EE 207

Chapter 3 The Fourier Series

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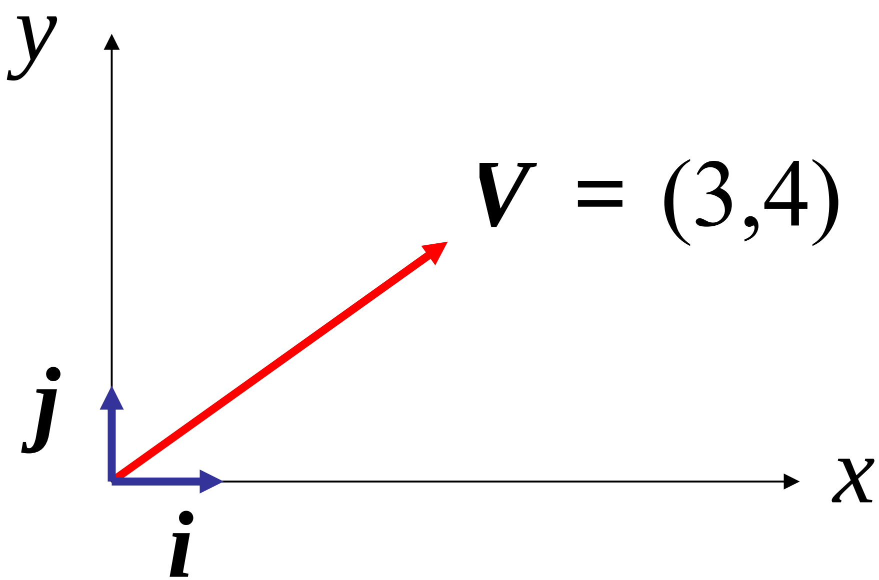


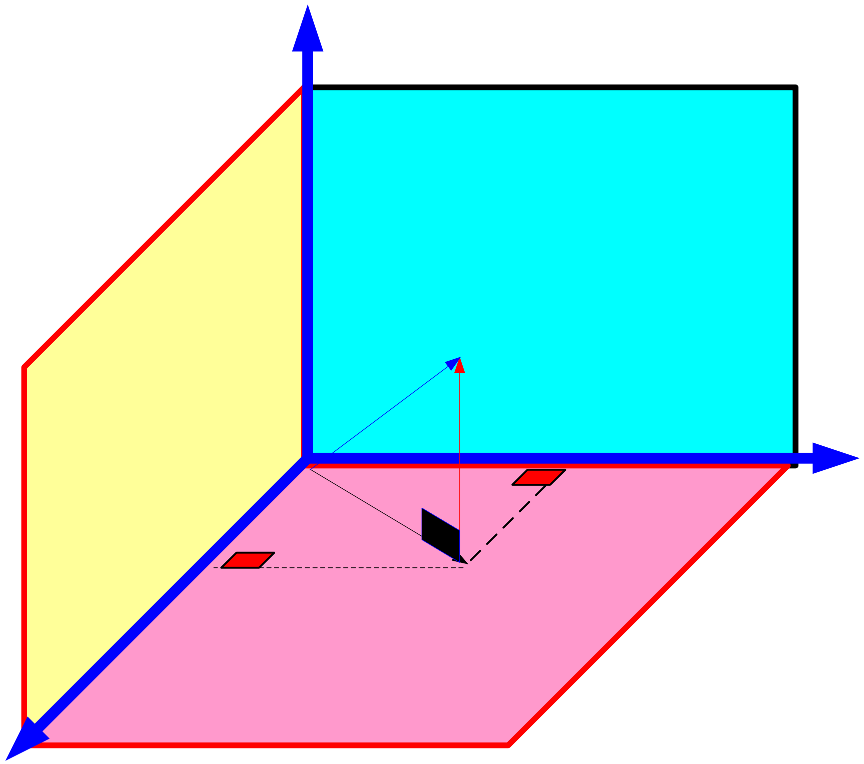
$$\boldsymbol{V} = (3,4)$$

$$\boldsymbol{V} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

$$\boldsymbol{V} = 3\boldsymbol{i} + 6\boldsymbol{j} + 4\boldsymbol{k}$$

$$\boldsymbol{V} = \alpha_1 \boldsymbol{i}_1 + \alpha_2 \boldsymbol{i}_2 + \alpha_3 \boldsymbol{i}_3 + \cdots + \alpha_N \boldsymbol{i}_N$$

$$\alpha_1 = \mathbf{V} \bullet \boldsymbol{i}_1 \quad \alpha_2 = \mathbf{V} \bullet \boldsymbol{i}_2 \quad \cdots \quad \alpha_N = \mathbf{V} \bullet \boldsymbol{i}_N$$



$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Integrating both side over one period

$$\int_{T_0} x(t)\,dt = \int_{T_0} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \right] dt$$

$$= \int_{T_0} a_0 dt + \int_{T_0} \sum_{n=1}^{\infty} a_n \cos n\omega_0 t dt + \int_{T_0} \sum_{n=1}^{\infty} b_n \sin n\omega_0 t dt$$

0 0

$$= \int_{T_0} a_0 dt + \sum_{n=1}^{\infty} \int_{T_0} a_n \sin n\omega_0 t dt + \sum_{n=1}^{\infty} \int_{T_0} b_n \cos n\omega_0 t dt$$

$$= \int_{T_0} a_0 dt = a_0 T_0$$

$$\Rightarrow \quad a_0 = \frac{1}{T_0} \int_{T_0} x(t)\,dt$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Multiplying both side by $\cos m\omega_0 t$ and Integrating over one period

$$\int_{T_0} x(t) \cos n\omega_0 t dt$$

$$= \int_{T_0} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \right] \cos m\omega_0 t dt$$

$$= \int_{T_0} a_0 \cos m\omega_0 t dt + \int_{T_0} \cos m\omega_0 t \sum_{n=1}^{\infty} a_n \cos n\omega_0 t dt$$

$$+ \int_{T_0} \cos m\omega_0 t \sum_{n=1}^{\infty} b_n \sin n\omega_0 t dt$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t$$

$$\int_{T_0} x(t)\cos n\omega_0 t dt$$

0

$$=\int_{T_0} a_0\cos m\omega_0 t dt+\int_{T_0}\cos m\omega_0 t\sum_{n=1}^{\infty}a_n\cos n\omega_0 t dt$$

$$+\int_{T_0}\cos m\omega_0 t\sum_{n=1}^{\infty}b_n\sin n\omega_0 t dt$$

$$\Rightarrow\quad \int_{T_0} x(t)\cos n\omega_0 t dt=\int_{T_0}\cos m\omega_0 t\sum_{n=1}^{\infty}a_n\cos n\omega_0 t dt$$

$$+\int_{T_0}\cos m\omega_0 t\sum_{n=1}^{\infty}b_n\sin n\omega_0 t dt$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\int_{T_0} x(t)\cos n\omega_0 t\,dt = \int_{T_0} \cos m\omega_0 t \sum_{n=1}^{\infty} a_n \cos n\omega_0 t\,dt$$

$$+ \int_{T_0} \cos m\omega_0 t \sum_{n=1}^{\infty} b_n \sin n\omega_0 t\,dt$$

$\sum_{n=1}^{\infty}$ Summation with respect to n

$$\Rightarrow \cos m\omega_0 t \sum_{n=1}^{\infty} a_n \cos n\omega_0 t = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t \cos m\omega_0 t$$

$$\Rightarrow \cos m\omega_0 t \sum_{n=1}^{\infty} b_n \sin n\omega_0 t = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \cos m\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\int_{T_0} x(t) \cos m\omega_0 t dt = \int_{T_0} \cos m\omega_0 t \sum_{n=1}^{\infty} a_n \cos n\omega_0 t dt$$

$$+ \int_{T_0} \cos m\omega_0 t \sum_{n=1}^{\infty} b_n \sin n\omega_0 t dt$$

$$= \int_{T_0} \sum_{n=1}^{\infty} a_n \cos n\omega_0 t \cos m\omega_0 t dt$$

$$+ \int_{T_0} \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \cos m\omega_0 t dt$$

$$\cos n\omega_0 t \cos m\omega_0 t = \frac{1}{2}\cos(n-m)\omega_0 t + \frac{1}{2}\cos(n+m)\omega_0 t$$

$$\sin n\omega_0 t \cos m\omega_0 t = \frac{1}{2}\sin(n-m)\omega_0 t + \frac{1}{2}\sin(n+m)\omega_0 t$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\int_{T_0} x(t)\cos m\omega_0 t dt = \int_{T_0} \sum_{n=1}^{\infty} a_n \cos n\omega_0 t \cos m\omega_0 t dt$$

$$+ \int_{T_0} \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \cos m\omega_0 t dt$$

$$= \int_{T_0} \sum_{n=1}^{\infty} a_n \left[\frac{1}{2}\cos(n-m)\omega_0 t + \frac{1}{2}\cos(n+m)\omega_0 t \right] dt$$

$$+ \int_{T_0} \sum_{n=1}^{\infty} b_n \left[\frac{1}{2}\sin(n-m)\omega_0 t + \frac{1}{2}\sin(n+m)\omega_0 t \right] dt$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\int_{T_0} x(t) \cos m\omega_0 t dt = \int_{T_0} \sum_{n=1}^{\infty} a_n \left[\frac{1}{2} \cos(n-m)\omega_0 t + \frac{1}{2} \cos(n+m)\omega_0 t \right] dt$$

$$+ \int_{T_0} \sum_{n=1}^{\infty} b_n \left[\frac{1}{2} \sin(n-m)\omega_0 t + \frac{1}{2} \sin(n+m)\omega_0 t \right] dt$$

since $\int_{T_0} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \int_{T_0}$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\int_{T_0} x(t)\cos m\omega_0 t\,dt = \int_{T_0} \sum_{n=1}^{\infty} a_n \left[\frac{1}{2}\cos(n-m)\omega_0 t + \frac{1}{2}\cos(n+m)\omega_0 t \right] dt$$

$$+ \int_{T_0} \sum_{n=1}^{\infty} b_n \left[\frac{1}{2}\sin(n-m)\omega_0 t + \frac{1}{2}\sin(n+m)\omega_0 t \right] dt$$

$$= \sum_{n=1}^{\infty} a_n \frac{1}{2} \left[\int_{T_0} \cos(n-m)\omega_0 t\,dt + \int_{T_0} \cos(n+m)\omega_0 t \right] dt$$

$$+ \sum_{n=1}^{\infty} b_n \frac{1}{2} \left[\int_{T_0} \sin(n-m)\omega_0 t\,dt + \int_{T_0} \sin(n+m)\omega_0 t \right] dt$$

