$$



$$\Longleftrightarrow \quad 2\mathrm{sinc}(\omega)$$

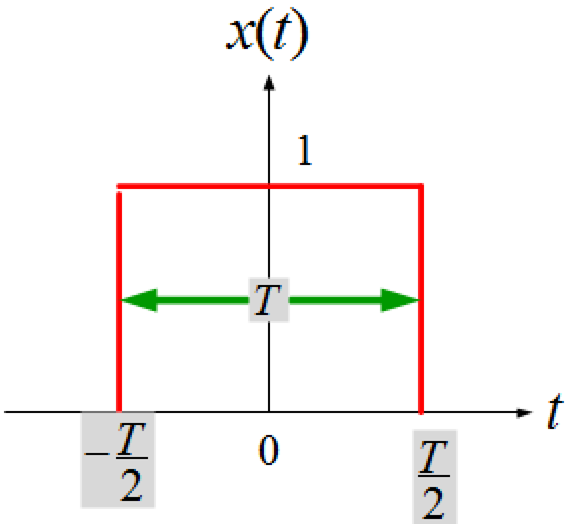


$$\Longleftrightarrow \quad 4\mathrm{sinc}(2\omega)$$

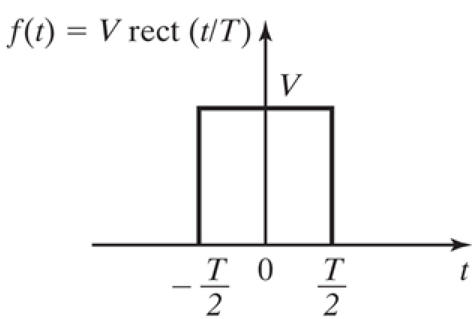


$$\Longleftrightarrow \quad A(2T)\mathrm{sinc}(T\omega)$$

rect$\left(\frac{t}{T}\right)$ function defintion



$$u\left(t+\frac{T}{2}\right)-u\left(t-\frac{T}{2}\right)$$



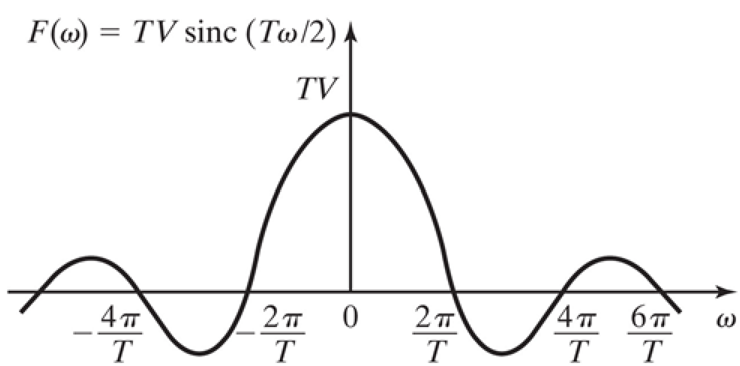


TABLE 5.2 Fourier Transform Pairs

Time Domain Signal	Fourier Transform
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$	$F(\omega)$
$\delta(t)$	1
$A\delta(t - t_0)$	$Ae^{-j\omega t_0}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
K	$2\pi K\delta(\omega)$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$\text{rect}(t/T)$	$T\text{sinc}\,(\omega T/2)$
$\cos\,(\omega_0 t)u(t)$	$\frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$

Example Find the Fourier Transform for the delta function $x(t) = \delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \underbrace{\left(e^{-j\omega t}\right)_{t=0} \delta(t)} dt = \int_{-\infty}^{\infty} (1)\delta(t) dt = 1$$

$$\Rightarrow \quad \delta(t) \Longleftrightarrow 1$$

Properties of the Fourier Transform

1-Linearity $F[a_1 x_1(t) + a_1 x_1(t)] = a_1 X_1(\omega) + a_2 X_2(\omega)$

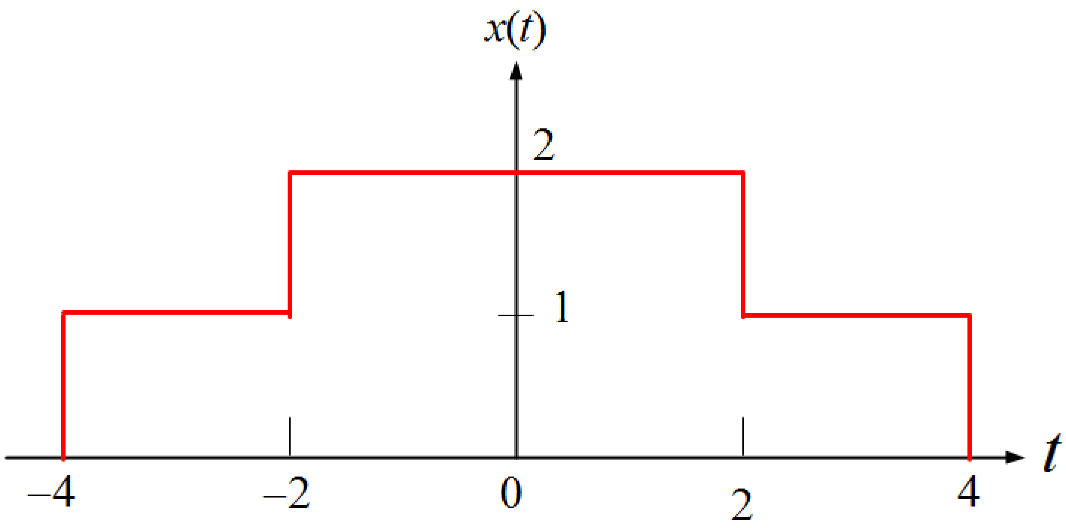
Proof

$$F[a_1 x_1(t) + a_1 x_1(t)] = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_1 x_1(t)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} a_1 x_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} a_2 x_2(t) e^{-j\omega t} dt$$

$$= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$$= a_1 X_1(\omega) + a_2 X_2(\omega)$$

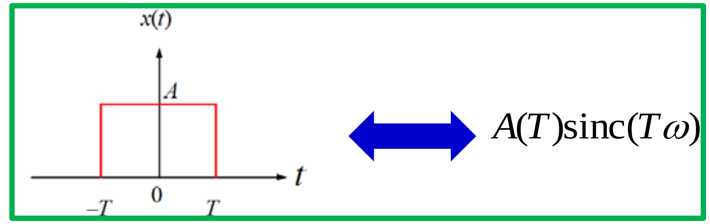


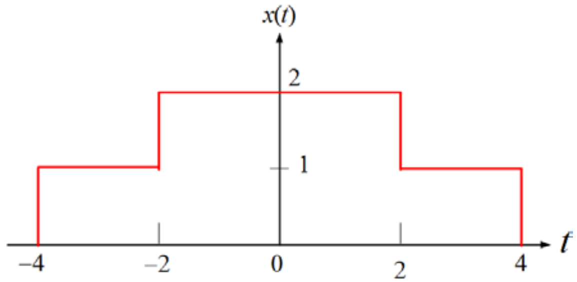
Direct Method

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-4}^{-2} (1) e^{-j\omega t} dt + \int_{-2}^{0} (2) e^{-j\omega t} dt + \int_{0}^{2} (2) e^{-j\omega t} dt + \int_{2}^{4} (1) e^{-j\omega t} dt$$

Using Fourier Transform Properties





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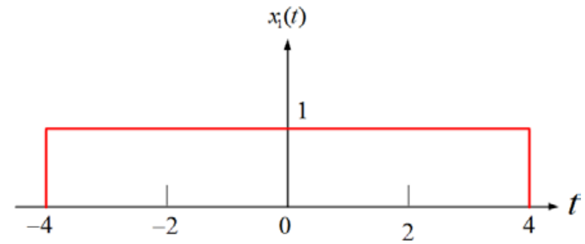

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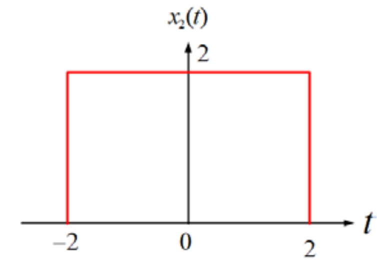


$$X(\omega) = X_1(\omega) + X_2(\omega)$$

$$= (1)(4)\text{sinc}(4\omega) + (2)(2)\text{sinc}(2\omega)$$

$$= 4\text{sinc}(4\omega) + 4\text{sinc}(2\omega)$$

2-Time-Scaling (compressing or expanding)

Let $x(t) \Leftrightarrow X(\omega)$ Then $x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

Proof

Let $a > 0$

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

Change of variable $t' = at$

$$F\{x(at)\} = \int_{-\infty}^{\infty} x(t') e^{-j\omega\left(\frac{t'}{a}\right)} \frac{dt'}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t') e^{-j\omega\left(\frac{t'}{a}\right)} dt'$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(t') e^{-j\left(\frac{\omega}{a}\right)t'} dt'$$

$$F\left\{x(at)\right\} = \frac{1}{a}\int_{-\infty}^{\infty} x(t')e^{-j\left(\frac{\omega}{a}\right)t'}\,dt'$$

Let $\omega' = \dfrac{\omega}{a}$ ⇒ $$F\left\{x(at)\right\} = \underbrace{\frac{1}{a}\int_{-\infty}^{\infty} x(t')e^{-j\omega'}\,dt'}_{X(\omega')}$$

⇒ $$F\left\{x(at)\right\} = \frac{1}{a}X(\omega') = \frac{1}{a}X\left(\frac{\omega}{a}\right)$$

Now Let $a < 0 \rightarrow at = -|a|t$

$$\Rightarrow \quad F\{x(at)\} = \int_{-\infty}^{\infty} x(-|a|t)e^{-j\omega t}dt$$

Change of variable $t' = -|a|t \Rightarrow dt = \frac{dt'}{-|a|}$ $\quad t = \infty \rightarrow t' = -\infty$, $\quad t = -\infty \rightarrow t' = \infty$

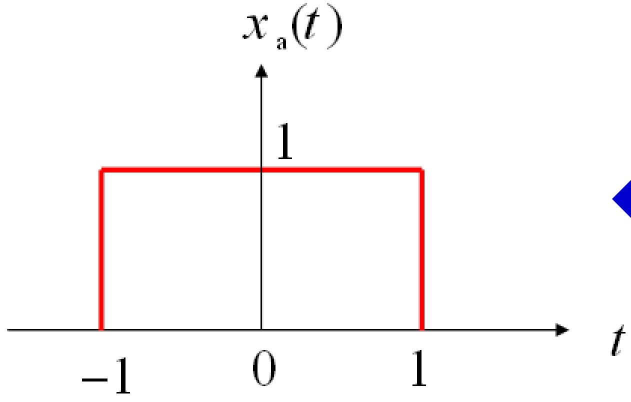
$$\Rightarrow \quad F\{x(at)\} = \int_{\infty}^{-\infty} x(t')e^{-j\omega\left(-\frac{t'}{|a|}\right)} \frac{dt'}{-|a|} = \frac{1}{|a|}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j\left(-\frac{\omega}{|a|}\right)t'}dt'}_{X\left(-\frac{\omega}{|a|}\right)}$$

$$F\{x(at)\} = \frac{1}{|a|}X\left(-\frac{\omega}{|a|}\right)$$

Since $a < 0 \rightarrow -|a| = a$ $\Rightarrow$

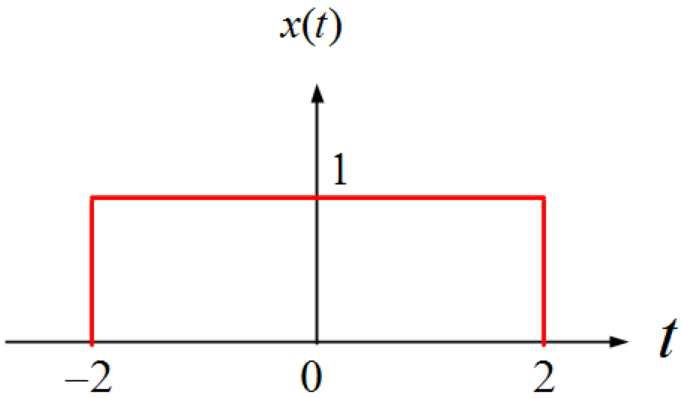
$$F\{x(at)\} = \frac{1}{|a|}X\left(\frac{\omega}{|a|}\right)$$

Let



$$X_a(\omega) = 2\text{sinc}(\omega)$$

what is the fourier transform of



since $x(t) = x_a\left(\frac{t}{1/2}\right)$

$$X(\omega) = \frac{1}{1/2} X_a\left(\frac{\omega}{1/2}\right) = 2X_a(2\omega)$$

$$= 2(2\text{sinc}(2\omega)) = 4\text{sinc}(2\omega)$$

3-Time-Shifting

Let $x(t) \Leftrightarrow X(\omega)$ Then $x(t-t_0) \Leftrightarrow X(\omega)e^{-j\omega t_0}$

