Formula Sheet

Algebra of Sets

1) $A \cap B = B \cap A$, $A \cup B = B \cup A$

2) $A \substack{\cap \\ \cup} \left(B \substack{\cup \\ \cap} C \right) = \left(A \substack{\cap \\ \cup} B \right) \substack{\cup \\ \cap} \left(A \substack{\cap \\ \cup} C \right)$

3) $A \substack{\cup \\ \cap} \left(B \substack{\cup \\ \cap} C \right) = \left(A \substack{\cup \\ \cap} B \right) \substack{\cup \\ \cap} C = A \substack{\cup \\ \cap} B \substack{\cup \\ \cap} C$

4) $\left(\overline{B \substack{\cup \\ \cap} C} \right) = \overline{A} \substack{\cap \\ \cup} \overline{B}$

Joint and Conditional Probability

5) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

6) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

7) For mutually exclusive events B_n with $\bigcup_{n=1}^{N} B_n = S$,

$$P(A) = \sum_{n=1}^{N} P(A|B_n)P(B_n)$$

8) Bayes' Rule $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

9) For Independent events *A* and *B*

$$P(A \cap B) = P(A)P(B)$$

$$P(A|B) = P(A), \quad P(B|A) = P(B)$$

Permutations and Combinations

10) No. of Permutations of *r* elements from *n*: $P_r^n = \dfrac{n!}{(n-r)!}$

11) No. Combinations of *r* elements from *n*: $\dbinom{n}{r} = \dfrac{n!}{(n-r)!r!}$

Bernoulli Trials

12) For an event *A* with $P(A) = p$

$$P\{\text{exactly } k \text{ occurances in } N \text{ trials}\} = \binom{N}{k} p^k (1-p)^{N-k}$$

Distribution Function (CDF)

13) $F_X(x) = P\{X \le x\}$

Properties of CDFs

14) $F_X(-\infty) = 0$

15) $F_X(\infty) = 1$

16) $0 \le F_X(x) \le 1$

17) $F_X(x_1) \le F_X(x_2)$ if $x_1 < x_2$

18) $P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$

19) $F_X(x^+) = F_X(x)$

Properties of PDFs

20) $0 \le f_X(x)$ for all x

21) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

22) $F_X(x) = \int_{-\infty}^{x} f_X(\alpha)d\alpha$

23) $P\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(\alpha)d\alpha$

CDF of a Discrete RV

24) For discrete X with values x_i with $P\{x_i\} = P\{X = x_i\}$

$$f_X(x) = \sum_{i=1}^{N} P\{x_i\}\delta(x - x_i)$$

$$F_X(x) = \sum_{i=1}^{N} P\{x_i\}u(x - x_i)$$

A Gaussian RV has

25) $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_X^2}}$,

26) $F_X(x) = \dfrac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-\frac{(\alpha-a_x)^2}{2\sigma_X^2}} d\alpha$

A Binomial RV has

27) $f_X(x) = \sum_{k=0}^{N} \dbinom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$

28) $F_X(x) = \sum_{k=0}^{N} \dbinom{N}{k} p^k (1-p)^{N-k} u(x-k)$

A Uniform RV has

29) $f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$

30) $F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & b \le x \end{cases}$

A Poisson RV with

31) $b = \lambda \cdot T$, λ = rate of arrivals, T = period of interest

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

An Exponential RV has

32) $f_X(x) = \begin{cases} 0 & x < a \\ \dfrac{1}{b} e^{-\frac{x-a}{b}} & x > a \end{cases}$

33) $F_X(x) = \begin{cases} 0 & x < a \\ 1 - e^{-\frac{x-a}{b}} & x > a \end{cases}$

An Rayleigh RV has

34) $f_X(x) = \begin{cases} 0 & x < a \\ \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} & x \ge a \end{cases}$

35) $F_X(x) = \begin{cases} 0 & x < a \\ 1 - e^{-\frac{(x-a)^2}{b}} & x \ge a \end{cases}$