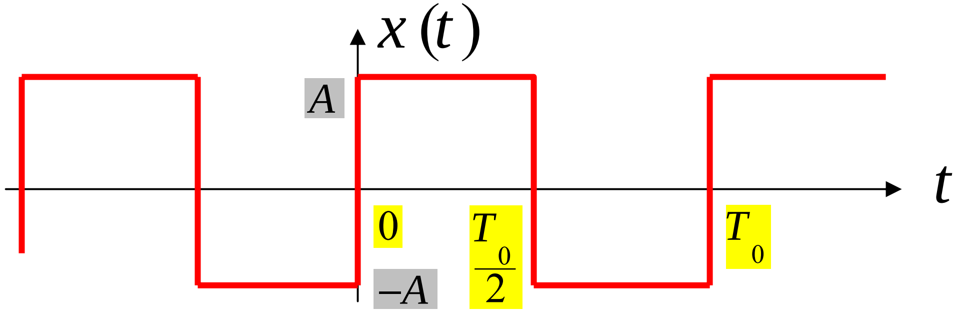
$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$ The average of x(t)

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t dt \qquad n \neq 0$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin n\omega_0 t dt$$

Example 3-4



The average value of x(t) = 0 $\Longrightarrow \quad a_0 = 0$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt \quad = \frac{2}{T_0}\int_0^{T_0/2} A\cos n\omega_0 t\,dt + \frac{2}{T_0}\int_{T_0/2}^{T_0} (-A)\cos n\omega_0 t\,dt$$

$$= \frac{2A}{T_0}\left[\left.\frac{\sin n\omega_0 t}{n\omega_0}\right|_0^{T_0/2} - \left.\frac{\sin n\omega_0 t}{n\omega_0}\right|_{T_0/2}^{T_0} \right] = 0$$

⇨ Thus all the a_n coefficients are zero

Note : $x(t)$ odd ⇨ $x(t)\cos n\omega_0 t$ is odd ⇨ $a_n = 0$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt \quad = \frac{2}{T_0}\int_0^{T_0/2} A\sin n\omega_0 t\,dt + \frac{2}{T_0}\int_{T_0/2}^{T_0} (-A)\sin n\omega_0 t\,dt$$

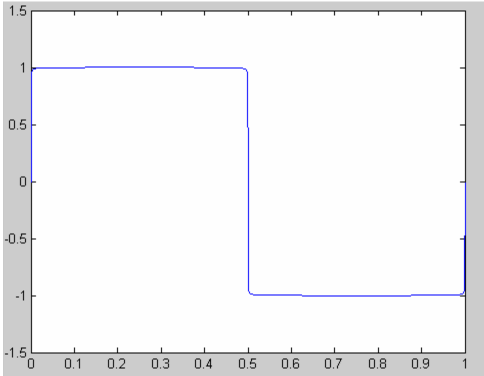
$$= \frac{2A}{n\pi}\left(1-\cos n\pi\right)$$

n odd $\rightarrow \cos n\pi = -1$

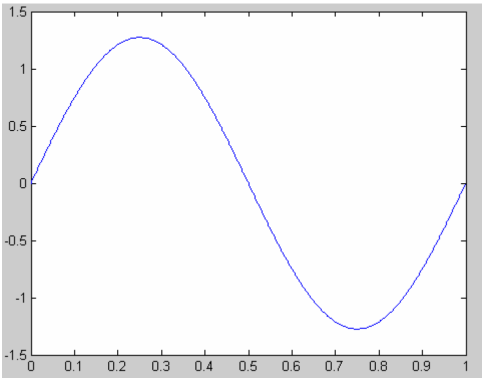
n even $\rightarrow \cos n\pi = +1$

$$b_n = \frac{2A}{n\pi}\left(1-\cos n\pi\right) = \begin{cases} \dfrac{2A}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

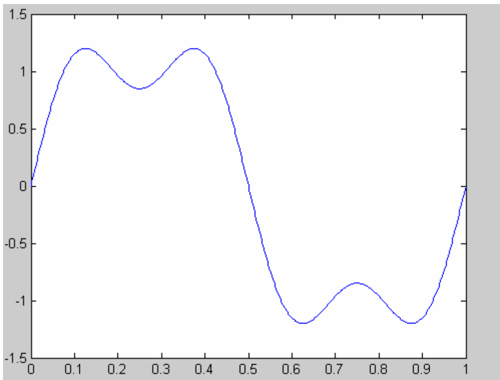
$$x(t) = \frac{4A}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right)$$


1.5
1
0.5
0
-0.5
-1
-1.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

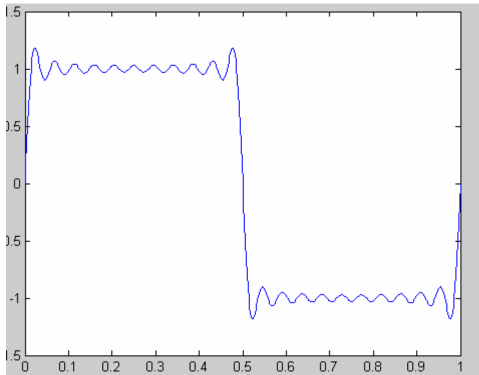
$n = 1$


1.5
1
0.5
0
-0.5
-1
-1.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

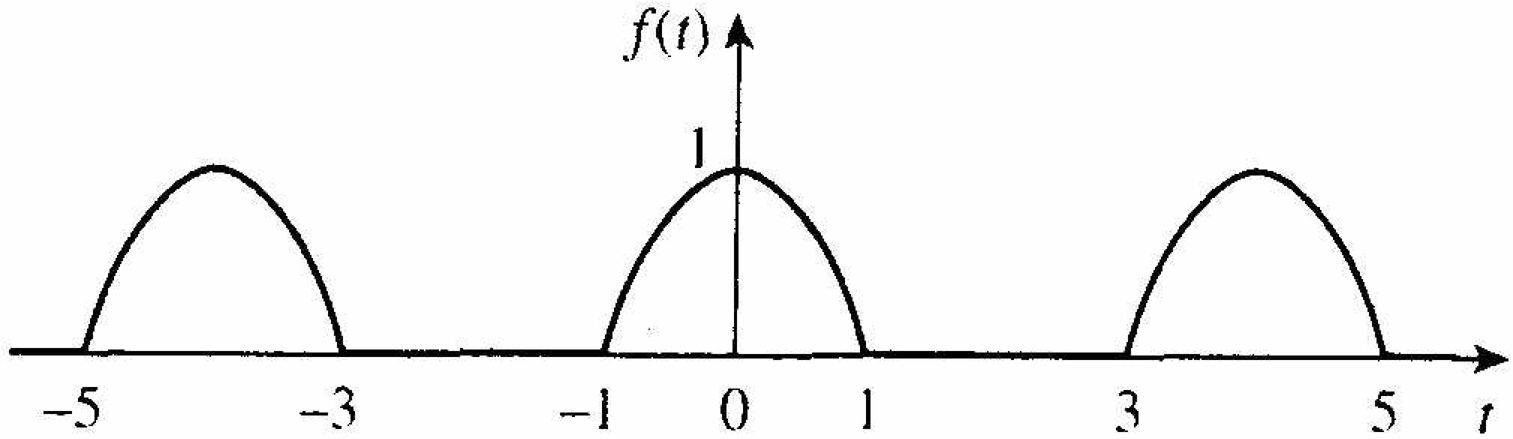
$n = 3$


1.5
1
0.5
0
-0.5
-1
-1.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

$n = 21$


1
0
-1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

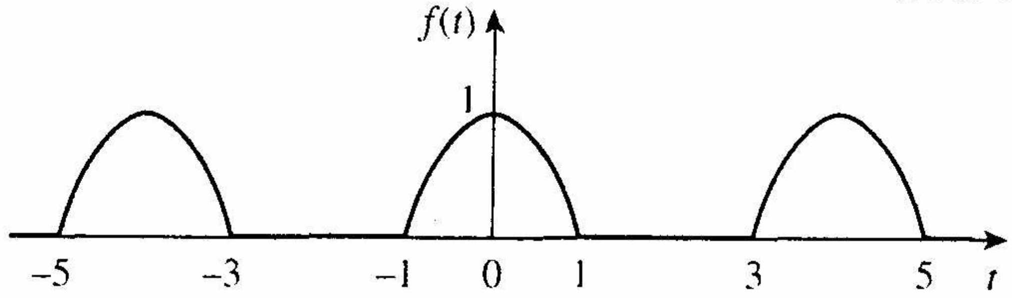
A half-wave rectified cosine function



This is an even function so that $b_n = 0$ $\quad T = 4, \omega_0 = 2\pi/T = \pi/2$

Over a period

$$f(t) = \begin{cases} 0, & -2 < t < -1 \\ \cos\frac{\pi}{2}t, & -1 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$



Over a period

$$f(t) = \begin{cases} 0, & -2 < t < -1 \\ \cos\frac{\pi}{2}t, & -1 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

$$b_n = 0 \quad T = 4, \omega_0 = 2\pi/T = \pi/2$$

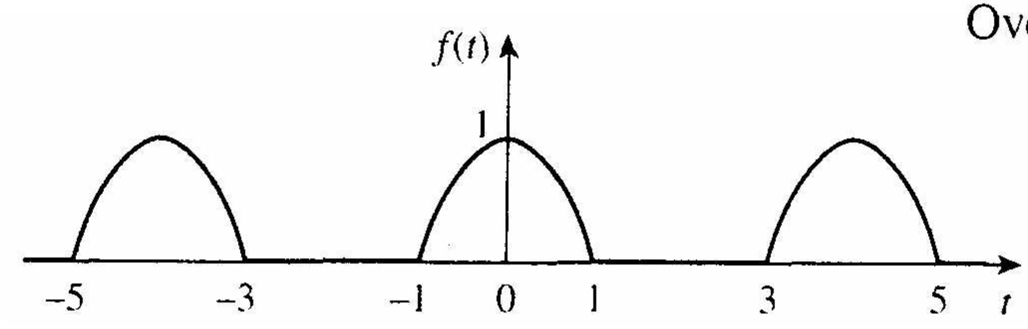
$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt = \frac{2}{4}\left[\int_0^1 \cos\frac{\pi}{2}t\,dt + \int_1^2 0\,dt\right] = \frac{1}{2}\frac{2}{\pi}\sin\frac{\pi}{2}t\Big|_0^1 = \frac{1}{\pi}$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt = \frac{4}{4}\left[\int_0^1 \cos\frac{\pi}{2}t\cos\frac{n\pi t}{2}\,dt + 0\right]$$

But $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$a_n = \frac{1}{2}\int_0^1\left[\cos\frac{\pi}{2}(n+1)t + \cos\frac{\pi}{2}(n-1)t\right]dt$$

$$= \frac{1}{\pi(n+1)}\sin\frac{\pi}{2}(n+1) + \frac{1}{\pi(n-1)}\sin\frac{\pi}{2}(n-1) \qquad n > 1$$