Proof

Let $y(t)=x(t-t_0)$

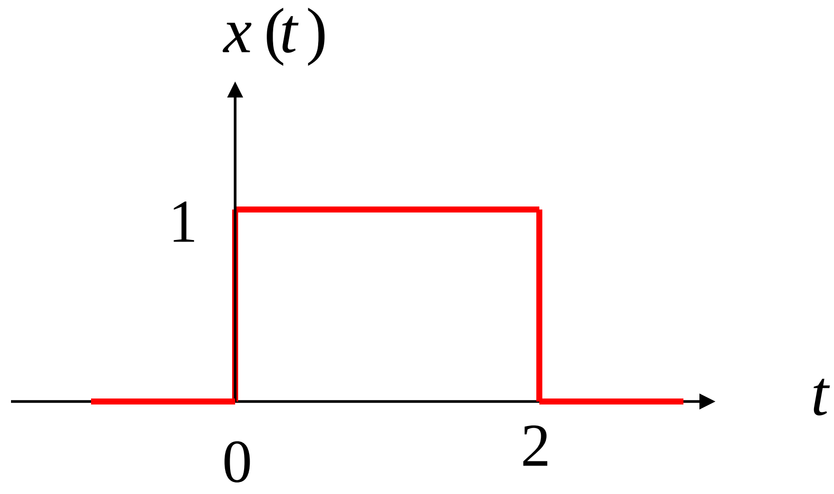
$$Y(\omega)=\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt$$

Change of variable $t' = t - t_0$ ⇒ $Y(\omega)=\int_{-\infty}^{\infty} x(t')e^{-j\omega(t'+t_0)}dt$

⇒ $$Y(\omega)=\int_{-\infty}^{\infty} x(t')e^{-j\omega t'}e^{-j\omega t_0}dt = e^{-j\omega t_0}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j\omega t'}dt}_{X(\omega)}$$

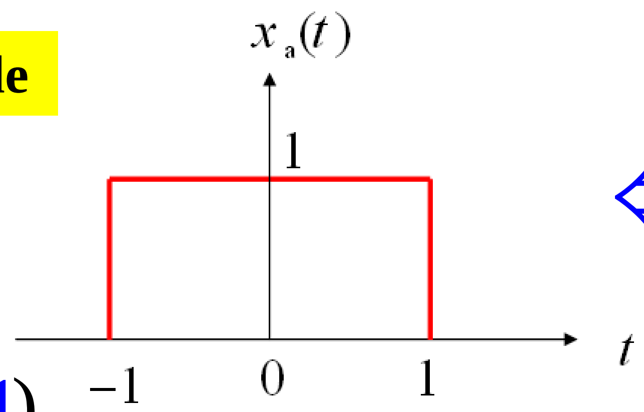
⇒ $x(t-t_0) \Leftrightarrow X(\omega)e^{-j\omega t_0}$

Example Find the Fourier Transform of the pulse function



Solution

From previous Example



$\Leftrightarrow \quad X_a(\omega) = 2\mathrm{sinc}(2\omega)$

Since $x(t) = x_a(t-1)$

$$X(\omega) = X_a(\omega)e^{-j\omega t_0} = 2\mathrm{sinc}(2\omega)\, e^{-j\omega(1)}$$

$$= 2\mathrm{sinc}(2\omega)\, e^{-j\omega}$$

4-Time Transformation

Let $x(t) \Leftrightarrow X(\omega)$ Then $x(at - t_0) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right) e^{-j\left(\frac{\omega}{a}\right)t_0}$

Proof

Let $a > 0$ $F\left\{x(at - t_0)\right\} = \int_{-\infty}^{\infty} x(at - t_0) e^{-j\omega t} dt$

Change of variable $t' = at - t_0$ ➡ $t = \frac{t' + t_0}{a}$ $dt = \frac{dt'}{a}$

$t = \infty \rightarrow t' = \infty$

$t = -\infty \rightarrow t' = -\infty$

$$F\left\{x(at - t_0)\right\} = \int_{-\infty}^{\infty} x(t') e^{-j\omega\left(\frac{t' + t_0}{a}\right)} \frac{dt'}{a} = \frac{e^{-j\omega\left(\frac{t_0}{a}\right)}}{a} \int_{-\infty}^{\infty} x(t') e^{-j\omega\left(\frac{t'}{a}\right)} dt'$$

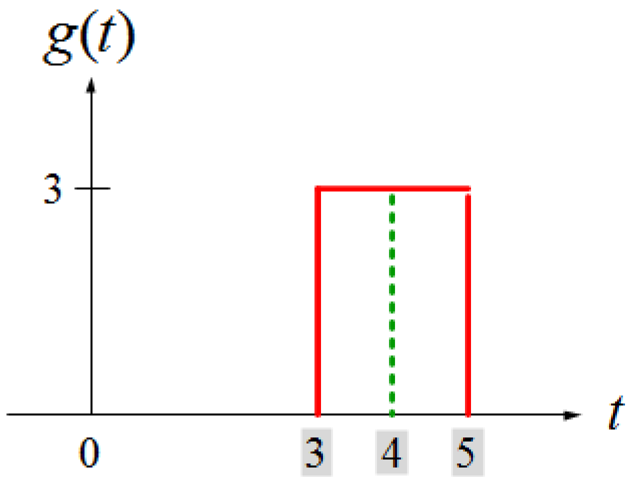
$$F\left\{x(at-t_0)\right\} = \int_{-\infty}^{\infty} x(t')\, e^{-j\omega\left(\frac{t'+t_0}{a}\right)} \frac{dt'}{a} = \frac{e^{-j\omega\left(\frac{t_0}{a}\right)}}{a} \int_{-\infty}^{\infty} x(t')\, e^{-j\omega\left(\frac{t'}{a}\right)} dt'$$

$$= \frac{e^{-j\left(\frac{\omega}{a}\right)t_0}}{a} \underbrace{\int_{-\infty}^{\infty} x(t')\, e^{-j\left(\frac{\omega}{a}\right)t'} dt'}_{X\left(\frac{\omega}{a}\right)} \qquad = \frac{e^{-j\left(\frac{\omega}{a}\right)t_0}}{a} X\left(\frac{\omega}{a}\right)$$

Similarly $a < 0$ $\quad F\left\{x(at-t_0)\right\} = \dfrac{e^{-j\left(\frac{\omega}{a}\right)t_0}}{|a|} X\left(-\dfrac{\omega}{a}\right)$

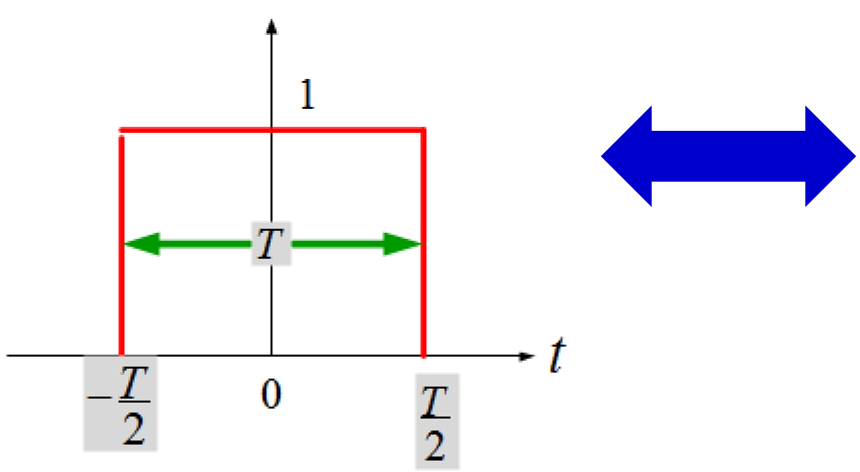
$$x(at-t_0) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right) e^{-j\left(\frac{\omega}{a}\right)t_0}$$

Find the Fourier Transform for the Rect function g(t)



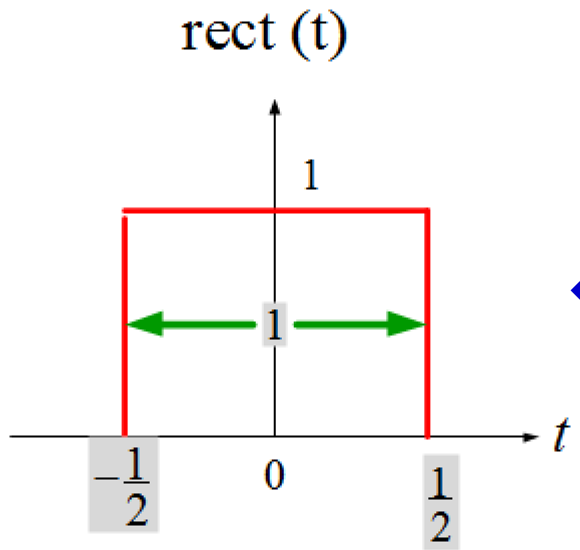
From Fourier Transform Pairs (Table 5.2)

$$\text{rect}\left(\frac{t}{T}\right)$$



$$T\text{sinc}\left(\frac{\omega T}{2}\right)$$

$$x(at-t_0) \Leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)e^{-j\left(\frac{\omega}{a}\right)t_0}$$

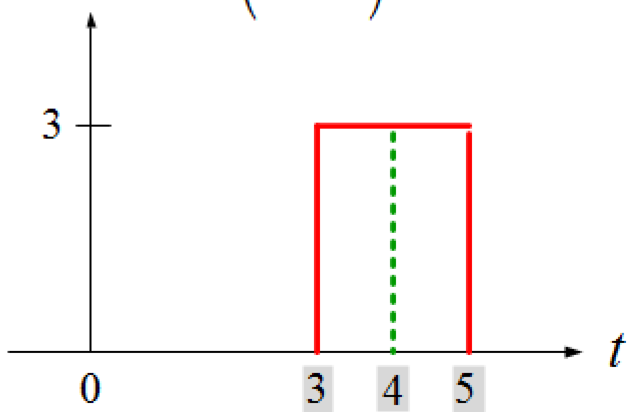


$$\Leftrightarrow \quad \text{sinc}\left(\frac{\omega}{2}\right)$$

$$g(t)=3\text{rect}\left(\frac{t-4}{2}\right)= 3\text{rect}\left(0.5t-2\right)$$

$$\Leftrightarrow \quad G(\omega)=3\left(\frac{1}{0.5}\right)\text{sinc}\left(\frac{\omega}{0.5}\right)e^{-j\left(\frac{\omega}{0.5}\right)(2)}$$

$$=6\text{sinc}\left(\frac{\omega}{0.5}\right)e^{-j4\omega}$$

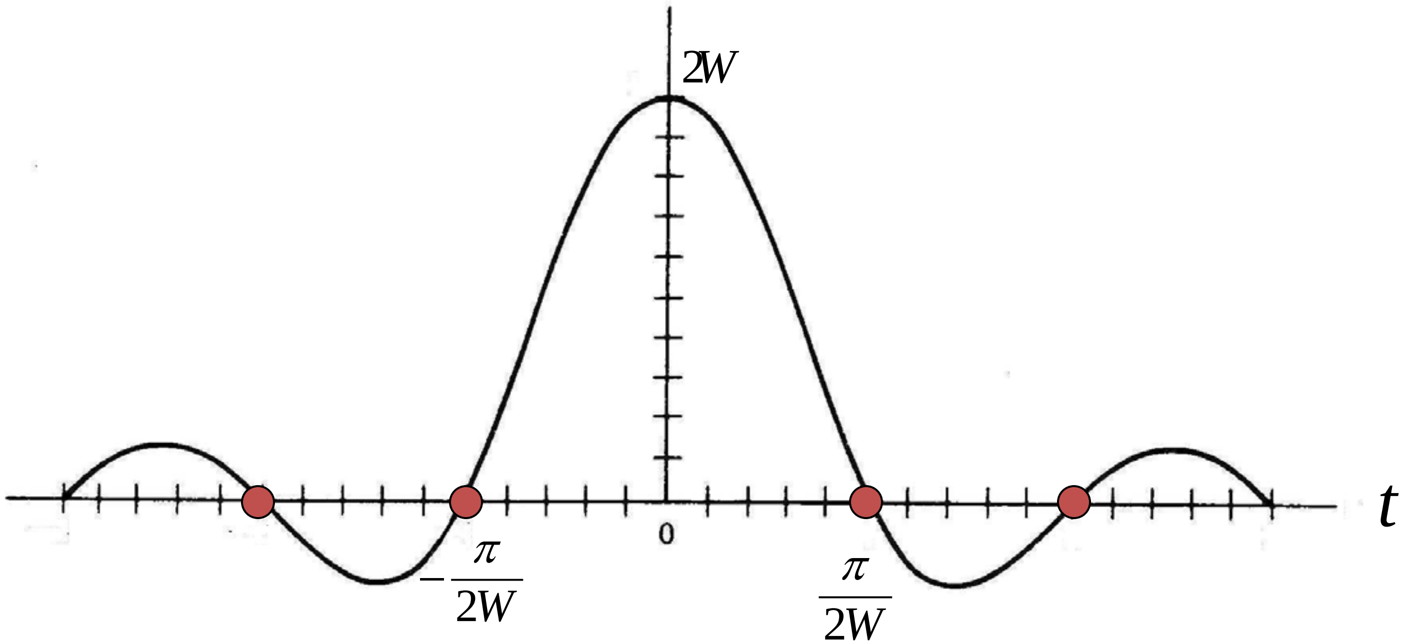


5-Duality ازدواجية

If $x(t) \Leftrightarrow X(\omega)$ **then** $X(t) \Leftrightarrow 2\pi x(-\omega)$