Conditional Distribution

36) $F_X(x \mid B) = P\{X \le x \mid B\} = \dfrac{P\{X \le x \cap B\}}{P(B)}$

37) $f_X(x \mid B) = \dfrac{dF_X(x \mid B)}{dx}$

Expected value of an RV or a function of an RV

38) $E[X] = \overline{X} = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$

39) $E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx$

Moments about the origin and Central Moments

40) $m_n = E[X^n]$ 41) $\mu_n = E\left[\left(X - \overline{X}\right)^n\right]$

42) Variance $\sigma_X^2 = \mu_2$

43) Skew $\mu_3 = E\left[\left(X - \overline{X}\right)^3\right]$

Characteristic Function

44) $\Phi_X(\omega) = E\left[e^{j\omega X}\right]$ 45)

$$m_n = (-j)^n \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$

Transformation of RVs

46) If $Y = T(X)$ and T is monotonic function

$$f_Y(y) = f_X\left(T^{-1}(y)\right)\left|\frac{dT^{-1}(y)}{dy}\right|$$

47) If $Y = T(X)$ and T is non–monotonic function

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{\left|\left.\frac{dT(x)}{dx}\right|_{x=x_n}\right|}$$

48) If $Y = T(X)$ is a constant y_1 over some period $x_1 \le X \le x_2$, the RV Y has discrete component with

$$P\{Y = y_1\} = F_X(x_1) - F_X(x_2) = \int_{x_1}^{x_2} f_X(x)dx$$

Joint Distributions

49) $F_{XY}(x,y) = P\{X \le x, Y \le y\}$

50) $f_{XY}(x,y) = \dfrac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y}$

Properties of Joint CDFs

51) $F_{XY}(-\infty,-\infty) = 0$, $F_{XY}(\infty,\infty) = 1$

52) $F_{XY}(-\infty,y) = 0$, $F_{XY}(x,-\infty) = 0$

53) $0 \le F_{XY}(x,y) \le 1$

54) $F_{XY}(x,y)$ is non–decreasing of x and y

55) $P\{x_1 \le X \le x_2, y_1 \le Y \le y_2\} = F_{XY}(x_2,y_2) + $
$+ F_{XY}(x_1,y_1) - F_{XY}(x_2,y_1) - F_{XY}(x_1,y_2)$

56) $F_{XY}(x,\infty) = F_X(x)$, $F_{XY}(\infty,y) = F_Y(y)$

Properties of Joint PDFs

57) $f_{XY}(x,y) \ge 0$

58) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$

59) $F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(\alpha,\beta)d\beta d\alpha$

$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(\alpha,\beta)d\alpha d\beta$

60) $P\{x_1 \le X \le x_2, y_1 \le Y \le y_2\} = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)dydx$

61) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$,

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$

Conditional Density and Distribution

62) $f_X(x \mid B) = \dfrac{dF_X(x \mid B)}{dx}$

63) For discrete RV X and Y such that X takes values x_i

$$f_X(x \mid Y = y_k) = \sum_{i=1}^{N} \frac{P(x_i,y_k)}{P(y_k)}\delta(x - x_i)$$

$$F_X(x \mid Y = y_k) = \sum_{i=1}^{N} \frac{P(x_i,y_k)}{P(y_k)}u(x - x_i)$$

64) For continuous RVs X and Y

$$f_X(x \mid Y = y) = f_X(x \mid y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Statistical Independence

65) X and Y are statistically independent if any is true

$F_{XY}(x,y) = F_X(x)F_Y(y)$,

$f_{XY}(x,y) = f_X(x)f_Y(y)$

Distribution of Sum of RVs

66) for $W = X + Y \Rightarrow F_W(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{w-x} f_{XY}(x,y)\,dy\,dx$

67) If X and Y are Statistically Independent

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x)\cdot f_Y(w-x)\,dx$$

Expectations of Multiple RVs

68) $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)\,dy\,dx$

69) $m_{nk} = E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\,dy\,dx$

70) $R_{XY} = m_{11} = E[XY]$

71) If X and Y are Statistically Independent

$R_{XY} = E[XY] = E[X]\cdot E[Y]$

72) X and Y orthogonal if $R_{XY} = E[XY] = 0$

73) $\mu_{nk} = E\left[\left(X-\overline{X}\right)^n \left(Y-\overline{Y}\right)^k\right]$

74) $C_{XY} = \mu_{11} = E\left[\left(X-\overline{X}\right)\left(Y-\overline{Y}\right)\right] = R_{XY} - \overline{X}\,\overline{Y}$

75) $\rho = \dfrac{C_{XY}}{\sigma_X \sigma_Y}, \qquad -1 \le \rho \le 1$

Characteristic Function of Joint RVs

76) $\Phi_{XY}(\omega_1,\omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right]$