Over a period

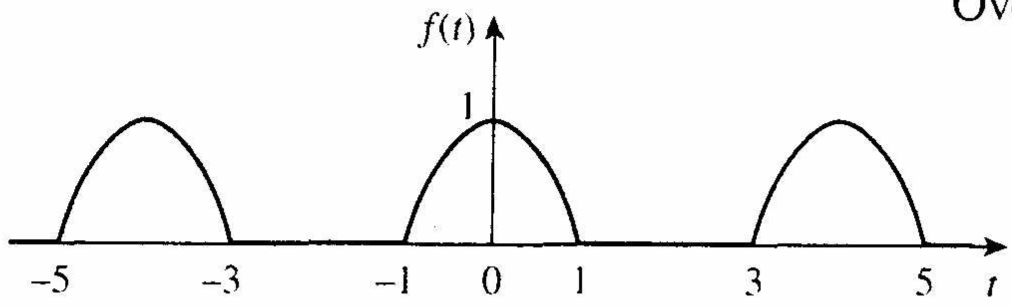
$$f(t) = \begin{cases} 0, & -2 < t < -1 \\ \cos\frac{\pi}{2}t, & -1 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

$T = 4, b_n = 0 \quad 2\pi/T = \pi/2 \qquad a_0 = \frac{1}{\pi}$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt = \frac{1}{\pi(n+1)}\sin\frac{\pi}{2}(n+1) + \frac{1}{\pi(n-1)}\sin\frac{\pi}{2}(n-1) \qquad \boxed{n > 1}$$

For n = odd ($n = 1, 3, 5, \ldots$) → $(n+1)$ and $(n-1)$ are both even

$$a_n = \frac{1}{\pi(n+1)}\underbrace{\sin\frac{\pi}{2}(n+1)}_{0} + \frac{1}{\pi(n-1)}\underbrace{\sin\frac{\pi}{2}(n-1)}_{0} = 0$$



Over a period

$$f(t) = \begin{cases} 0, & -2 < t < -1 \\ \cos\frac{\pi}{2}t, & -1 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

$T = 4, b_n = 0 \quad 2\pi/T = \pi/2 \qquad a_0 = \dfrac{1}{\pi}$

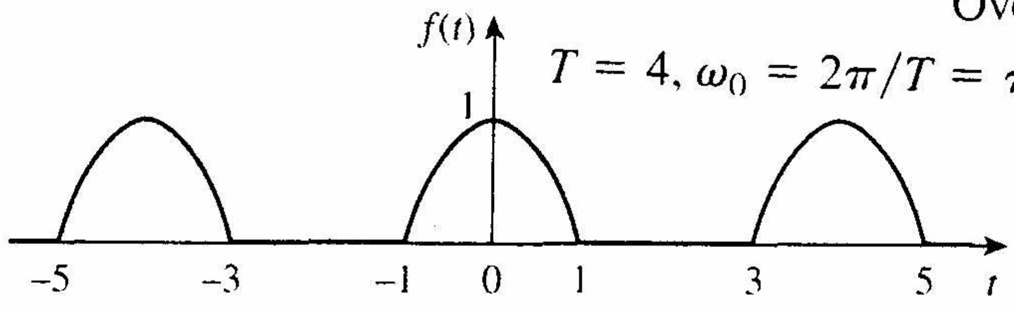
$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt = \frac{1}{\pi(n+1)}\sin\frac{\pi}{2}(n+1) + \frac{1}{\pi(n-1)}\sin\frac{\pi}{2}(n-1) \qquad \boxed{n > 1}$$

For n = odd ($n = 1, 3, 5, \ldots$) → $a_n = 0$

For n = even ($n = 2, 4, 6, \ldots$) → $(n+1)$ and $(n-1)$ are both odd

$$\sin\frac{\pi}{2}(n+1) = (-1)^{n/2} \qquad \sin\frac{\pi}{2}(n-1) = -(-1)^{n/2}$$

$$a_n = \frac{(-1)^{n/2}}{\pi(n+1)} + \frac{-(-1)^{n/2}}{\pi(n-1)} = \frac{-2(-1)^{n/2}}{\pi(n^2-1)} \qquad n = \text{even}$$



$T = 4,\ \omega_0 = 2\pi/T = \pi/2$

Over a period

$$f(t) = \begin{cases} 0, & -2 < t < -1 \\ \cos\frac{\pi}{2}t, & -1 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

$b_n = 0 \quad a_0 = \frac{1}{\pi}$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\, dt = \frac{1}{\pi(n+1)}\sin\frac{\pi}{2}(n+1) + \frac{1}{\pi(n-1)}\sin\frac{\pi}{2}(n-1) \qquad n > 1$$

$a_n = 0 \qquad n = \text{odd}$

$$a_n = \frac{-2(-1)^{n/2}}{\pi(n^2-1)} \qquad n = \text{even}$$

Now for n =`1 $\quad a_1 = \frac{1}{2}\int_0^1 [\cos\pi t + 1]\, dt = \frac{1}{2}\left[\frac{\sin\pi t}{\pi} + t\right]\Bigg|_0^1 = \frac{1}{2}$

$$a_n = \begin{cases} \frac{1}{2} & n = 1 \\ \frac{-2(-1)^{n/2}}{\pi(n^2-1)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$e^{-jn\omega_0 t} = \cos n\omega_0 t - j\sin n\omega_0 t$$

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j\sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

$$\sin \omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2}+\sum_{n=1}^{\infty}b_n\frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2j}$$

$$x(t)=a_0+\sum_{n=1}^{\infty}\frac{1}{2}(a_n-jb_n)e^{jn\omega_0 t}+\sum_{n=1}^{\infty}\frac{1}{2}(a_n+jb_n)e^{-jn\omega_0 t}$$

$$x(t)=a_0+\sum_{n=1}^{\infty}\underbrace{\frac{1}{2}(a_n-jb_n)e^{jn\omega_0 t}}_{1}+\sum_{n=1}^{\infty}\underbrace{\frac{1}{2}(a_n+jb_n)e^{-jn\omega_0 t}}_{2}$$

term 1 and term 2 are complex conjugate of each other

Then we can write x(t) as

$$x(t)=\{\cdots+X_{-2}\,e^{-j2\omega_0 t}+X_{-1}\,e^{-j\omega_0 t}\}+X_0+\{X_1\,e^{j\omega_0 t}+X_2\,e^{j2\omega_0 t}+\cdots\}$$

Where

$$X_0=a_0$$

$$X_1=\frac{1}{2}(a_1-jb_1)\quad X_2=\frac{1}{2}(a_2-jb_2)\cdots\Rightarrow X_n=\frac{1}{2}(a_n-jb_n)$$

$$X_{-1}=\frac{1}{2}(a_1+jb_1)\quad X_{-2}=\frac{1}{2}(a_2+jb_2)\cdots\Rightarrow X_{-n}=\frac{1}{2}(a_n+jb_n)$$

$$x(t)=a_0+\sum_{n=1}^{\infty}\frac{1}{2}\underbrace{(a_n-jb_n)e^{jn\omega_0 t}}_{1}+\sum_{n=1}^{\infty}\frac{1}{2}\underbrace{(a_n+jb_n)e^{-jn\omega_0 t}}_{2}$$

$$x(t)=\underbrace{\{\cdots+X_{-2}e^{-j2\omega_0 t}+X_{-1}e^{-j\omega_0 t}\}}_{\sum_{n=1}^{\infty}X_{-n}e^{-jn\omega_0 t}=\sum_{n=-1}^{-\infty}X_n e^{jn\omega_0 t}}+X_0+\underbrace{\{X_1 e^{j\omega_0 t}+X_2 e^{j2\omega_0 t}+\cdots\}}_{\sum_{n=1}^{\infty}X_n e^{jn\omega_0 t}}$$

$$x(t)=\sum_{n=-1}^{-\infty}X_n e^{jn\omega_0 t}+X_0+\sum_{n=1}^{\infty}X_n e^{jn\omega_0 t}=\sum_{n=-\infty}^{\infty}X_n e^{jn\omega_0 t}$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t=\sum_{n=-\infty}^{\infty}X_n e^{jn\omega_0 t}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

How to find X_n ?

Since $X_n = \frac{1}{2}(a_n - jb_n)$

$$X_n = \frac{1}{2}\left[\frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt - j\frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt\right]$$

$$= \frac{1}{T_0}\int_{T_0} x(t)\left[\cos n\omega_0 t - j\sin n\omega_0 t\right]dt = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t=\sum_{n=-\infty}^{\infty}X_n e^{jn\omega_0 t}$$

$$X_n=\frac{1}{2}(a_n-jb_n)=\frac{1}{T_0}\int_{T_0}x(t)e^{-jn\omega_0 t}dt$$

Another method to find X_n ?

Multiplying both side of x(t) by $e^{-jm\omega_0 t}$ and integrating over T_0

$$\int_{T_0}x(t)e^{-jm\omega_0 t}dt=\int_{T_0}\left(\sum_{n=-\infty}^{\infty}X_n e^{jn\omega_0 t}\right)e^{jm\omega_0 t}dt$$

$$=\sum_{n=-\infty}^{\infty}X_n\int_{T_0}e^{j(n-m)\omega_0 t}dt$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad X_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$$\int_{T_0} x(t) e^{-jm\omega_0 t} dt = \int_{T_0} \left(\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \right) e^{jm\omega_0 t} dt$$

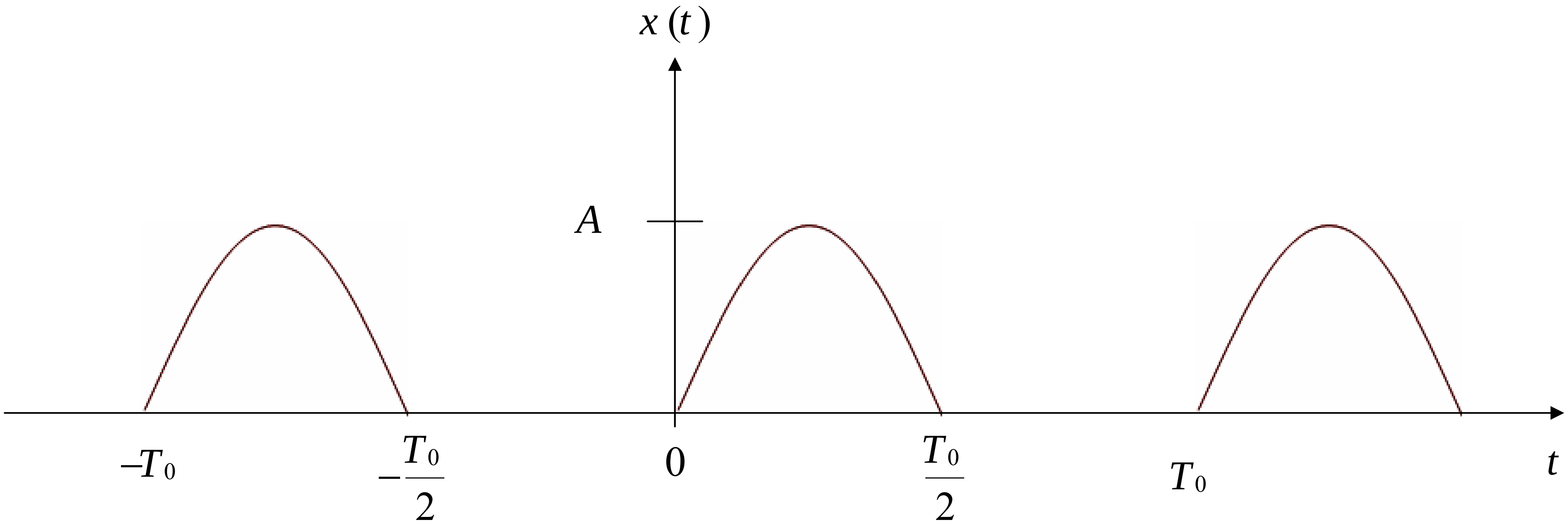
$$= \sum_{n=-\infty}^{\infty} X_n \int_{T_0} e^{j(n-m)\omega_0 t} dt$$

$$\int_{T_0} e^{j(n-m)\omega_0 t} dt = \begin{cases} 0 & \text{if } m \neq n \\ T_0 & \text{if } m = n \end{cases}$$