Find the **F.T** of $\{2W \operatorname{sinc}(2Wt)\}$

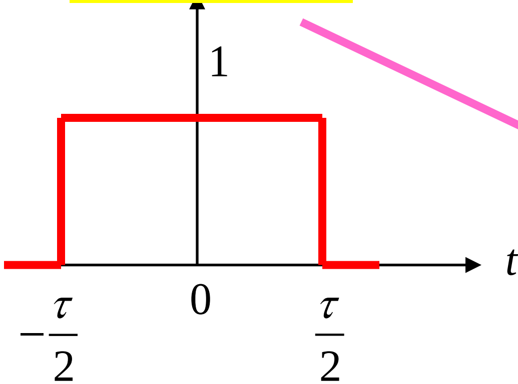
$$2W \operatorname{sinc}(2Wt)$$



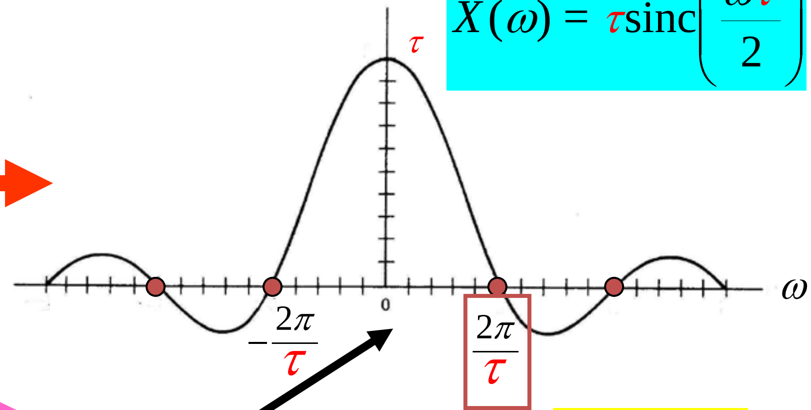
$$X(t) \Leftrightarrow 2\pi x(-\omega)$$

Step 1 from Known transform from the F.T Table

$$x(t) = \text{rect}\left(\frac{t}{\tau}\right)$$



$$X(\omega) = \tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$$

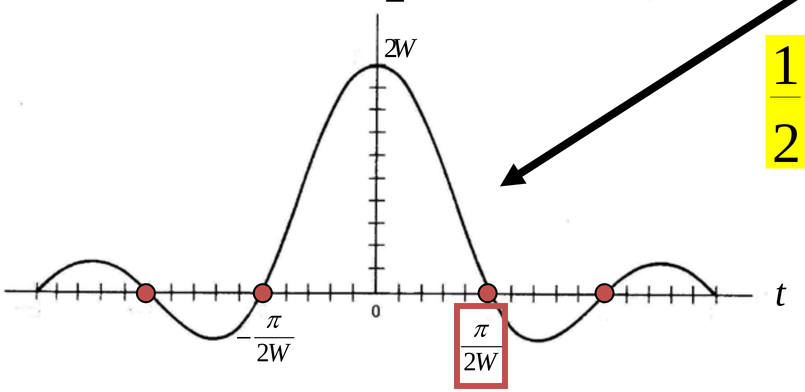


Even Function

$$\pi \text{rect}\left(\frac{-\omega}{4W}\right) = \pi \text{rect}\left(\frac{\omega}{4W}\right)$$

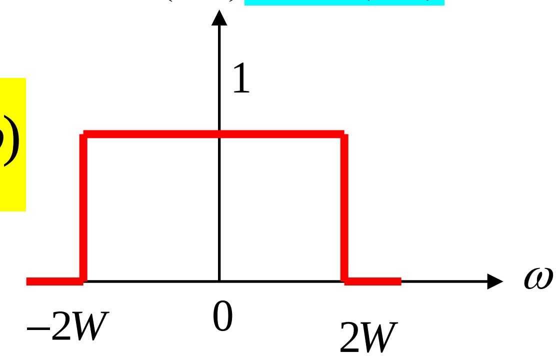
Step 2

$$2W\text{sinc}(2Wt) = \frac{1}{2}X(t) \qquad (\tau = 4W)$$



$$\frac{1}{2}\left\{2\pi x(-\omega)\right\} = \pi x(-\omega)$$

$$t = -\omega$$



6- The convolution Theorem

$$x_1(t)*x_2(t) \Leftrightarrow X_1(\omega)X_2(\omega)$$

Convolution in Time ⟺ **Multiplication in Frequency**

Proof

$$x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda$$

$$x_2(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t}d\omega \Rightarrow x_2(t-\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega(t-\lambda)}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t}e^{-j\omega\lambda}d\omega$$

$$x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)x_2(t-\lambda)d\lambda$$

$$x_2(t-\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t}e^{-j\omega\lambda}d\omega$$

Now substitute $x_2(t\text{-}\lambda)$ (as the inverse Fourier Transform) in the convolution integral

$$x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t}e^{-j\omega\lambda}d\omega\right\}d\lambda$$

$$x_1(t)*x_2(t)=\int_{-\infty}^{\infty}x_1(\lambda)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(\omega)e^{j\omega t}e^{-j\omega\lambda}d\omega\right\}d\lambda$$

Exchanging the order of integration , we have

$$x_1(t)*x_2(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(\omega)\underbrace{\left\{\int_{-\infty}^{\infty}x_1(\lambda)e^{-j\omega\lambda}d\lambda\right\}}_{X_1(\omega)}e^{j\omega t}d\omega$$

$$x_1(t)*x_2(t)=\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(\omega)\,X_1(\omega)\,e^{j\omega t}d\omega}_{\text{Inverse Fourier Transform}}$$

$$x_1(t)*x_2(t)\Leftrightarrow X_1(\omega)X_2(\omega)$$

The multiplication Theorem

$$x_1(t)x_2(t) \Leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

Proof

Similar to the convolution theorem , left as an exercise

Applying the multiplication Theorem

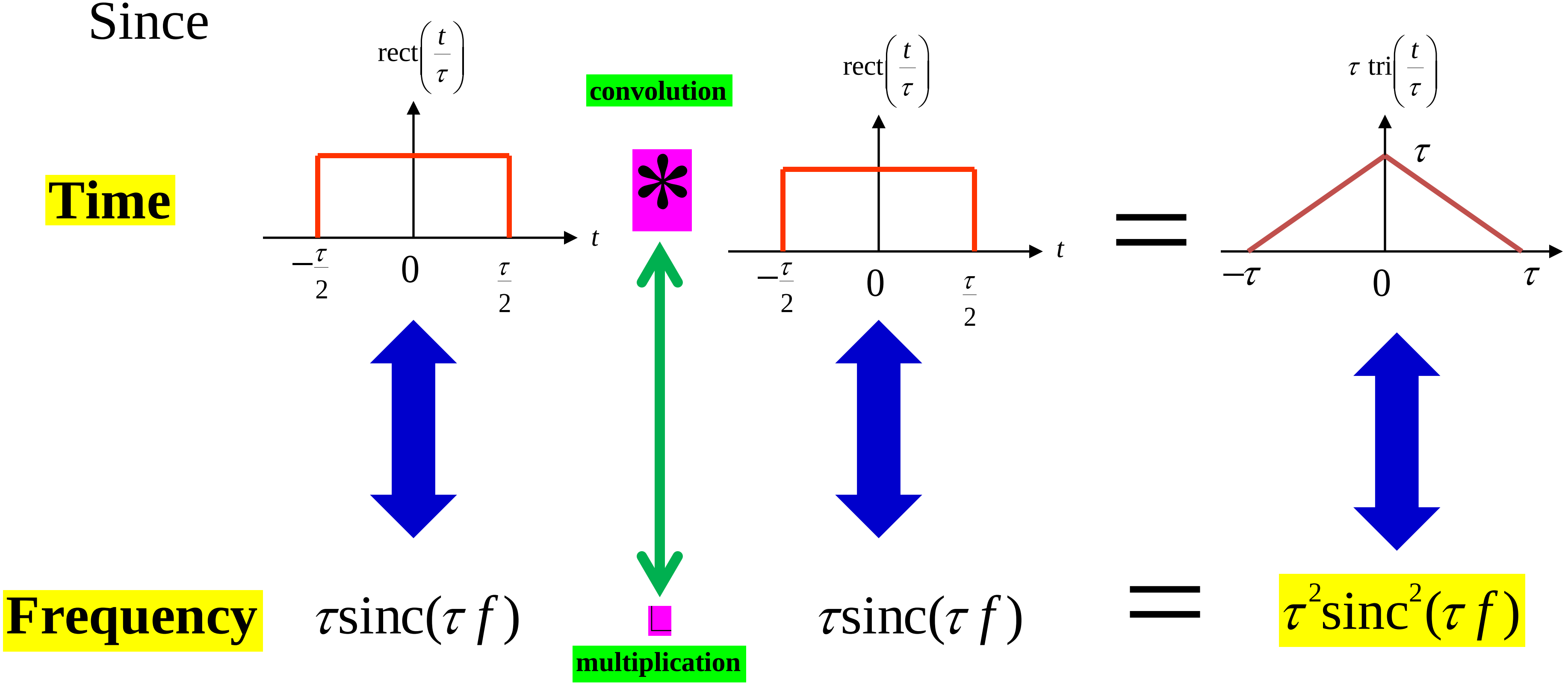
Applying the multiplication Theorem $x_1(t)x_2(t) \Leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

Find the Fourier Transform of following

$$\text{rect}\left(\frac{t}{\tau}\right)*\text{rect}\left(\frac{t}{\tau}\right)$$

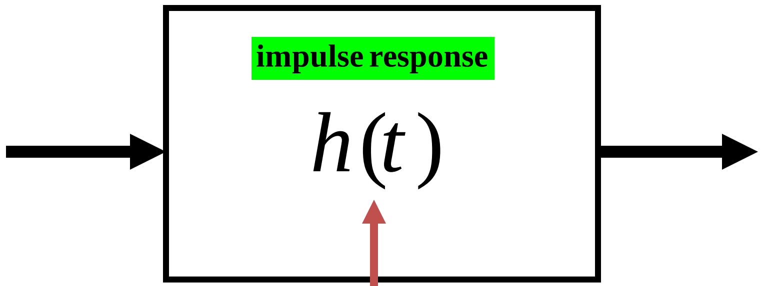
Solution

Since



System Analysis with Fourier Transform

$x(t)$



$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda)\, d\lambda$$

convolution in time

multiplication in Frequency

$X(\omega)$ $H(\omega)$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$y(t) = x(t) * h(t) \Longleftrightarrow Y(\omega) = X(\omega) \cdot H(\omega)$$

convolution in time

multiplication in Frequency

6- Frequency Shifting

$$\text{Let } x(t) \Leftrightarrow X(\omega) \quad \text{Then} \quad x(t)e^{j\omega_0 t} \Leftrightarrow X(\omega-\omega_0)$$

Proof Let $y(t)=x(t)e^{j\omega_0 t}$

$$Y(\omega)=\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt$$

Change of variable $\omega' = \omega - \omega_0$

$$\Rightarrow \quad Y(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega' t}dt = X(\omega') = X(\omega-\omega_0)$$

$$\text{Let } x(t) \Leftrightarrow X(\omega) \quad \text{Then} \quad x(t)e^{j\omega_0 t} \Leftrightarrow X(\omega-\omega_0)$$

$$\text{Similarly} \quad x(t)e^{-j\omega_0 t} \Leftrightarrow X(\omega+\omega_0)$$

$$\Rightarrow \quad x(t)e^{\pm j\omega_0 t} \Leftrightarrow X(\omega \mp \omega_0)$$

Example Find the Fourier Transform for $x(t) = A\cos(2\pi\omega_0 t)$

$$x(t) = \cos(2\pi\omega_0 t) \qquad = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \qquad = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$X(\omega) = F.T\left[\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right] = \left(\frac{1}{2}\right)F.T[e^{j\omega_0 t}] + \left(\frac{1}{2}\right)F.T[e^{-j\omega_0 t}]$$

Since $\delta(t) \Leftrightarrow 1$ ➡ $1 \Leftrightarrow 2\pi\delta(\omega)$ **duality**

Since $x(t)e^{\pm j\omega_0 t} \Leftrightarrow X(\omega \mp \omega_0)$

➡ $F.T[(1)e^{\pm j\omega_0 t}] = 2\pi\delta(\omega \mp \omega_0)$

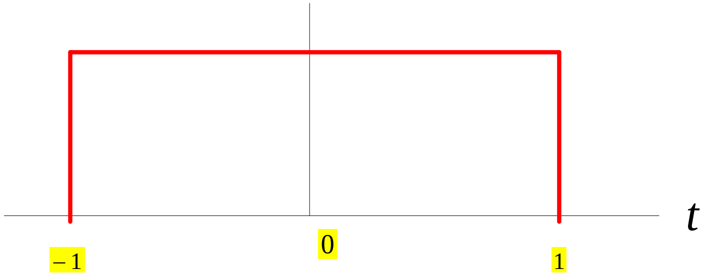
➡ $$X(\omega) = \left(\frac{1}{2}\right)\left(2\pi\delta(\omega - \omega_0)\right) + \left(\frac{1}{2}\right)\left(2\pi\delta(\omega + \omega_0)\right)$$