77) $m_{nk} = (-j)^{n+k} \left.\dfrac{\partial^{n+k}\Phi_{XY}(\omega_1,\omega_2)}{\partial\omega_1^n\,\partial\omega_2^k}\right|_{\omega_1=0,\ \omega_2=0}$

Jointly Gaussian Random Varibles

78)

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\overline{X})^2}{\sigma_X^2} - \frac{2\rho(x-\overline{X})(y-\overline{Y})}{\sigma_X\sigma_Y} + \frac{(y-\overline{Y})^2}{\sigma_Y^2}\right]}$$

Estimation of Mean, Power, and Variance

79) $\widehat{X}_N = \dfrac{1}{N}\sum_{n=1}^{N} X_n, \qquad \widehat{X_N^2} = \dfrac{1}{N}\sum_{n=1}^{N} X_n^2$

$$\widehat{\sigma_N^2} = \frac{1}{N-1}\sum_{n=1}^{N}\left(X_n - \widehat{X}_N\right)^2$$

Density and Distribution Function of Random of Processes

80) $F_X(x_1;t_1) = P\{X(t_1) \le x_1\}$

$f_X(x_1;t_1) = dF_X(x_1;t_1)/dx_1$

81) $F_X(x_1,x_2;t_1,t_2) = P\{X(t_1)\le x_1, X(t_2)\le x_2\}$

$f_X(x_1,x_2;t_1,t_2) = \partial^2 F_X(x_1,x_2;t_1,t_2)/(\partial x_1 \partial x_2)$

Stationarity of Random Processes

82) 1st–order stat: $f_X(x_1;t_1) = f_X(x_1;t_1+\Delta)$

83) 2nd–order stat:

$f_X(x_1,x_2;t_1,t_2) = f_X(x_1,x_2;t_1+\Delta,t_2+\Delta)$

84) If $X(t)$ is 1st and 2nd–order stat. ➔ $X(t)$ is strictly stat.

85) $R_{XX}(t_2,t_2) = E\left[X(t_1)X(t_2)\right]$

86) $C_{XX}(t_1,t_2) = R_{XX}(t_2,t_2) - E\left[X(t_1)\right]E\left[X(t_2)\right]$

87) $X(t)$ is wide–sense stationary if

$E\left[X(t)\right] = \overline{X} = \text{constant}$

$R_{XX}(t,t+\tau) = E\left[X(t)X(t+\tau)\right] = R_{XX}(\tau)$

Time Averages and Ergodicity

88) Time average is given by $A[\cdot] = \lim_{T\to\infty}\dfrac{1}{2T}\int_{-T}^{T}[\cdot]\,dt$

89) $\overline{x} = A\left[x(t)\right]$

90) $\Re_{xx}(\tau) = A\left[x(t)x(t+\tau)\right]$

91) If $\overline{x} = \overline{X}$ and $\Re_{xx}(\tau) = R_{XX}(\tau)$ ➔ $X(t)$ is Ergodic

Properties of $R_{XX}(\tau)$

92) $R_{XX}(0) = E\left[X^2(t)\right]$, 93) $R_{XX}(\tau) = R_{XX}(-\tau)$

94) $\left|R_{XX}(\tau)\right| \le R_{XX}(0)$

Transmission of a random process through a linear filter

If a w.s.s $X(t)$ is fed to a filter $h(t)$ to produce $Y(t)$

95) $Y(t) = h(t) * X(t)$

96) $\mu_Y = \mu_X H(0)$, $\quad R_{YY}(\tau) = h(\tau) * h(-\tau) * R_{XX}(\tau)$

97) $S_{XX}(f) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j2\pi f\tau}\,d\tau$,

$R_{XX}(\tau) = \int_{-\infty}^{+\infty} S_{XX}(f) e^{j2\pi f\tau}\,df$

98) $S_{YY}(f) = \left|H(f)\right|^2 S_{XX}(f)$

Properties of $S_{XX}(f)$

99) $S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)\,d\tau$ = DC power

100) $R_{XX}(0) = E\left[X^2(t)\right] = \int_{-\infty}^{\infty} S_{XX}(f)\,df$ = Total power

101) $S_{XX}(f) \ge 0$ for all f

102) $S_{XX}(f) = S_{XX}(-f)$ 103) $S_{XX}(f)$ is real