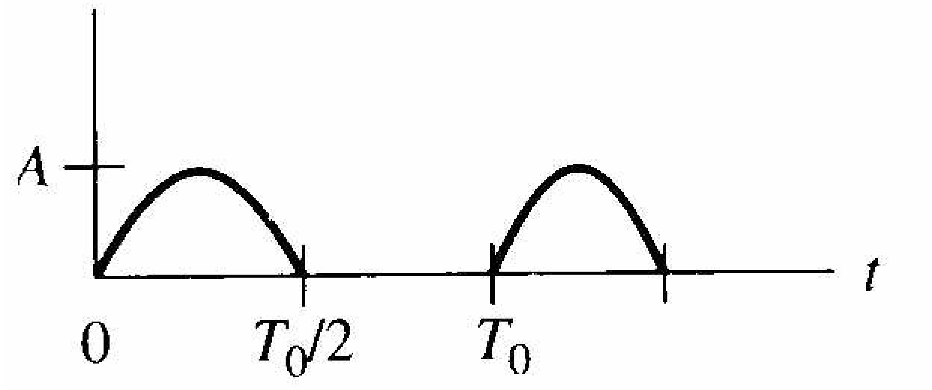
$$\Rightarrow \int_{T_0} x(t) e^{-jnm\omega_0 t} dt = X_m (T_0) \Rightarrow X_m = \frac{1}{T_0}\int_{T_0} x(t) e^{-jm\omega_0 t} dt$$

Since it is true for all m then it is true for all n $\Rightarrow$ $X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$

Example 3-6 Find the complex Fourier series coefficients for A half-rectified sine wave



$$x(t) = \begin{cases} A \sin \omega_0 t & 0 \le t \le \dfrac{T_0}{2} \\ 0 & \dfrac{T_0}{2} \le t \le T_0 \end{cases}$$



$$x(t) = \begin{cases} A \sin \omega_0 t & 0 \le t \le \frac{T_0}{2} \\ 0 & \frac{T_0}{2} \le t \le T_0 \end{cases}$$

$$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_0^{T_0/2} A \sin \omega_0 t \, e^{-jn\omega_0 t} \, dt$$

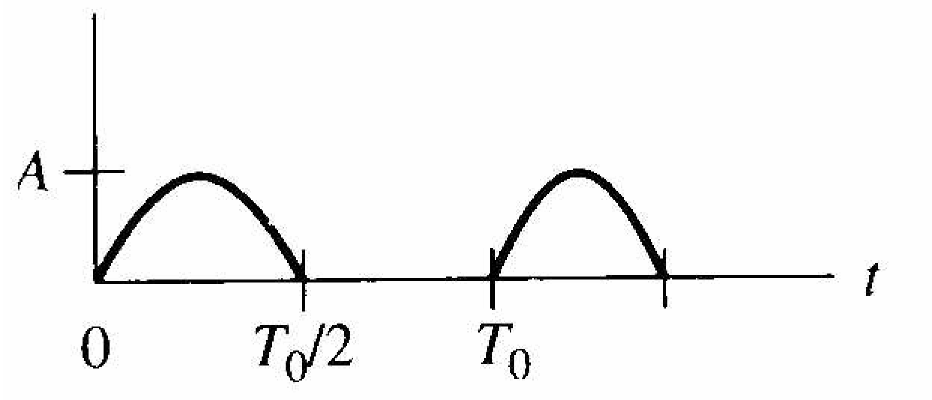
since $\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$X_n = \frac{A}{2jT_0} \left[\int_0^{T_0/2} (e^{j\omega_0 t} - e^{-j\omega_0 t}) e^{-jn\omega_0 t} \, dt \right] = \frac{A}{2jT_0} \left[\int_0^{T_0/2} e^{j\omega_0 (1-n)t} \, dt - \int_0^{T_0/2} e^{-j\omega_0 (1+n)t} \, dt \right]$$

$\omega_0 = 2\pi / T_0$

$$X_n = -\frac{A}{4\pi} \left[\frac{e^{j(1-n)\pi} - 1}{1 - n} + \frac{e^{j(1+n)\pi} - 1}{1 + n} \right]$$

$n \neq 1$ $\qquad$ $n \neq -1$



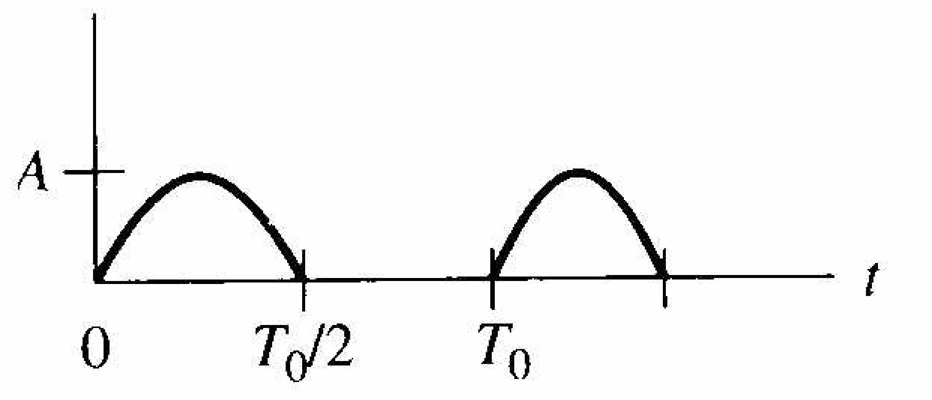
$$x(t)=\begin{cases} A\sin\omega_0 t & 0\le t\le \frac{T_0}{2} \\ 0 & \frac{T_0}{2}\le t\le T_0 \end{cases}$$

$$X_n = -\frac{A}{4\pi}\left[\frac{e^{j(1-n)\pi}-1}{1-n}+\frac{e^{j(1+n)\pi}-1}{1+n}\right] \qquad n\neq 1 \text{ or } -1$$

since $e^{j(1\pm n)\pi} = e^{j\pi}\, e^{\pm j1n\pi} = -(-1)^n$

$e^{j\pi} \rightarrow -1$ $\qquad$ $e^{\pm j1n\pi} \rightarrow (-1)^n$

$$\Rightarrow \quad X_n=\begin{cases} 0 & n \text{ odd} \\ \dfrac{A}{\pi(1-n^2)} & n \text{ even} \end{cases} \qquad n\neq \pm 1$$



$$x(t)=\begin{cases} A\sin\omega_0 t & 0\le t\le \frac{T_0}{2} \\ 0 & \frac{T_0}{2}\le t\le T_0 \end{cases}$$

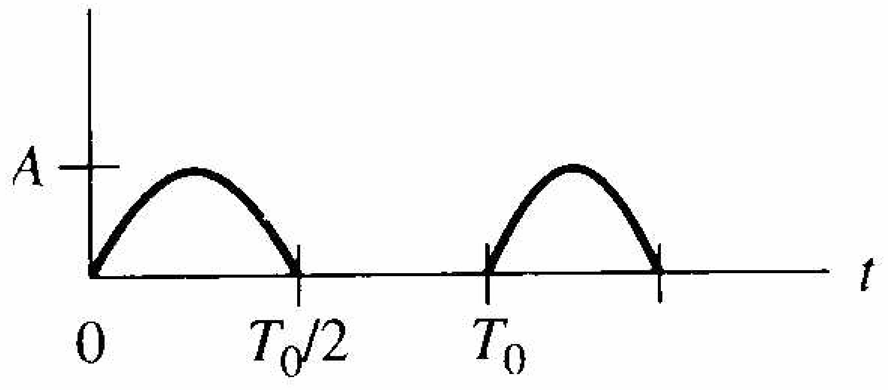
$$X_n=\frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$$

$$X_n=\begin{cases} 0 & n\ \text{odd} \\ \frac{A}{\pi(1-n^2)} & n\ \text{even} \end{cases} \qquad n\neq\pm1$$

$$X_1=\frac{A}{2jT_0}\int_0^{T_0/2}\left(e^{j\omega_0 t}-e^{-j\omega_0 t}\right)e^{-j\omega_0 t}dt=\frac{A}{2jT_0}\int_0^{T_0/2}\left(1-e^{-j2\omega_0 t}\right)dt=\frac{A}{4j}$$

Similarly

$$X_{-1}=-\frac{A}{4j}$$



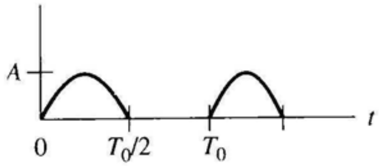
$$x(t) = \begin{cases} A\sin\omega_0 t & 0 \le t \le \dfrac{T_0}{2} \\ 0 & \dfrac{T_0}{2} \le t \le T_0 \end{cases}$$

$$X_n = \begin{cases} \dfrac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n \text{ odd and } \neq \pm 1 \\ -\frac{1}{4} jnA, & n = \pm 1 \end{cases}$$

First Entry in Table 3-1

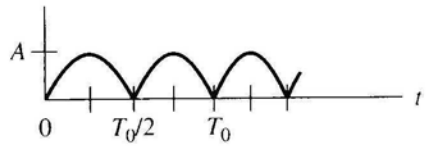
Table 3-1

1. Half-rectified sine wave



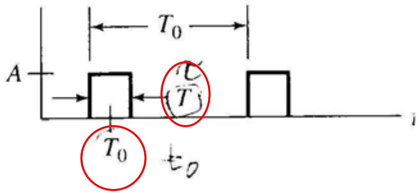
$$X_n = \begin{cases} \dfrac{A}{\pi(1-n^2)}, & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n \text{ odd and } \neq \pm 1 \\ -\frac{1}{4} jnA, & n = \pm 1 \end{cases}$$