➡ $\cos(2\pi\omega_0 t) \Leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$

Find the Fourier Transform of the function

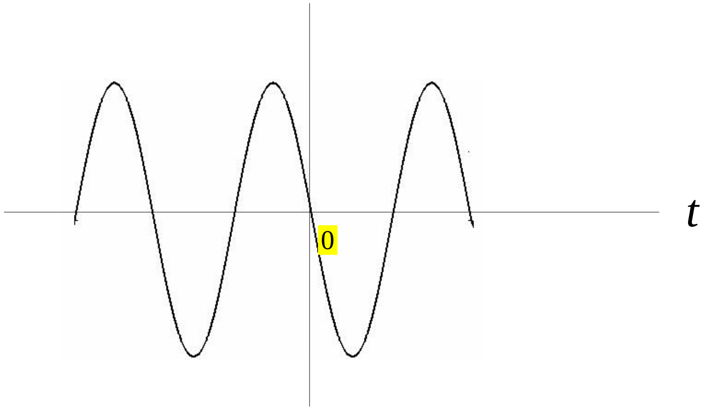
$$x(t) = \mathrm{rect}\left(\frac{t}{2}\right)\cos(\omega_0 t)$$

$\mathrm{rect}\left(\frac{t}{2}\right)$

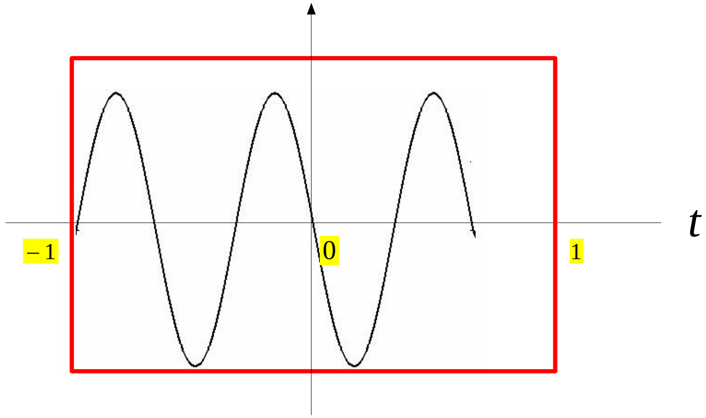


X

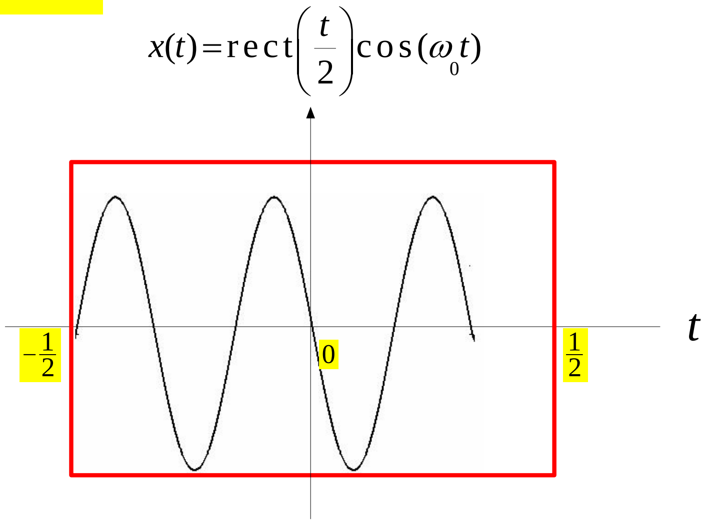
$\cos(\omega_0 t)$



$x(t) = \mathrm{rect}\left(\frac{t}{2}\right)\cos(\omega_0 t)$



Method 1



$$x(t)=\mathrm{rect}\left(\frac{t}{2}\right)\cos(\omega_0 t)$$

$$=\mathrm{rect}\left(\frac{t}{2}\right)\left\{\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right\}$$

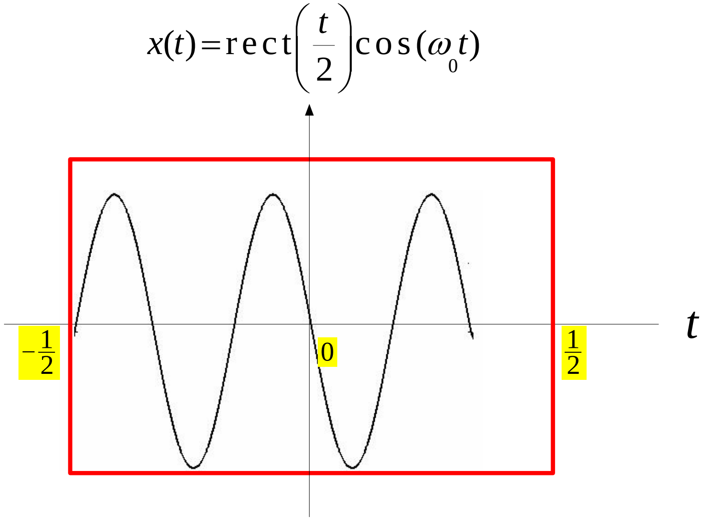
$$x(t)=\frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right)e^{j\omega_0 t}+\frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right)e^{-j\omega_0 t}$$

Since $x(t)e^{\pm j\omega_0 t} \Leftrightarrow X(\omega \mp \omega_0)$ **and** $\mathrm{rect}\left(\frac{t}{2}\right) \Leftrightarrow 2\mathrm{sinc}(2\omega)$

Therefore

$$x(t)=\mathrm{rect}\left(\frac{t}{2}\right)\cos(\omega_0 t) \Leftrightarrow \mathrm{sinc}(2(\omega-\omega_0))+\mathrm{sinc}(2(\omega+\omega_0))$$

Method 2



$$x(t)=\text{rect}\left(\frac{t}{2}\right)\cos(\omega_0 t)$$

Since $x_1(t)x_2(t) \Leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

$$\text{rect}\left(\frac{t}{2}\right) \Leftrightarrow 2\text{sinc}(2\omega)$$

$$\cos(2\pi\omega_0 t) \Leftrightarrow \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$

$$\Rightarrow x(t)=\text{rect}\left(\frac{t}{2}\right)\cos(\omega_0 t) \Rightarrow X(\omega)= F.T\left[\text{rect}\left(\frac{t}{2}\right)\right]*F.T\left[\cos(\omega_0 t)\right]$$

$$\Rightarrow X(\omega)=\frac{1}{2\pi}\left[2\,\text{sinc}(2\omega)\right]*\left[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)\right]$$

$$\Rightarrow X(\omega)=\text{sinc}(2(\omega-\omega_0))+\text{sinc}(2(\omega+\omega_0))$$

7-Differentiation

Let $x(t) \Leftrightarrow X(\omega)$ $\qquad \frac{d}{dt}x(t) \Leftrightarrow (j\omega)\,X(\omega)$

in general $\qquad \frac{d^n}{dt^n}x(t) \Leftrightarrow (j\omega)^n\,X(\omega)$

Proof

$$F\left[\frac{dx(t)}{dt}\right] = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t} dt$$

$$\frac{d}{dt}x(t) \Leftrightarrow (j\omega)\,X(\omega) \qquad \frac{d^n}{dt^n}x(t) \Leftrightarrow (j\omega)^n\,X(\omega)$$

Proof

$$F\left[\frac{dx(t)}{dt}\right] = \int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-j\omega t} dt$$

Using integration by parts

$$\int_a^b u\,dv = uv\,\Big|_a^b - \int_a^b v\,du$$

$$u = e^{-j\omega t}$$

$$du = -j\omega e^{-j\omega t}$$

$$dv = \frac{dx(t)}{dt}dt = dx(t)$$

$$v = x(t)$$

$$F\left[\frac{dx(t)}{dt}\right] = x(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} + j\omega\int_{-\infty}^{\infty} x(t)e^{-j\omega t} d\,t$$

$$F\left[\frac{dx(t)}{dt}\right] = x(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} + j\omega\int_{-\infty}^{\infty} x(t)e^{-j\omega t}d\,t$$

$$= \left\{x(\infty)e^{-j\omega(\infty)} - x(-\infty)e^{-j\omega(-\infty)}\right\} + j\omega f\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Since x(t) is absolutely integrable $\int_{-\infty}^{\infty}|x(t)|dt < \infty$ ⇒ $x(\infty) \to 0$, $x(-\infty) \to 0$

⇒ $\left\{x(\infty)e^{-j\omega(\infty)} - x(-\infty)e^{-j\omega(-\infty)}\right\} \longrightarrow 0$

⇒ $F\left[\frac{dx(t)}{dt}\right] = j\omega\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ $= j\omega X(\omega)$

⇒ $\frac{dx(t)}{dt} \Leftrightarrow j\omega X(\omega)$ $\frac{d^n x(t)}{dt^n} \Leftrightarrow (j\omega)^n X(\omega)$

7- Integration

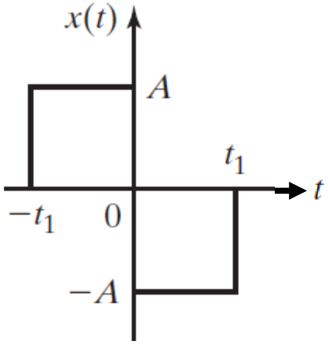
$$\int_{-\infty}^{t} x(\tau)d\tau \Leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$$

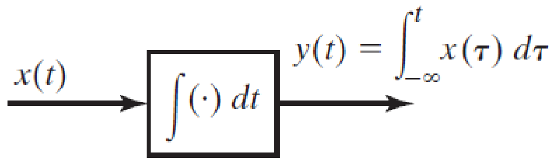
Example Find the Fourier Transform of the unit step function $u(t)$

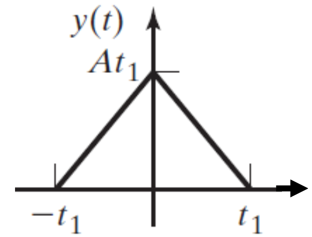
$$\delta(t) \Leftrightarrow 1$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \Leftrightarrow \frac{1}{j\omega}(1) + \pi(1)\Big|_{\omega=0}\delta(\omega)$$

$$u(t) \Leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$







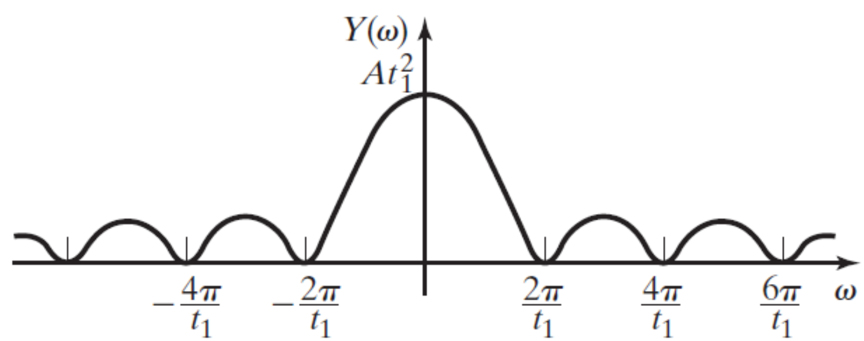
$$X(\omega) = j\omega A t_1^2 \,\mathrm{sinc}^2(t_1\omega/2)$$

$Y(\omega)$ **?**

0

$$Y(\omega) = \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$$

$$Y(\omega) = \frac{1}{j\omega}X(\omega) = A t_1^2 \,\mathrm{sinc}^2(t_1\omega/2)$$



Frequency Differentiation

The time-differentiation property has a dual for the case of differentiation in the frequency domain

$$\text{If} \qquad f(t) \stackrel{\mathscr{F}}{\longleftrightarrow} F(\omega) \qquad \text{then} \qquad (-jt)^n f(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{d^n F(\omega)}{d\omega^n}$$

Proof

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

by differentiating both sides of the equation

$$\frac{dF(\omega)}{d\omega} = \int_{-\infty}^{\infty} [(-jt) f(t)] e^{-j\omega t}\, dt$$

$$(-jt) f(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{d[F(\omega)]}{d\omega}$$

If $\delta(t) \Leftrightarrow X(\omega) = ?$

$$F.T\,[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left(e^{-j\omega t}\right)_{t=0} \delta(t) dt$$

$$= \int_{-\infty}^{\infty} (1)\delta(t) dt = 1$$

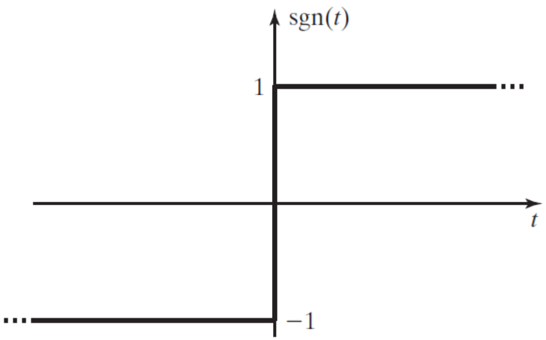
$$\delta(t) \Leftrightarrow 1$$

From duality ➡ 1 (DC) $\Leftrightarrow 2\pi\delta(\omega)$

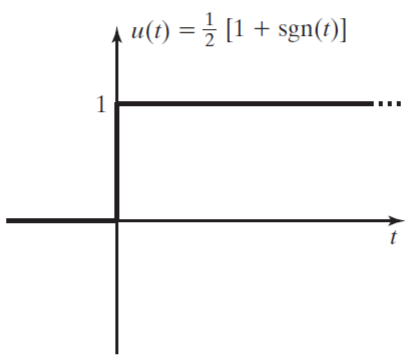
$$k \Leftrightarrow 2\pi k\delta(\omega)$$

Unit Step Function

The Fourier transform of the unit step function can be derived



$$\operatorname{sgn}(t) \xleftrightarrow{\mathcal{F}} \frac{2}{j\omega}$$



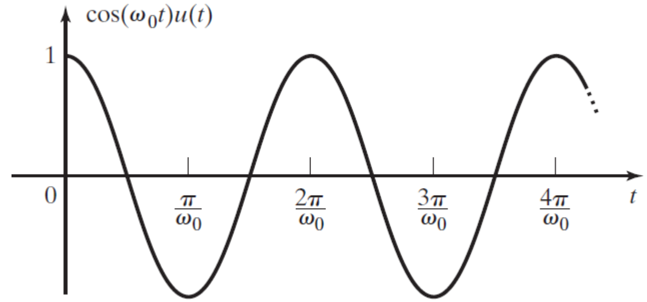
$$u(t) = \frac{1}{2}[1 + \operatorname{sgn}(t)] \Longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$1 \rightarrow 2\pi\delta(\omega)$

$\operatorname{sgn}(t) \rightarrow \frac{2}{j\omega}$

Switched Cosine

$$f(t) = \cos(\omega_0 t)u(t)$$



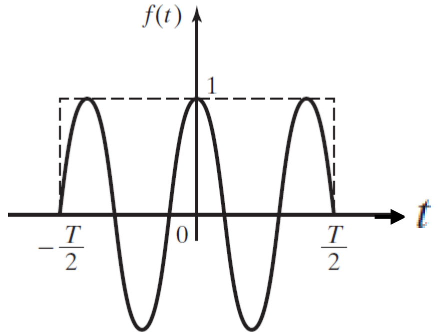
$$f(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}u(t) = \tfrac{1}{2}e^{j\omega_0 t}u(t) + \tfrac{1}{2}e^{-j\omega_0 t}u(t)$$

We now apply the linearity property and the frequency, shifting property

to the Fourier transform of the unit step function to yield

$$\cos(\omega_0 t)u(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$$

Pulsed Cosine



$$f(t) = \mathrm{rect}(t/T)\cos(\omega_0 t)$$

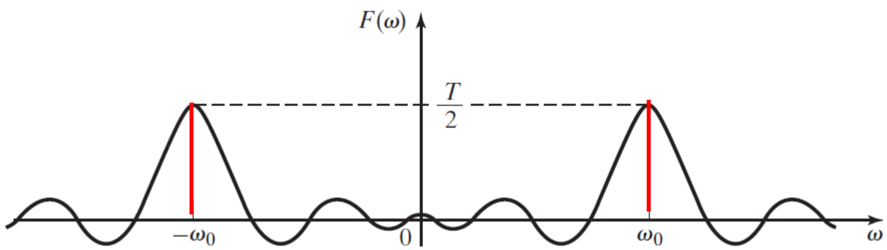
$$\mathrm{rect}(t/T) \overset{\mathscr{F}}{\longleftrightarrow} T\,\mathrm{sinc}(\omega T/2)$$

$$\cos(\omega_0 t) \overset{\mathscr{F}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

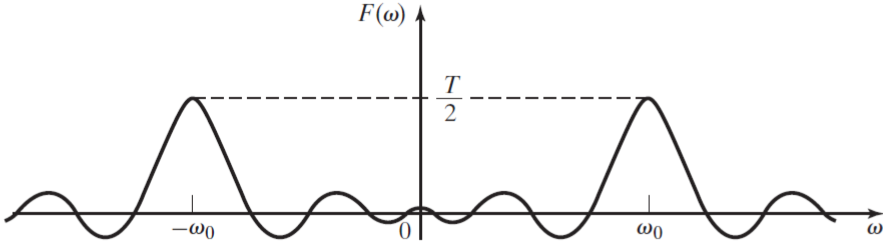
Applying the multiplication Theorem $x_1(t)x_2(t) \Leftrightarrow \dfrac{1}{2\pi} X_1(\omega) * X_2(\omega)$

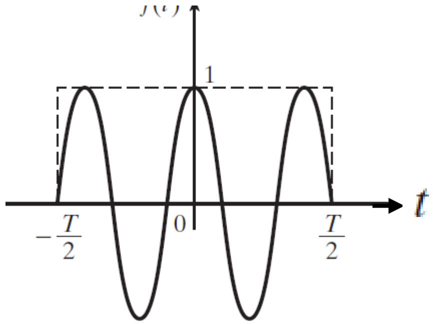
and $G(\omega) * \delta(\omega \pm \omega_0) = G(\omega \pm \omega_0)$ we have

$$\mathrm{rect}(t/T)\cos(\omega_0 t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{T}{2}\left[\mathrm{sinc}\,\frac{(\omega - \omega_0)T}{2} + \mathrm{sinc}\,\frac{(\omega + \omega_0)T}{2}\right]$$



Pulsed Cosine





$$f(t) = \text{rect}(t/T)\cos(\omega_0 t)$$

$$\text{rect}(t/T) \overset{\mathcal{F}}{\longleftrightarrow} T\,\text{sinc}(\omega T/2)$$

$$G(\omega) * \delta(\omega \pm \omega_0) = G(\omega \pm \omega_0)$$

$$\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

applying the convolution equation we have

$$F(\omega) = \frac{T}{2}\int_{-\infty}^{\infty} [\delta(\omega - \lambda - \omega_0) + \delta(\omega - \lambda + \omega_0)]\text{sinc}(\lambda T/2)d\lambda$$

$$\text{rect}(t/T)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{T}{2}\left[\text{sinc}\frac{(\omega - \omega_0)T}{2} + \text{sinc}\frac{(\omega + \omega_0)T}{2}\right]$$

Fourier Transforms of Periodic Functions

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \qquad C_k = \frac{1}{T_0}\int_{T_0} f(t) e^{-jk\omega_0 t} dt$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \right] e^{-j\omega t} dt$$

$$= \sum_{k=-\infty}^{\infty} C_k \underbrace{\int_{-\infty}^{\infty} (e^{jk\omega_0 t}) e^{-j\omega t} dt}_{\delta(\omega - k\omega_0)}$$

Since $1 \Leftrightarrow 2\pi\delta(\omega)$ ⇒ $e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$

Then $$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)$$

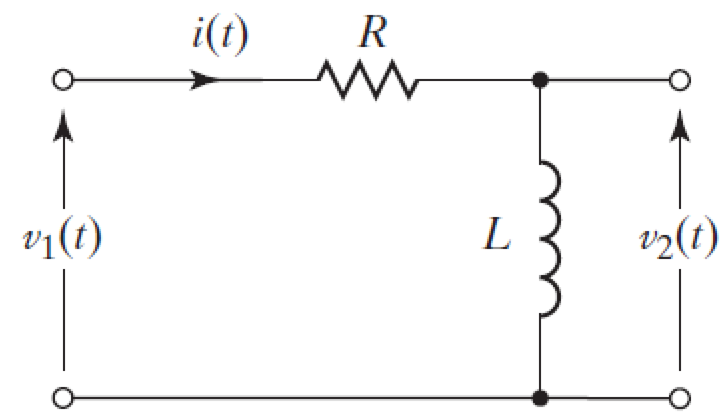
5.5 APPLICATION OF THE FOURIER TRANSFORM

Fourier transforms can be used to simplify the calculation of the response of linear systems to input signals.

For example, Fourier transforms allow the use of algebraic equations to analyze systems

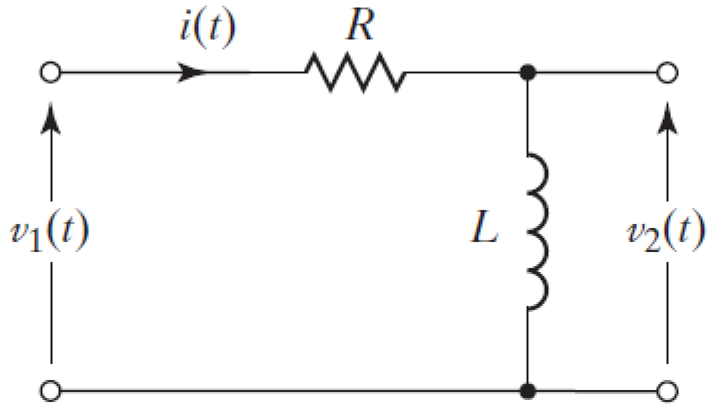
that are described by linear, time-invariant differential equations

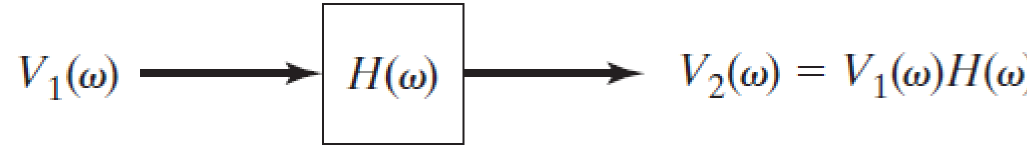
Consider the simple circuit shown



This circuit can be described by the differential equations

$$v_1(t) = Ri(t) + L\frac{di(t)}{dt} \quad \text{and} \quad v_2(t) = L\frac{di(t)}{dt}$$





$$v_1(t) = Ri(t) + L\frac{di(t)}{dt} \quad \text{and} \quad v_2(t) = L\frac{di(t)}{dt}$$

If we take the Fourier transform of each equation, using the properties

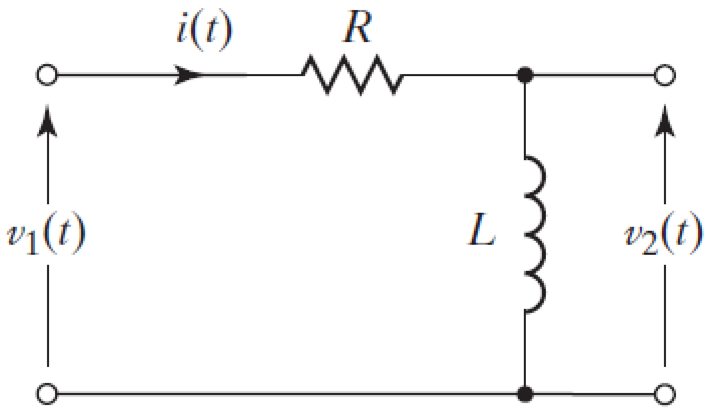
$$V_1(\omega) = RI(\omega) + j\omega LI(\omega) \quad \text{and} \quad V_2(\omega) = j\omega LI(\omega)$$

From the first equation, we solve algebraically for $I(\omega)$:

$$I(\omega) = \frac{1}{R + j\omega L}V_1(\omega)$$

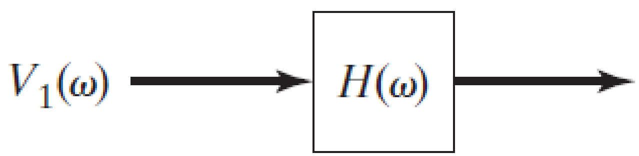
Substituting this result into the second equation yields

$$V_2(\omega) = \frac{j\omega L}{R + j\omega L}V_1(\omega)$$



$$v_1(t) = Ri(t) + L\frac{di(t)}{dt} \quad \text{and} \quad v_2(t) = L\frac{di(t)}{dt}$$

$$V_2(\omega) = \frac{j\omega L}{R + j\omega L}V_1(\omega)$$

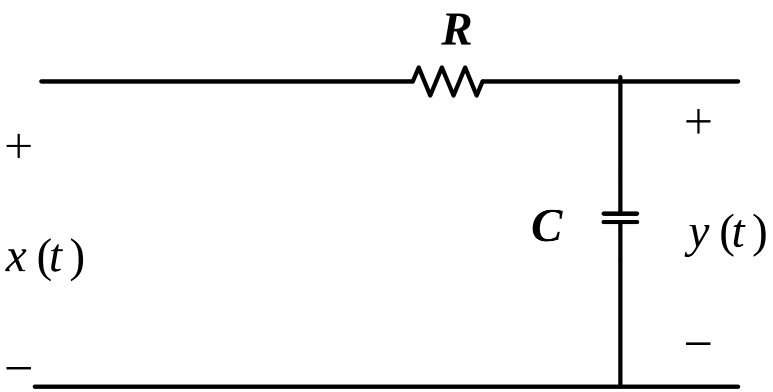


$$V_2(\omega) = V_1(\omega)H(\omega) \qquad H(\omega) = \frac{V_2(\omega)}{V_1(\omega)} \quad \textit{transfer function}$$

$$H(\omega) = \frac{j\omega L}{R + j\omega L}$$

$$H(\omega) = |H(\omega)|\angle\phi(\omega) = \frac{|V_2(\omega)|\angle V_2}{|V_1(\omega)|\angle V_1} = \frac{|V_2(\omega)|}{|V_1(\omega)|}\angle V_2 - V_1$$