2. Full-rectified sine wave*



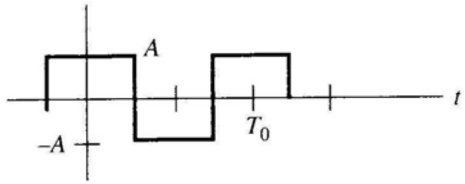
$$X_n = \begin{cases} \dfrac{2A}{\pi(1-n^2)}, & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

3. Pulse-train signal



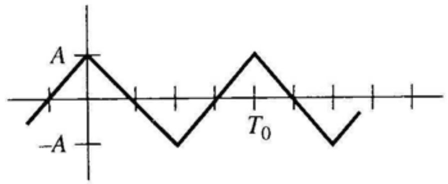
$$X_n = \frac{A\tau}{T_0} \operatorname{sinc} nf_0\tau \, e^{-j2\pi n f_0 t_0}, \quad f_0 = T_0^{-1}$$

4. Square wave



$$X_n = \begin{cases} \dfrac{2A}{|n|\pi}, & n = \pm 1, \pm 5, \ldots \\ \dfrac{-2A}{|n|\pi}, & n = \pm 3, \pm 7, \ldots \\ 0, & n \text{ even} \end{cases}$$

5. Triangular wave



$$X_n = \begin{cases} \dfrac{4A}{\pi^2 n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Symmetry Properties of Fourier Series coefficients

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \qquad = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t) dt \qquad X_n = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jn\omega_0 t}$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t dt \qquad n \neq 0$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin n\omega_0 t dt$$

$$X_0 = a_0$$

$$X_n = \frac{1}{2}(a_n - jb_n) \qquad X_{-n} = \frac{1}{2}(a_n + jb_n) = X_n^*$$

$$a_n = 2\mathrm{Re}[X_n] \qquad b_n = -2\mathrm{Im}[X_n]$$

$$a_n = X_n + X_n^* \qquad b_n = \frac{X_n^* - X_n}{j}$$

3.6 Parsevals Thm

From **ch1**, the average power defined as

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{T_0}\int_{T_0} x(t)x(t)^* dt$$

Now we would like to express P_{av} in terms of the Fourier Coefficients of *x(t)*

$$P_{av} = \frac{1}{T_0}\int_{T_0} x(t)x(t)^* dt \qquad = \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}\right)^* dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}\right) dt$$

$$P_{av} = \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}\right)^* dt = \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}\right) dt$$

The order of integration and summation can be inter changed

$$= \sum_{n=-\infty}^{\infty} X_n^* \underbrace{\left[\frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt\right]}_{X_n} = \sum_{n=-\infty}^{\infty} X_n^* X_n = \sum_{n=-\infty}^{\infty} |X_n|^2$$

Parsevals Thm

$$\underbrace{P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt}_{\text{Time domain}} = \underbrace{\sum_{n=-\infty}^{\infty} |X_n|^2}_{\text{Frequency domain}} = \underbrace{X_0^2}_{\text{DC power}} + \underbrace{2\sum_{n=1}^{\infty} |X_n|^2}_{\text{Harmonic Power}}$$

$$Note \quad = |X_n| = |X_{-n}|$$

Average power is the sum of DC power and harmonics power

Example

Let $x(t)=A\cos\omega_0 t$ (real signal)

Then,

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{T_0}\int_{T_0} x(t)x(t)^* dt = \frac{1}{T_0}\int_{T_0} x^2(t) dt$$

$$= \frac{1}{T_0}\int_0^{T_0} A^2 \cos^2 \omega_0 t \; dt = \frac{A^2}{T_0}\int_0^{T_0} \frac{1}{2}[1+\cos 2\omega_0 t]\, dt$$

Remember from EE 201 in 1 Ω resistor

$$= \frac{A^2}{2T_0}\int_0^{T_0} dt + \frac{A^2}{2T_0}\int_0^{T_0} \cos 2\omega_0 t \; dt = \frac{A^2}{2} = I_{rms}^2 = V_{rms}^2 = \left(\frac{A}{\sqrt{2}}\right)^2$$

The same result can be shown for $x(t)=A\sin\omega_0 t$

Now let us apply Parsavel Thm next

$$x(t)=A\cos\omega_0 t = A\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2} = \frac{A}{2}e^{j\omega_0 t}+\frac{A}{2}e^{-j\omega_0 t} = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

Without evaluating the Fourier Series complex coefficient $X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}$

We conclude

$$X_n = \begin{cases} \frac{A}{2} & n=\pm 1 \\ 0 & \text{else} \end{cases} \Rightarrow X_1 = \frac{A}{2} \quad X_{-1} = \frac{A}{2}$$

$$\Rightarrow P_{av} = X_0^2 + 2\sum_{n=1}^{\infty}|X_n|^2 = 0 + 2\left(\frac{A}{2}\right)^2 = \frac{A^2}{2}$$

Note here that x(t) contain one harmonic

Example

Let $x(t) = A\cos\omega_{01}t + B\cos\omega_{02}t$

We can find X_n Without evaluating the Fourier Series complex coefficient

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}$$

Where ω_0 is the fundamental frequency which can be found as

$$\omega_{01} = n_1\omega_0 \qquad \omega_{02} = n_2\omega_0$$

Since x(t) is periodical , then n_1 and n_2 are integers which can be determined as follows

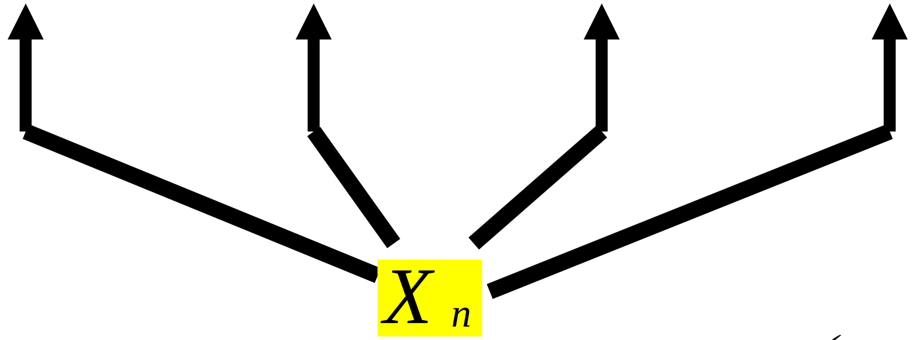
Now let us apply Parsavel Thm next

Let $x(t) = A\cos\omega_{01}t + B\cos\omega_{02}t \qquad \omega_{01} = n_1\omega_0 \qquad \omega_{02} = n_2\omega_0$

n_1 and n_2 can be determined as follows

$$x(t) = \frac{A}{2}[e^{j\omega_{01}t} + e^{-j\omega_{01}t}] + \frac{B}{2}[e^{j\omega_{02}t} + e^{-j\omega_{02}t}]$$

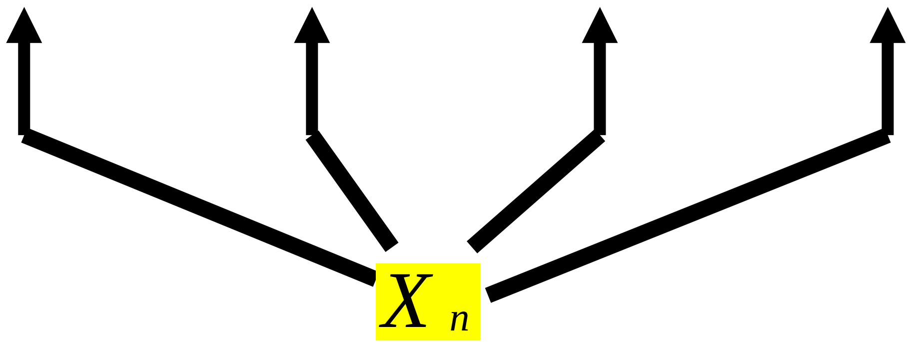
$$= \frac{A}{2}e^{j\omega_{01}t} + \frac{A}{2}e^{-j\omega_{01}t} + \frac{B}{2}e^{j\omega_{02}t} + \frac{B}{2}e^{-j\omega_{02}t}$$



$$P_{av} = X_0^2 + 2\sum_{n=1}^{\infty}|X_n|^2 = 0 + 2\left(\frac{A^2}{4} + \frac{B^2}{4}\right) = \frac{A^2 + B^2}{2}$$

$$x(t) = A\cos\omega_{01}t + B\cos\omega_{02}t \qquad \omega_{01} = n_1\omega_0 \qquad \omega_{02} = n_2\omega_0$$

$$= \frac{A}{2}e^{j\omega_{01}t} + \frac{A}{2}e^{-j\omega_{01}t} + \frac{B}{2}e^{j\omega_{02}t} + \frac{B}{2}e^{-j\omega_{02}t}$$



$$P_{av} = X_0^2 + 2\sum_{n=1}^{\infty}|X_n|^2 = 0 + 2\left(\frac{A^2}{4} + \frac{B^2}{4}\right) = \frac{A^2 + B^2}{2}$$

Try to verify this by computing $P_{av} = \frac{1}{T_0}\int_{T_0} x^2(t)\,dt$ **Where T_0 is the fundamental period**

Line Spectra

$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$=\{\cdots+X_{-2}e^{-j2\omega_0 t}+X_{-1}e^{-j\omega_0 t}\}+X_0+\{X_1 e^{j\omega_0 t}+X_2 e^{j2\omega_0 t}+\cdots\}$$

where

$$X_n=|X_n|\underline{\angle \theta_n}$$

In general a complex number that can be represented as a phasor

$X_n e^{jn\omega_0 t}$ **Is a rotating phasor of frequency** $n\omega_0$

Therefore, $x(t)$ consists of a summation of rotating phasors

Recall from chapter 1 (phasor signals and spectra p12) , that is $x(t)$ is a sinusoidal,

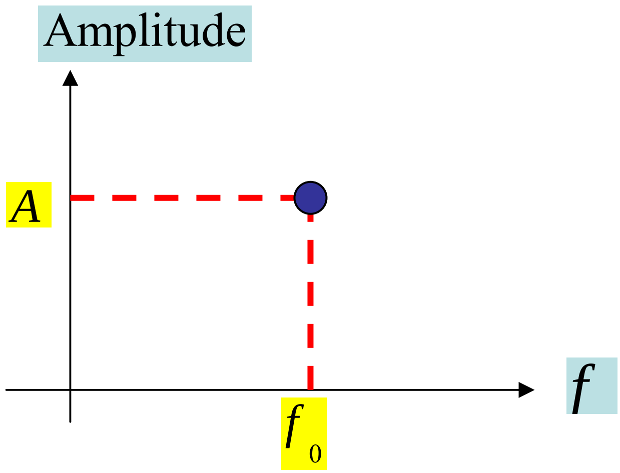
$$x(t) = A\cos(\omega_0 t + \theta) = \mathrm{Re}[Ae^{j(\omega_0 t + \theta)}] = \mathrm{Re}[\tilde{x}(t)]$$

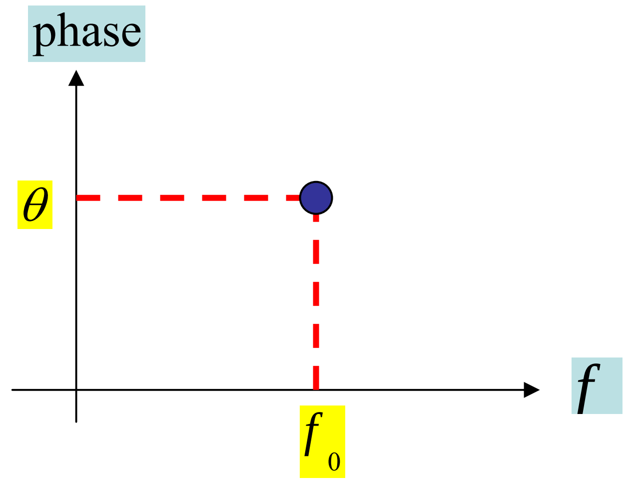
where $\tilde{x}(t) = Ae^{j(\omega_0 t + \theta)}$

$x(t)$ also can be written as

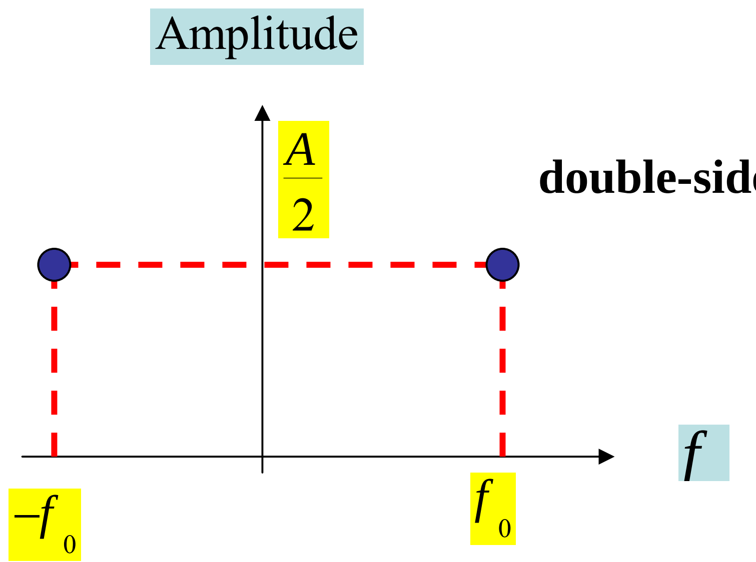
$$x(t) = A\cos(\omega_0 t + \theta) = \frac{1}{2}Ae^{j(\omega_0 t + \theta)} + \frac{1}{2}Ae^{-j(\omega_0 t + \theta)} = \frac{1}{2}\tilde{x}(t) + \frac{1}{2}\tilde{x}^*(t)$$

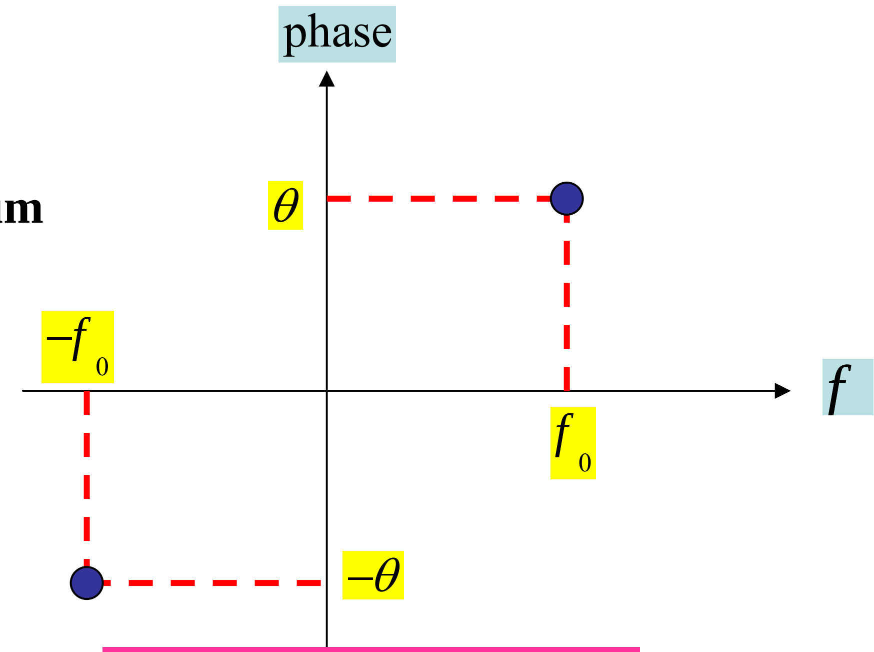
single-sided spectrum





double-sided spectrum





Amplitude is an even function

Phase is an odd function

Now let $x(t)$ be

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$= \{\cdots + X_{-2} e^{-j2\omega_0 t} + X_{-1} e^{-j\omega_0 t}\} + X_0 + \{X_1 e^{j\omega_0 t} + X_2 e^{j2\omega_0 t} + \cdots\}$$

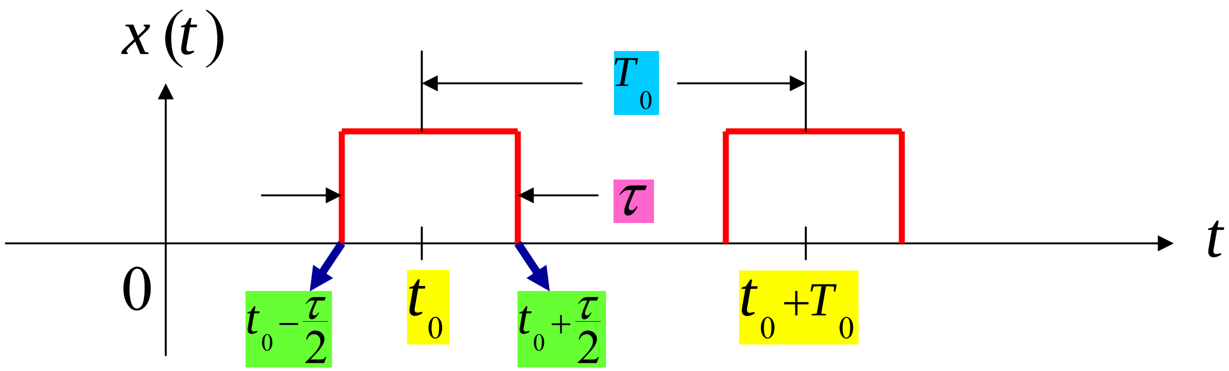
For each $nf_0 \quad (-\infty < n < \infty)$, $\quad X_n = |X_n| \underline{|\theta_n}$

$|X_n| = |X_{-n}|$ **even function**

$\theta_n = -\theta_{-n}$ **odd function**

Example 3.5

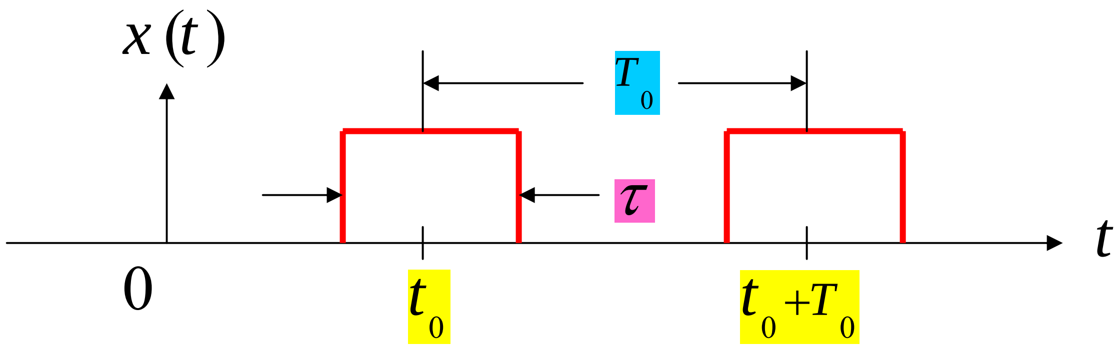
Find the complex Fourier Series coefficients



$$X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt \quad = \frac{1}{T_0}\int_{t_0-\frac{\tau}{2}}^{t_0+\frac{\tau}{2}} A e^{-jn\omega_0 t}dt \ = \frac{-A}{jn\omega_0 T_0} e^{-jn\omega_0 t}\Big|_{t_0-\frac{\tau}{2}}^{t_0+\frac{\tau}{2}}$$

$$= \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0}\left(\frac{e^{jn\omega_0\frac{\tau}{2}} - e^{-jn\omega_0\frac{\tau}{2}}}{2j}\right) \qquad n \neq 0$$

$$= \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0}\sin\left(n\omega_0\frac{\tau}{2}\right) \qquad n \neq 0$$

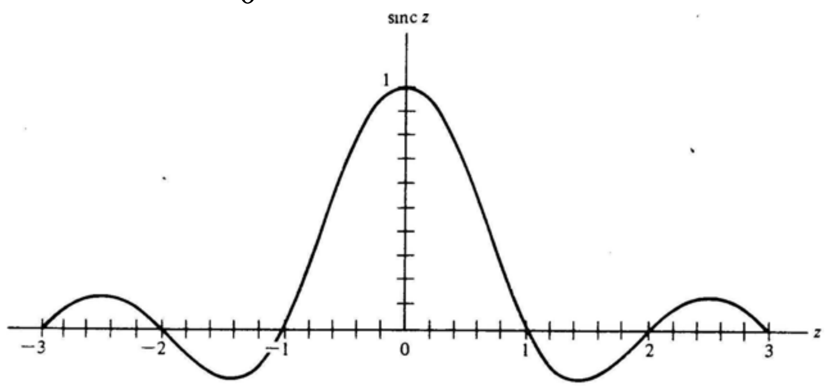


$$X_n = \frac{2A}{n\omega_0 T_0} e^{-jn\omega_0 t_0} \sin\left(n\omega_0 \frac{\tau}{2}\right) \qquad n \neq 0$$

since $\omega_0 = 2\pi f_0$ $\Rightarrow$ $X_n = \frac{A\tau}{T_0} e^{-j2\pi n f_0 t_0} \frac{\sin(\pi n f_0 \tau)}{\pi n f_0 \tau}$