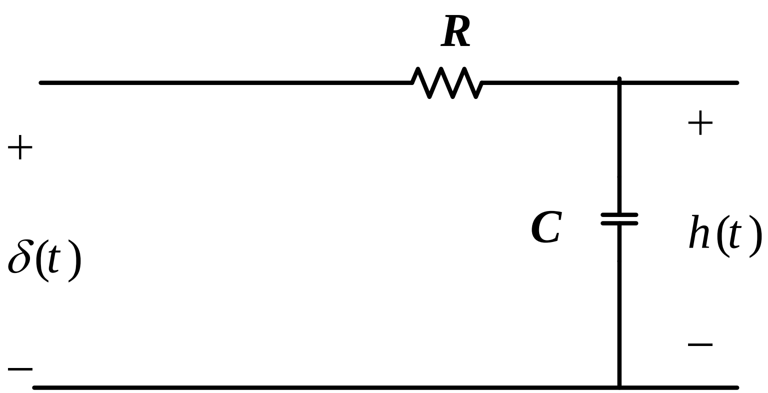
Find the Transfer Function for the following *RC* circuit



$$RC\frac{dy(t)}{dt}+y(t)=x(t)\qquad -\infty<t<\infty$$

Method 1

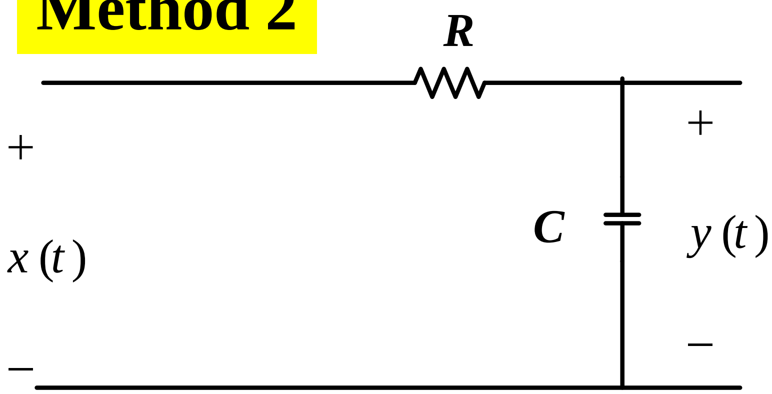
we can find *h*(t) by solving differential equation as follows



$$RC\frac{dh(t)}{dt}+h(t)=\delta(t)$$

$$h(t)=\frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

Method 2



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

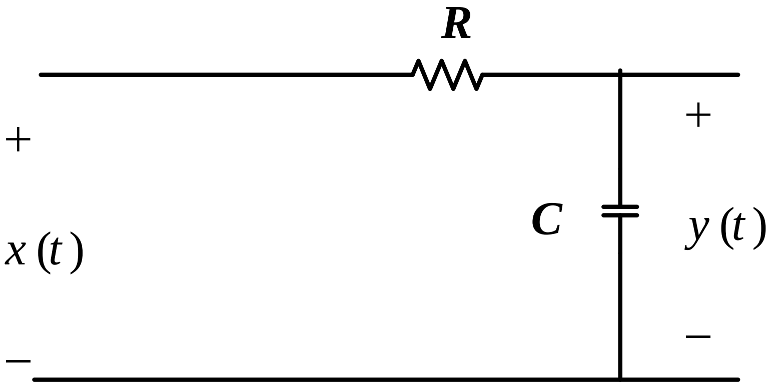
We will find *h*(t) using Fourier Transform Method rather than solving differential equation as follows

$$\mathrm{FT}\left[RC\frac{dy(t)}{dt} + y(t)\right] = \mathrm{FT}\left[x(t)\right]$$

$$RC(j\omega)\,Y(\omega) + Y(\omega) = X(\omega)$$

$$\left[(j\omega RC) + 1\right] Y(\omega) = X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega RC) + 1}$$



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega RC) + 1} = \frac{(1/RC)}{(j\omega) + (1/RC)}$$

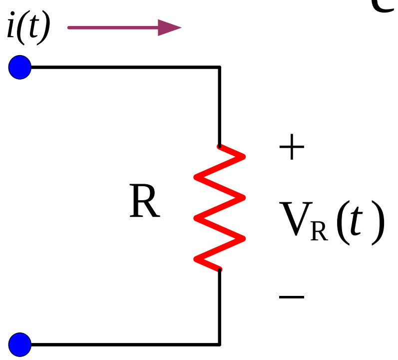
From Table 4-2

$$x(t) = e^{-\alpha t}u(t) \quad \alpha > 0 \quad \Leftrightarrow \quad \frac{1}{\alpha + j\omega}$$

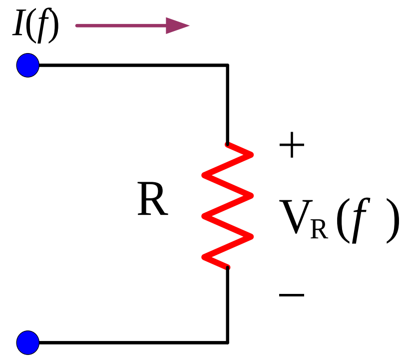
$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

Method 3

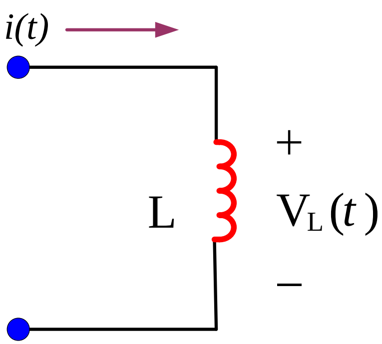
In this method we are going to transform the circuit to the Fourier domain . However we first see the FT on Basic elements



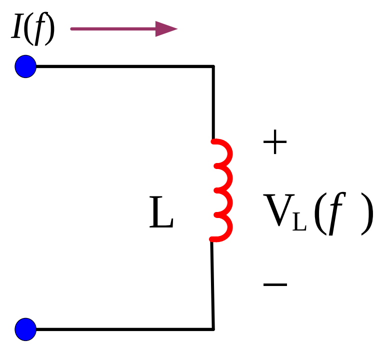
$$V_R(t) = R\,i(t)$$



$$V_R(\omega) = RI(\omega)$$

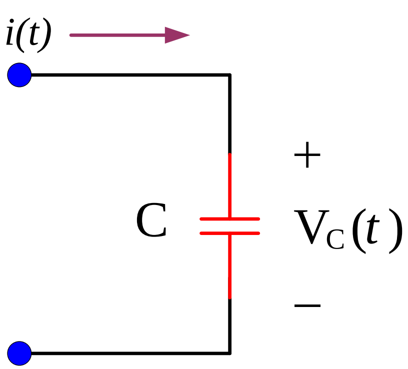


$$V_L(t) = L\frac{di(t)}{dt}$$

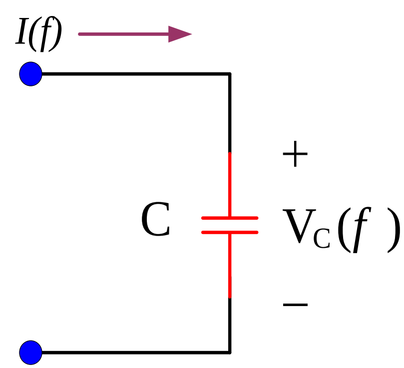


$$V_L(\omega) = L(j\omega)I(\omega)$$

$$= (j\omega L)I(\omega)$$



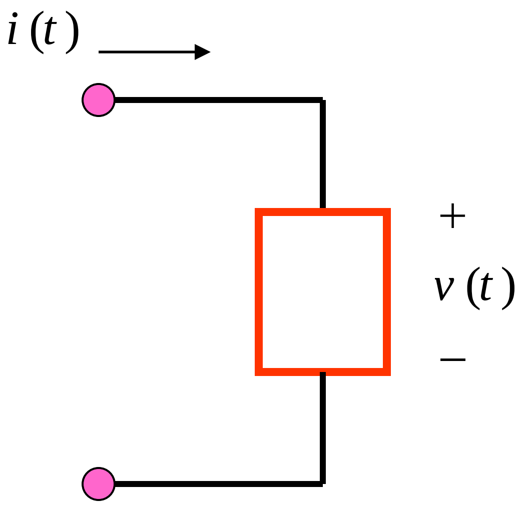
$$i(t) = C\frac{dV_C(t)}{dt}$$

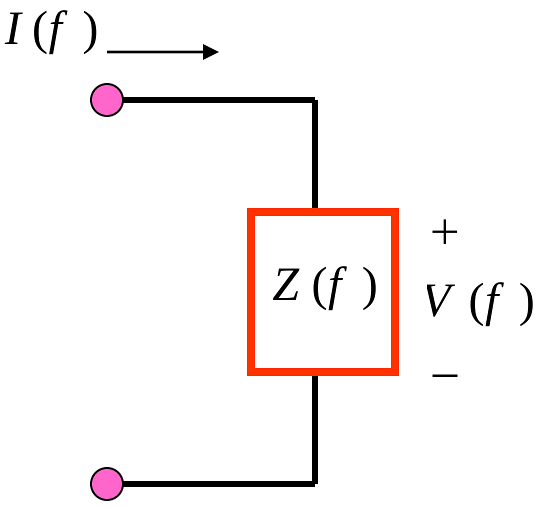


$$I(\omega) = C(j\omega)V_C(\omega)$$

$$V_C(f) = \frac{1}{(j\omega C)}I(f)$$

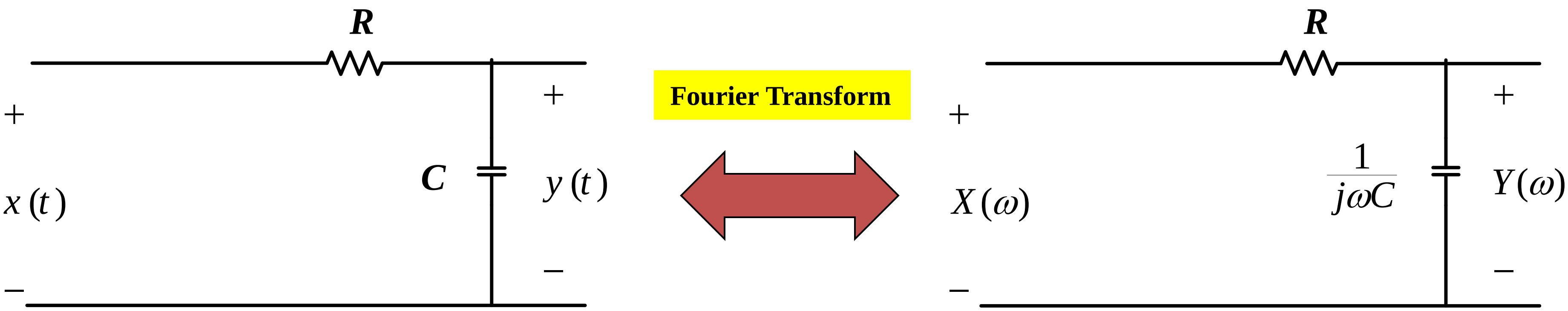
Method 3





$$Z(f) = \begin{cases} R & \text{Resistor} \\ j\omega L & \text{Inductor} \\ \dfrac{1}{j\omega C} & \text{Capacitor} \end{cases}$$