Define The sinc function

$$\text{sinc } z = \frac{\sin \pi z}{\pi z}$$



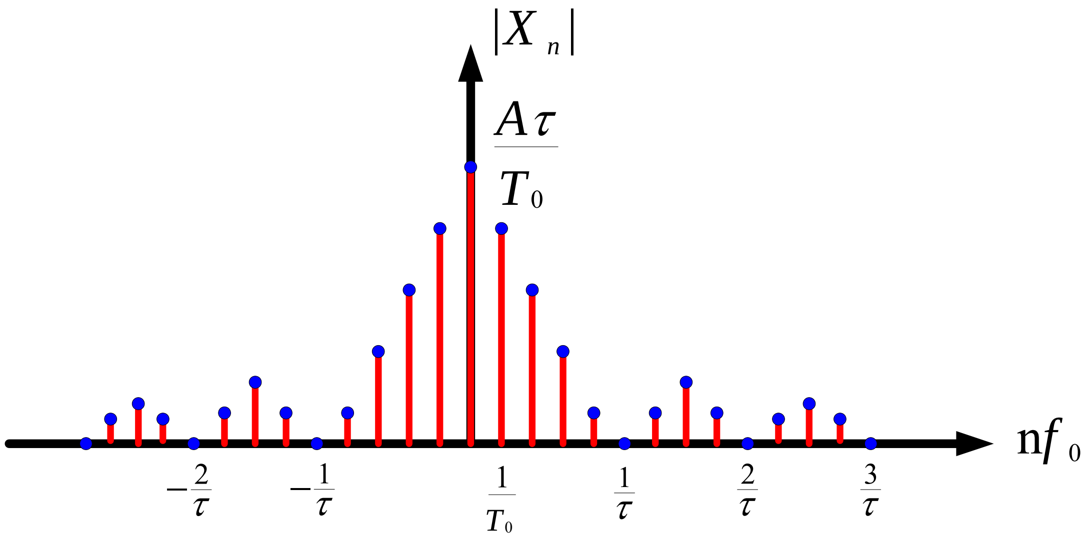
$$\Rightarrow X_n = \frac{A\tau}{T_0} e^{-jn\omega_0 t_0} \frac{\sin(\pi n f_0 \tau)}{\pi n f_0 \tau} = \frac{A\tau}{T_0} \text{sinc}\left(n f_0 \tau\right) e^{-j2\pi n f_0 t_0}$$

$$X_n = \frac{A\tau}{T_0} \operatorname{sinc}\left(nf_0\tau\right) e^{-j2\pi n f_0 t_0}$$

If $t_0 = \frac{\tau}{2}$ $\Rightarrow$ $$X_n = \frac{A\tau}{T_0} \operatorname{sinc}\left(nf_0\tau\right) e^{-j\pi n f_0 \tau}$$

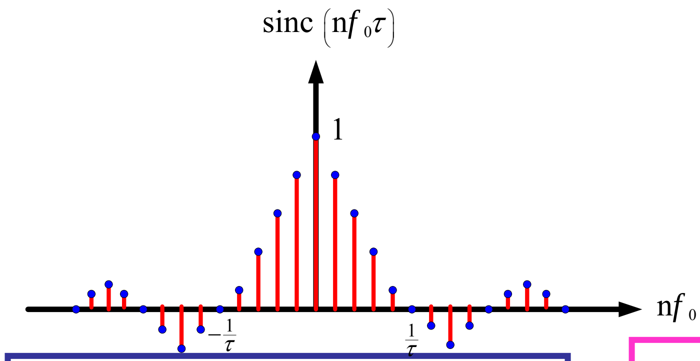
Since $X_n = \left|X_n\right| \angle \theta_n$

Magnitude $$\left|X_n\right| = \frac{A\tau}{T_0} \left|\operatorname{sinc}\left(nf_0\tau\right)\right|$$



$$X_n = \frac{A\tau}{T_0}\,\text{sinc}\left(nf_0\tau\right)e^{-j\pi nf_0\tau} \qquad X_n = \left|X_n\right| \angle \theta_n$$

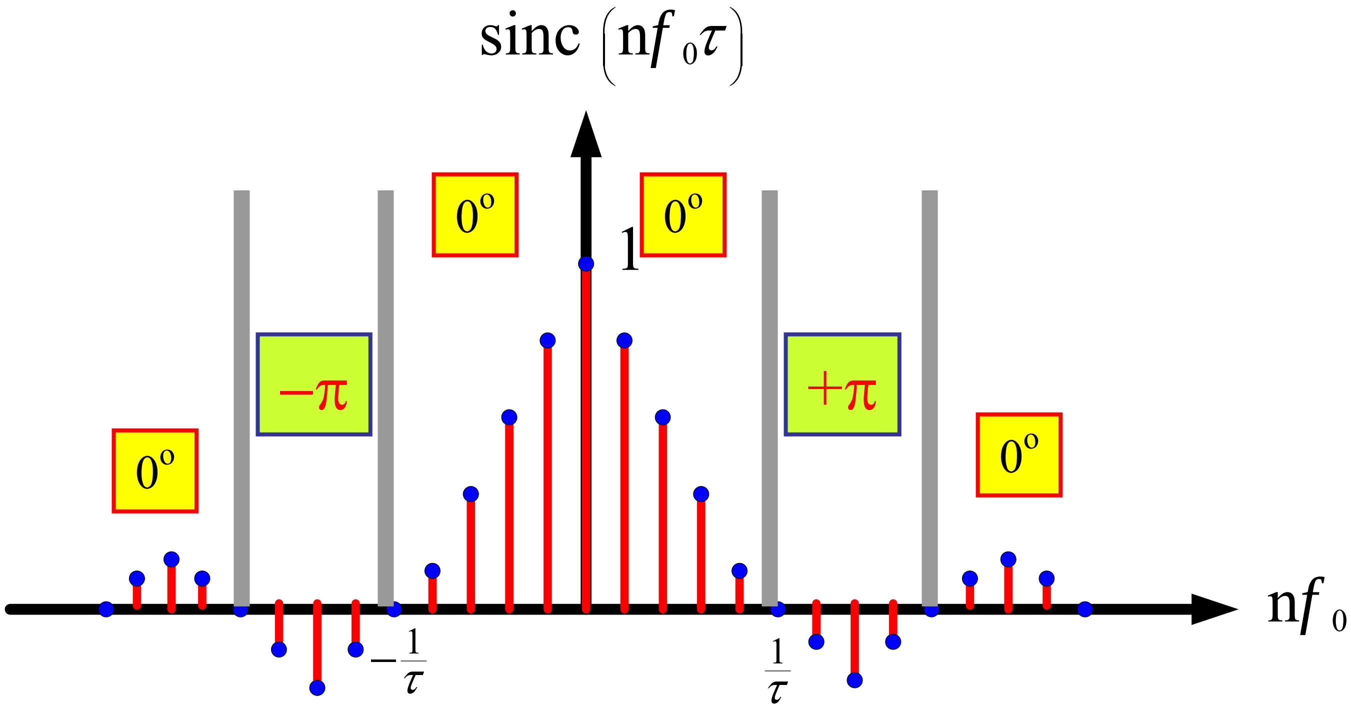
Phase θ_n



$$X_n = \boxed{\frac{A\tau}{T_0}}\;\boxed{\text{sinc}\left(nf_0\tau\right)}\,e^{\boxed{-j\pi nf_0\tau}}$$

Angle is $-\pi nf_0\tau$

- $\frac{A\tau}{T_0}$: Always positive → Do not add any angle to the phase
- $\text{sinc}\left(nf_0\tau\right)$: Can be positive and negative
  - When positive it do not add any angle to the phase
  - When negative it add $\pm\pi$ to the phase



$$X_n = \frac{A\tau}{T_0}\,\mathrm{sinc}\left(nf_0\tau\right)e^{-j\pi n f_0\tau}$$

$\frac{A\tau}{T_0}$ → 0°

$\mathrm{sinc}\left(nf_0\tau\right)$ → $\pm\pi$

$-j\pi n f_0\tau$ → $-\pi n f_0\tau$

the angle alternate sign to insure phase is odd

$$X_n = \frac{A\tau}{T_0}\,\operatorname{sinc}\left(nf_0\tau\right) e^{-j\pi n f_0 \tau}$$

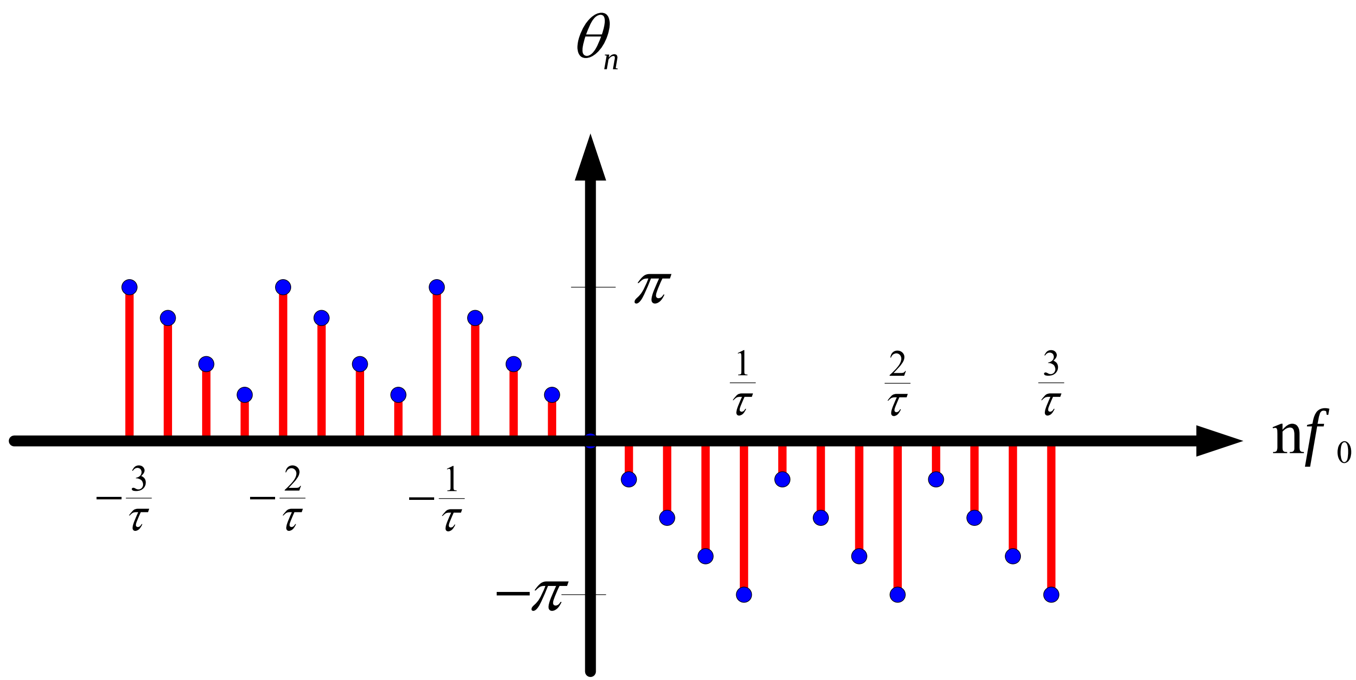
$\frac{A\tau}{T_0} \rightarrow 0^{o}$

$\operatorname{sinc}\left(nf_0\tau\right) \rightarrow \pm\pi$

$-j\pi n f_0\tau \rightarrow -\pi n f_0 \tau$

$$\theta_n = \begin{cases} -\pi n f_0 \tau & \text{if } \operatorname{sinc}\left(nf_0\tau\right) > 0 \\ -\pi n f_0 \tau + \pi & \text{if } nf_0 > 0 \text{ and } \operatorname{sinc}\left(nf_0\tau\right) < 0 \\ -\pi n f_0 \tau - \pi & \text{if } nf_0 < 0 \text{ and } \operatorname{sinc}\left(nf_0\tau\right) < 0 \end{cases}$$