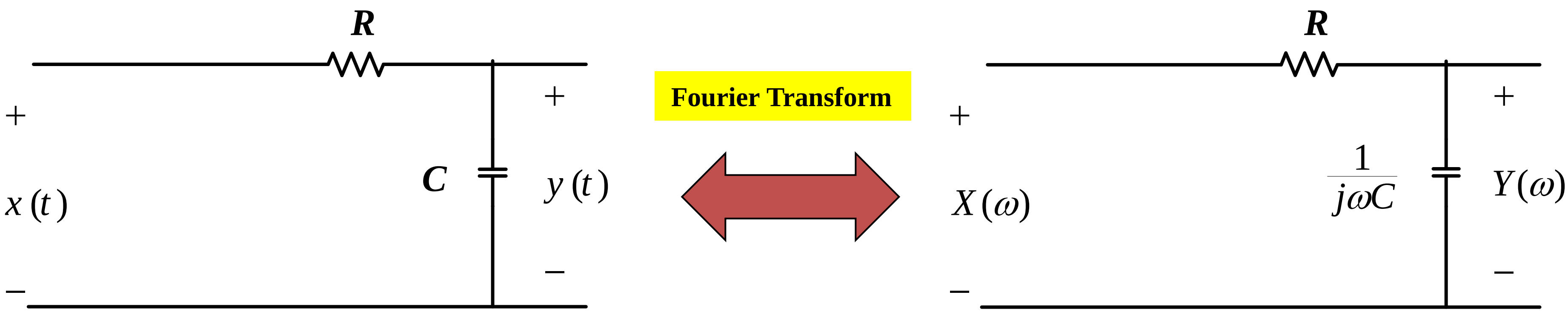
Method 3



$$Y(f)=\frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}}X(\omega) \qquad =\frac{1}{1+j\omega RC}X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)}=\frac{1}{1+j\omega RC}=H(\omega)$$

$$h(t)=\frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$

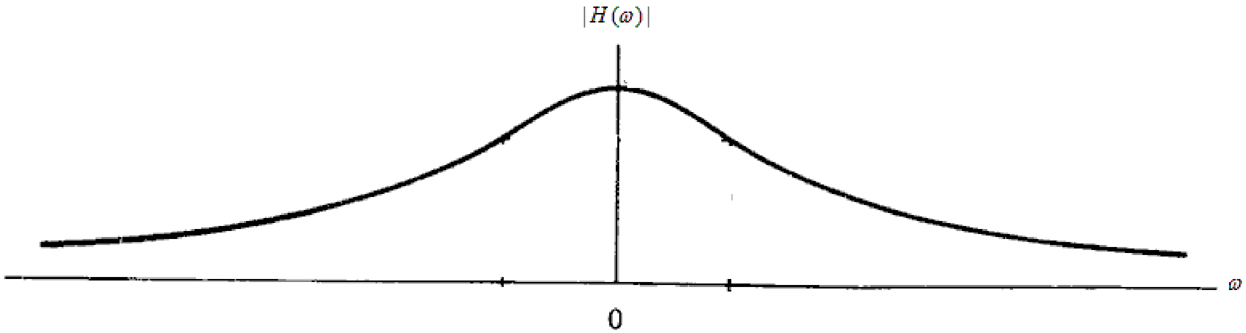


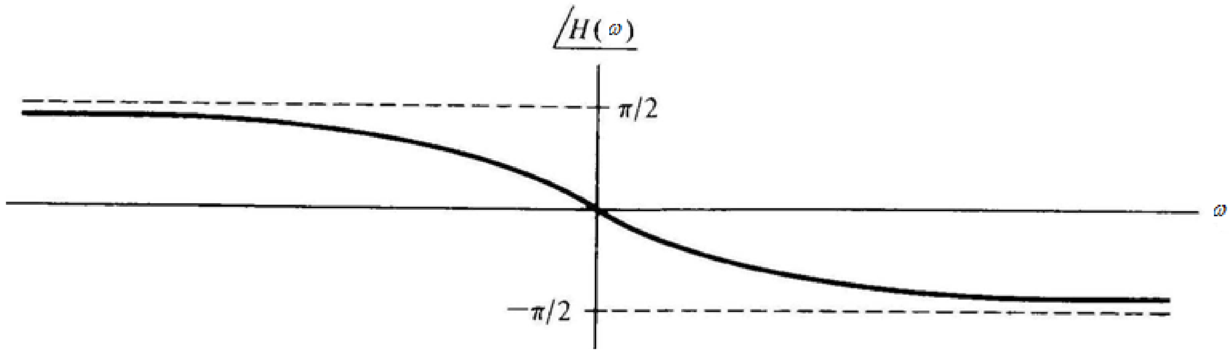
$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC} = H(\omega) = |H(\omega)| \angle H(\omega)$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \qquad \angle H(\omega) = -\tan^{-1}(\omega RC)$$

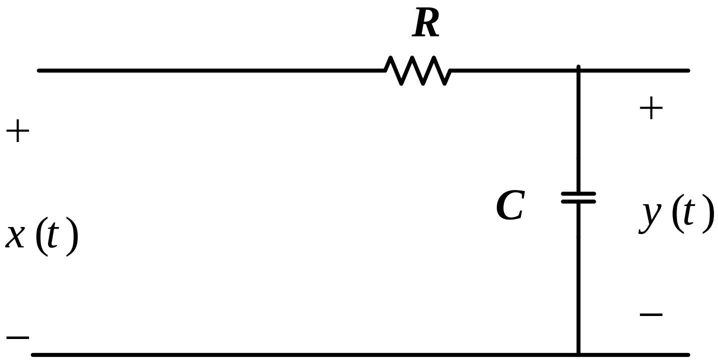
$$H(\omega) = \frac{1}{1 + j\omega RC}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \qquad \angle H(\omega) = -\tan^{-1}(\omega RC)$$





Example



Find y(t) if the input x(t) is

$$x(t) = A e^{-\alpha t} u(t)$$

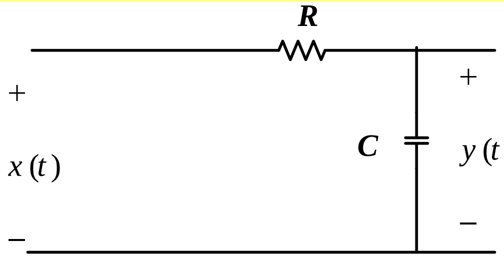
$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

Method 1 **(convolution method)**

Using the time domain (convolution method , Chapter 3)

$$y(t) = x(t) * h(t)$$

Method 2 **Fourier Transform**



$$x(t) = A e^{-\alpha t} u(t) \qquad X(\omega) = \frac{A}{\alpha + j\omega}$$

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t) \qquad H(\omega) = \frac{(1/RC)}{(1/RC) + (j\omega)}$$

$$Y(\omega) = X(\omega) H(\omega) \quad = \frac{A/RC}{(\alpha + j\omega)(1/RC + j\omega)}$$

Sine $Y(\omega)$ is not on the Fourier Transform Table 5-2

Using partial fraction expansion (will be shown later)

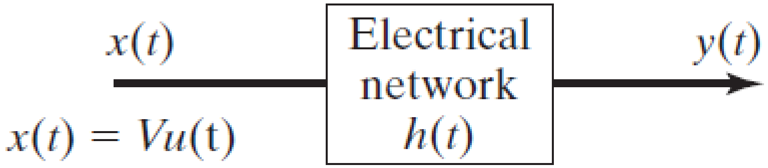
$$Y(\omega) = \frac{A}{\alpha RC - 1} \left[\frac{1}{1/RC + j\omega} - \frac{1}{\alpha + j\omega} \right]$$

From Table 5-2

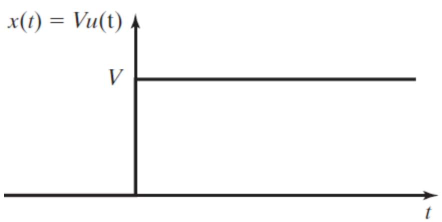
$$y(t) = \frac{A}{\alpha RC - 1} \left[e^{-t/RC} - e^{-\alpha t} \right] u(t)$$

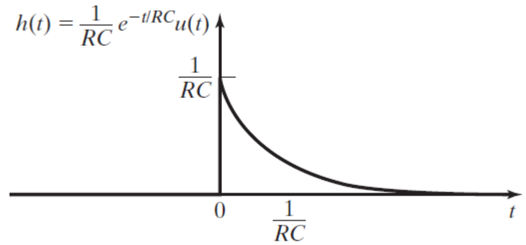
Example

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$



Find $y(t)$



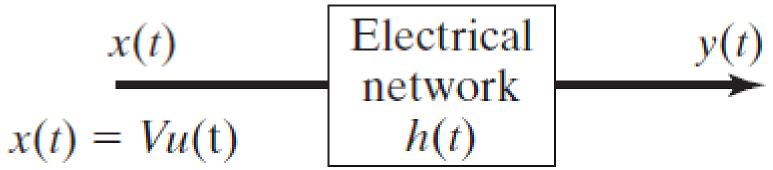


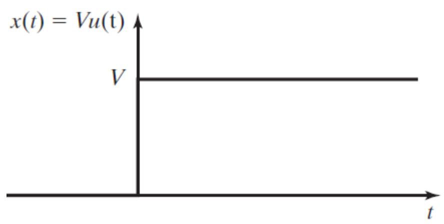
Method 1 **(convolution method)**

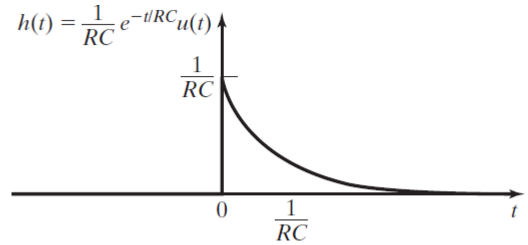
$$y(t) = x(t) * h(t)$$

Method 2 **Fourier Transform**

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$







From Table 5.2, the Fourier transforms of $h(t)$ and $x(t)$ are found to be

$$H(\omega) = \mathscr{F}\{(1/RC)e^{-t/RC}u(t)\} = \frac{1}{1 + j\omega RC} \qquad X(\omega) = \mathscr{F}\{Vu(t)\} = V\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

Therefore, $$Y(\omega) = X(\omega)H(\omega) = V\left[\frac{1}{1 + j\omega RC}\right]\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= V\left[\frac{1}{j\omega(1 + j\omega RC)} + \frac{\pi\delta(\omega)}{1 + j\omega RC}\right]$$

$$Y(\omega) = V\left[\frac{1}{j\omega(1 + j\omega RC)} + \frac{\pi\delta(\omega)}{1 + j\omega RC}\right]$$

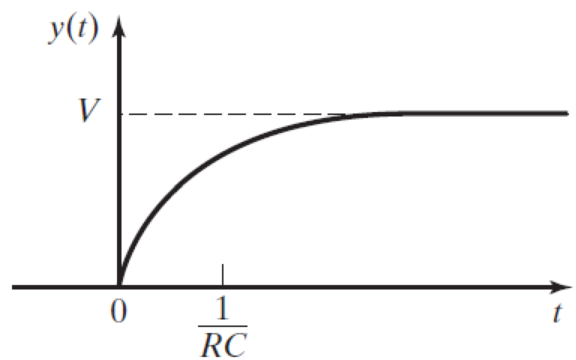
Partial fraction expansion of the first term in brackets (see Appendix F) gives

$$Y(\omega) = V\left[\frac{-RC}{1 + j\omega RC} + \frac{1}{j\omega} + \frac{\pi\delta(\omega)}{1 + j\omega RC}\right] = V\left[\frac{-RC}{1 + j\omega RC} + \frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$\frac{-RC}{1 + j\omega RC} \leftrightarrow e^{-t/RC}u(t) \qquad \frac{1}{j\omega} + \pi\delta(\omega) \leftrightarrow u(t)$$

$$\updownarrow$$

$$y(t) = V(1 - e^{-t/RC})u(t)$$



5.6 ENERGY AND POWER DENSITY SPECTRA

An energy signal is defined in Section 5.1 as a waveform, $f(t)$, for which

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty$$ where E is the energy associated with the signal

If the signal is written in terms of its Fourier transform $$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega$$

its energy equation can be rewritten as $$E = \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega\right]dt$$

The order of integration can be rearranged so that $$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\underbrace{\left[\int_{-\infty}^{\infty} f(t)e^{j\omega t} dt\right]}_{F(-\omega)}d\omega$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F(-\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F^*(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$ *Parseval's theorem*

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{\pi}\int_{0}^{\infty} |F(\omega)|^2 d\omega$$

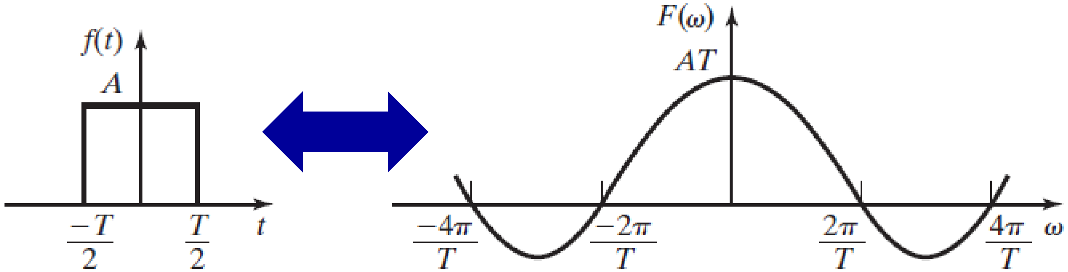
The *energy spectral density* function of the signal $f(t)$ is defined as

$$\mathscr{E}_f(\omega) \equiv \frac{1}{\pi}|F(\omega)|^2 = \frac{1}{\pi} F(\omega)F(\omega)^* \quad \Rightarrow \quad E = \int_0^{\infty} \mathscr{E}_f(\omega)d\omega$$

Therefor , to obtain the energy spectral density function $\mathscr{E}_f$ of the signal $f(t)$:

$$f(t) \Rightarrow F(\omega) \Rightarrow |F(\omega)|^2 \Rightarrow \frac{|F(\omega)|^2}{2\pi} \Rightarrow \text{fold about } \omega=0 \Rightarrow \frac{|F(\omega)|^2}{\pi}$$

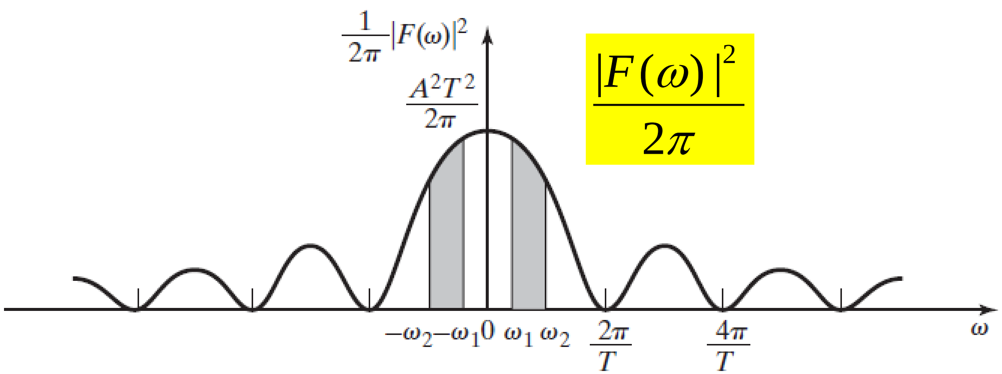
EXAMPLE 5.19 Energy spectral density of a rectangular pulse

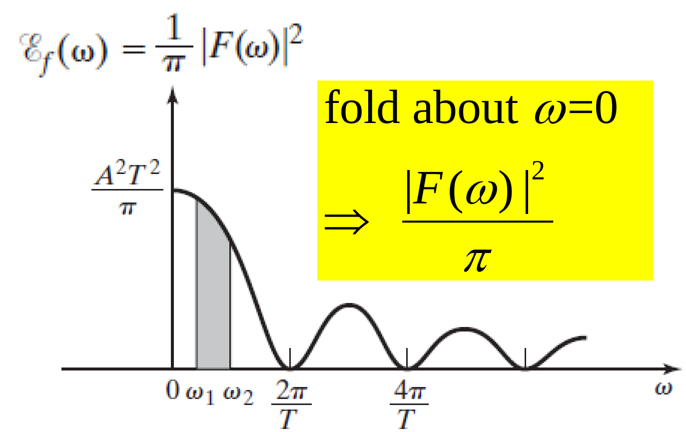


For the rectangular waveform the frequency spectrum sinc function

We now find the energy spectral density

$$\mathscr{E}_f(\omega) \equiv \frac{1}{\pi}|F(\omega)|^2 = \frac{1}{\pi}F(\omega)F(\omega)^*$$





We find the energy contained in some band of frequencies of particular interest by finding the area under the energy spectral density curve over that band of frequencies

$$E_B = \int_{\omega_1}^{\omega_2} \mathscr{E}_f(\omega)\,d\omega$$

Power Density Spectrum

we consider signals that have infinite energy but contain a finite amount of power

For these signals, the normalized average signal power is finite:

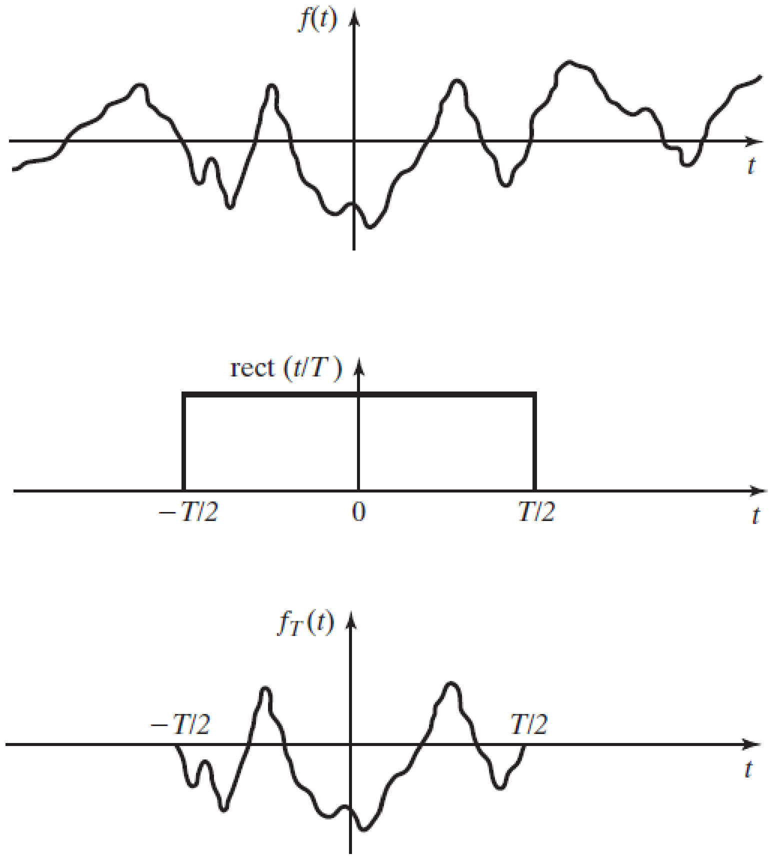
$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt < \infty$$

Such signals are called *power signals*

The step function, the signum function, and all periodic functions are examples of power signals

A problem with working in the frequency domain in the case of power signals arises from the fact that power signals have infinite energy and, therefore, may not be Fourier transformable.

To overcome this problem, a version of the signal that is truncated in time is employed

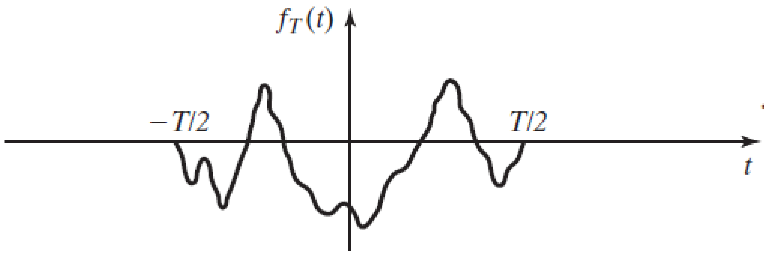


The signal $f_T(t)$ is a truncated

$$f_T(t) = f(t)\,\mathrm{rect}(t/T)$$

This signal meets the other Dirichlet conditions and, therefore, has a Fourier transform:

$$f_T(t) \overset{\mathscr{F}}{\longleftrightarrow} F_T(\omega)$$



$$f_T(t) = f(t)\,\mathrm{rect}(t/T)$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |f_T(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$

Because $f_T(t)$ has finite energy, the integral term can be recognized as the total energy contained in the truncated signal:

$$E = \int_{-\infty}^{\infty} |f_T(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2\, d\omega$$ By Parseval's theorem

$$\Rightarrow \quad P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |f_T(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} |F_T(\omega)|^2\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} \frac{1}{T} |F_T(\omega)|^2}_{\mathcal{P}_f(\omega)} d\omega$$

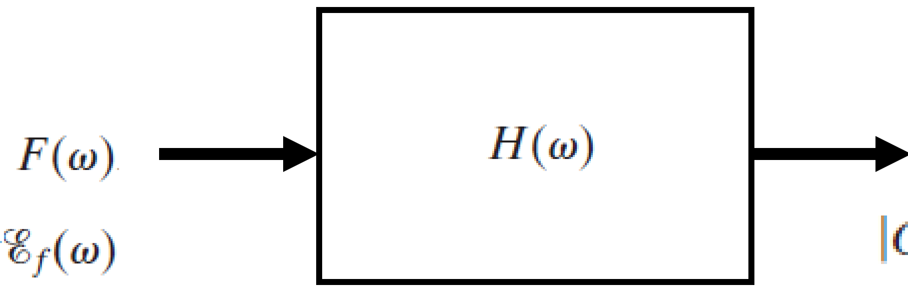
$\mathcal{P}_f(\omega)$

power spctral density (PSD)

$$\mathcal{P}_f(\omega) \equiv \lim_{T\to\infty} \frac{1}{T} |F_T(\omega)|^2 \quad \Rightarrow \quad P = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{P}_f(\omega) d\omega = \frac{1}{\pi} \int_{0}^{\infty} \mathcal{P}_f(\omega) d\omega$$

$\mathcal{P}_f(\omega)$ is an even function

Power and Energy Transmission The input–output relationship of a system



$$G(\omega) = H(\omega)F(\omega)$$

$$|G(\omega)|^2 = |H(\omega)|^2|F(\omega)|^2$$

$$\mathscr{E}_g(\omega) = |H(\omega)|^2\mathscr{E}_f(\omega)$$

can also be expressed in terms of the energy or power spectral densities of the input and output signals.

$$G^*(\omega) = [H(\omega)F(\omega)]^* = H^*(\omega)F^*(\omega)$$

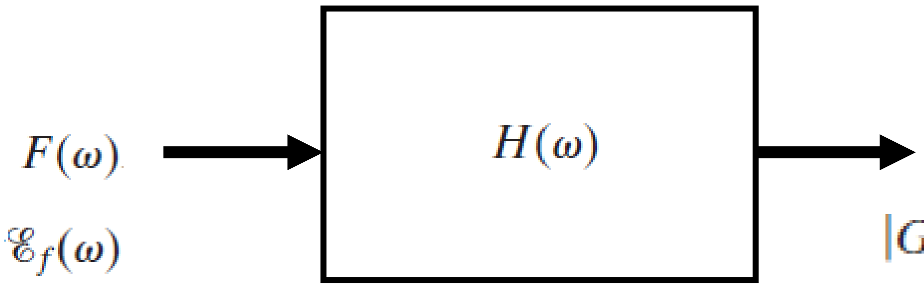
$$G(\omega)G^*(\omega) = H(\omega)H^*(\omega)F(\omega)F^*(\omega)$$

$$|G(\omega)|^2 = |H(\omega)|^2|F(\omega)|^2$$

If both sides on this expression are divided by π and the equivalents from

$$\mathscr{E}_f(\omega) \equiv \frac{1}{\pi}|F(\omega)|^2 = \frac{1}{\pi}F(\omega)F(\omega)^*$$

$$\mathscr{E}_g(\omega) = |H(\omega)|^2\mathscr{E}_f(\omega)$$



$$G(\omega) = H(\omega)F(\omega)$$

$$|G(\omega)|^2 = |H(\omega)|^2|F(\omega)|^2$$

$$\mathcal{E}_g(\omega) = |H(\omega)|^2\mathcal{E}_f(\omega)$$

$$\mathcal{P}_g(\omega) = |H(\omega)|^2\mathcal{P}_f(\omega)$$

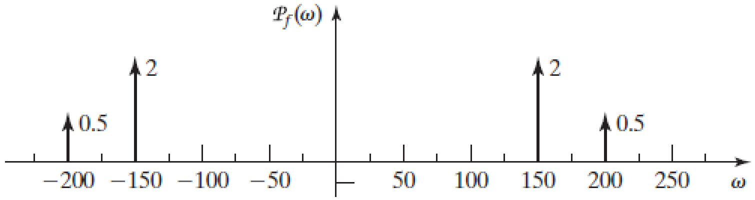
For the case that the input to a system is a power signal, the time-averaging tion can be applied to both sides of $|G(\omega)|^2 = |H(\omega)|^2|F(\omega)|^2$

$$\lim_{T\to\infty}\frac{1}{T}|G_T(\omega)|^2 = |H(\omega)|^2\lim_{T\to\infty}\frac{1}{T}|F_T(\omega)|^2$$

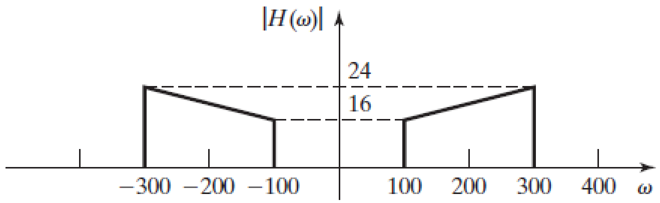
from $\mathcal{P}_f(\omega) \equiv \lim_{T\to\infty}\frac{1}{T}|F_T(\omega)|^2$

$$\mathcal{P}_g(\omega) = |H(\omega)|^2\mathcal{P}_f(\omega)$$

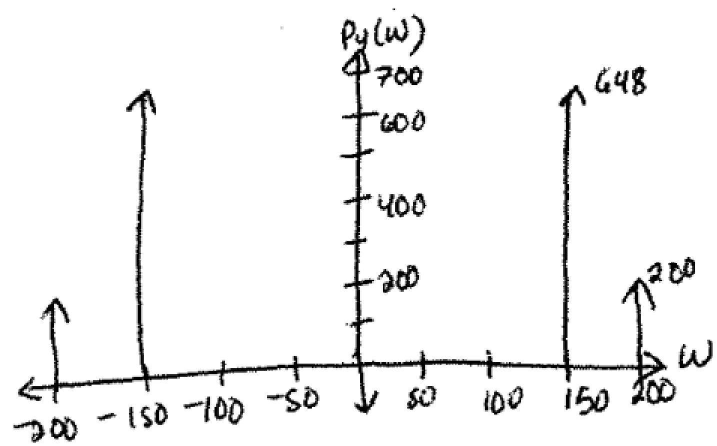
5.28. **(a)** A power signal with the power spectral density shown in Figure P5.28(a) is the input to a linear system with the frequency response shown in Figure P5.28(b). Calculate and sketch the power spectral density of the system's output signal.



(a)



(b)



(a) $P_y(\omega) = P_f(\omega)|H(\omega)|^2$

$$H(150) = 8/200(150 - 100) + 16 = 18$$

$$H(200) = 8/200(200 - 100) + 16 = 20$$

$$P_y(\omega) = (20^2)0.5\delta(\omega + 200) + (18^2)2\delta(\omega + 150) + (18^2)2\delta(\omega - 150) + (20^2)0.5\delta(\omega - 200)$$