$$\theta_n = \begin{cases} -\pi n f_0 \tau & \text{if } \mathrm{sinc}\left(nf_0\tau\right) > 0 \\ -\pi n f_0 \tau + \pi & \text{if } nf_0 > 0 \text{ and } \mathrm{sinc}\left(nf_0\tau\right) < 0 \\ -\pi n f_0 \tau - \pi & \text{if } nf_0 < 0 \text{ and } \mathrm{sinc}\left(nf_0\tau\right) < 0 \end{cases}$$



Another look at the Fourier Series Expansion

$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \quad =\sum_{n=-\infty}^{-1} X_n e^{jn\omega_0 t}+X_0+\sum_{n=1}^{\infty} X_n e^{jn\omega_0 t}$$

$$=X_0+\sum_{n=1}^{\infty}|X_{-n}|e^{j(-n\omega_0 t+\theta_{-n})}+\sum_{n=1}^{\infty}|X_n|e^{j(n\omega_0 t+\theta_n)}$$

$$\theta_{-n}=-\theta_n \qquad |X_{-n}|=|X_n|$$

$$=X_0+\sum_{n=1}^{\infty}|X_n|e^{j(-n\omega_0 t-\theta_n)}+\sum_{n=1}^{\infty}|X_n|e^{j(n\omega_0 t+\theta_n)}$$

$$=X_0+\sum_{n=1}^{\infty}|X_n|\left[e^{j(n\omega_0 t+\theta_n)}+e^{j(-n\omega_0 t-\theta_n)}\right]$$

$$=X_0+\sum_{n=1}^{\infty}|X_n|\underbrace{\left[e^{j(n\omega_0 t+\theta_n)}+e^{-j(n\omega_0 t+\theta_n)}\right]}_{2\cos(n\omega_0 t+\theta_n)}$$

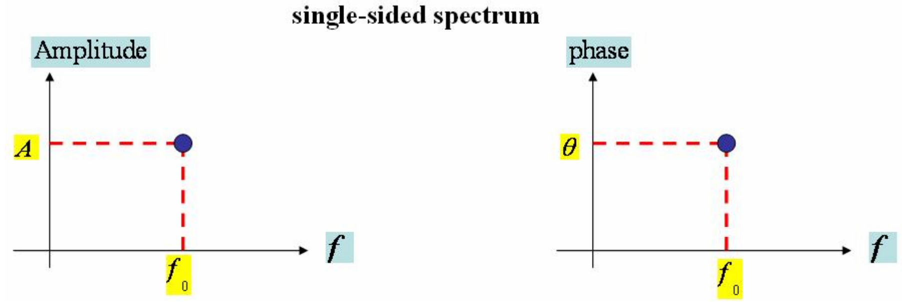
$$\Rightarrow \quad x(t)=X_0+2\sum_{n=1}^{\infty}|X_n|\cos(n\omega_0 t+\theta_n)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$= X_0 + 2\sum_{n=1}^{\infty} |X_n| \cos(n\omega_0 t + \theta_n)$$

$$x(t) = A\cos(\omega_0 t + \theta)$$



Since $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

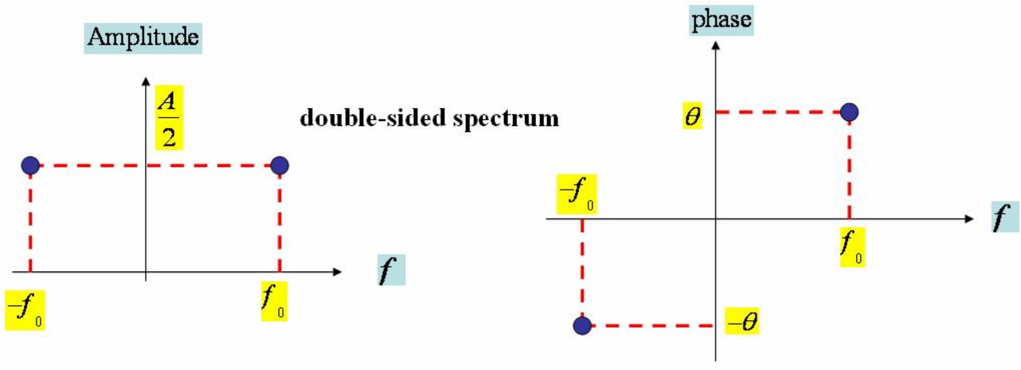
$$x(t) = A[\cos(\omega_0 t)\cos(\theta) - \sin(\omega_0 t)\sin(\theta)]$$

$$= \underbrace{A\cos(\theta)}_{a_1}\cos(\omega_0 t) + \underbrace{A(-\sin(\theta))}_{b_1}\sin(\omega_0 t)$$

$$a_1 = A\cos(\theta) \qquad b_1 = A(-\sin(\theta))$$

$$x(t) = A\cos(\omega_0 t + \theta) = A\frac{e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}}{2}$$

$$= \frac{A}{2}e^{j(\omega_0 t+\theta)} + \frac{A}{2}e^{-j(\omega_0 t+\theta)}$$

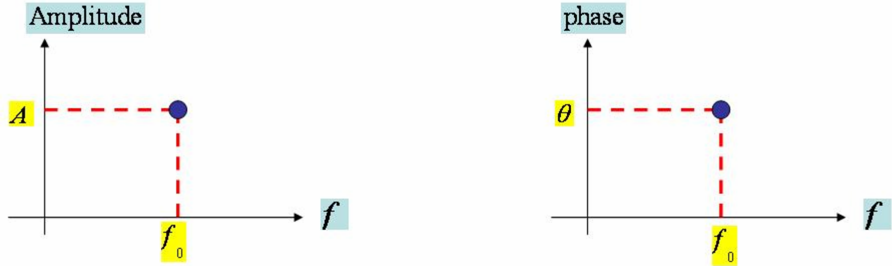


$$= \underbrace{\frac{A}{2}e^{j\theta}}_{X_1} e^{j\omega_0 t} + \underbrace{\frac{A}{2}e^{-j\theta}}_{X_{-1}} e^{-j\omega_0 t}$$

$$X_1 = \frac{A}{2}e^{j\theta} \qquad X_{-1} = \frac{A}{2}e^{-j\theta}$$

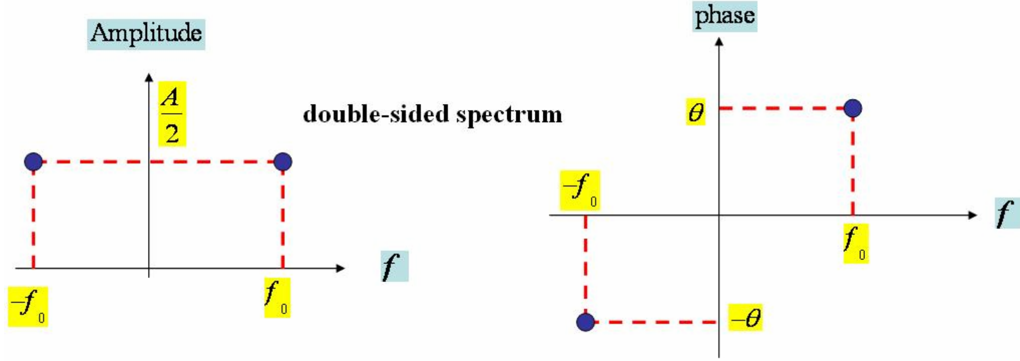
$$x(t) = A\cos(\omega_0 t + \theta)$$

single-sided spectrum



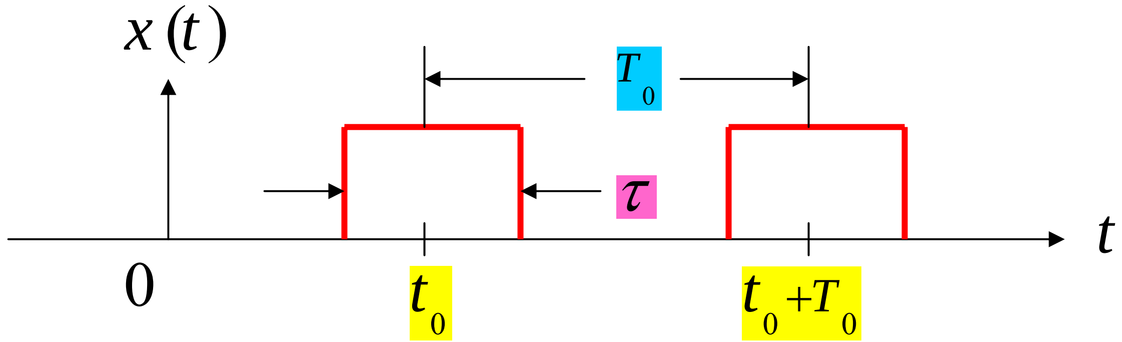
$$= \underbrace{A\cos(\theta)}_{a_1 = A\cos(\theta)}\cos(\omega_0 t) + \underbrace{A(-\sin(\theta))}_{b_1 = A(-\sin(\theta))}\sin(\omega_0 t)$$

$$x(t) = A\cos(\omega_0 t + \theta)$$



$$= \underbrace{\frac{A}{2}e^{j\theta}}_{X_1 = \frac{A}{2}e^{j\theta}} e^{j\omega_0 t} + \underbrace{\frac{A}{2}e^{-j\theta}}_{X_{-1} = \frac{A}{2}e^{-j\theta}} e^{-j\omega_0 t}$$

$$X_1 = \frac{A}{2}e^{j\theta} \qquad X_{-1} = \frac{A}{2}e^{-j\theta}$$

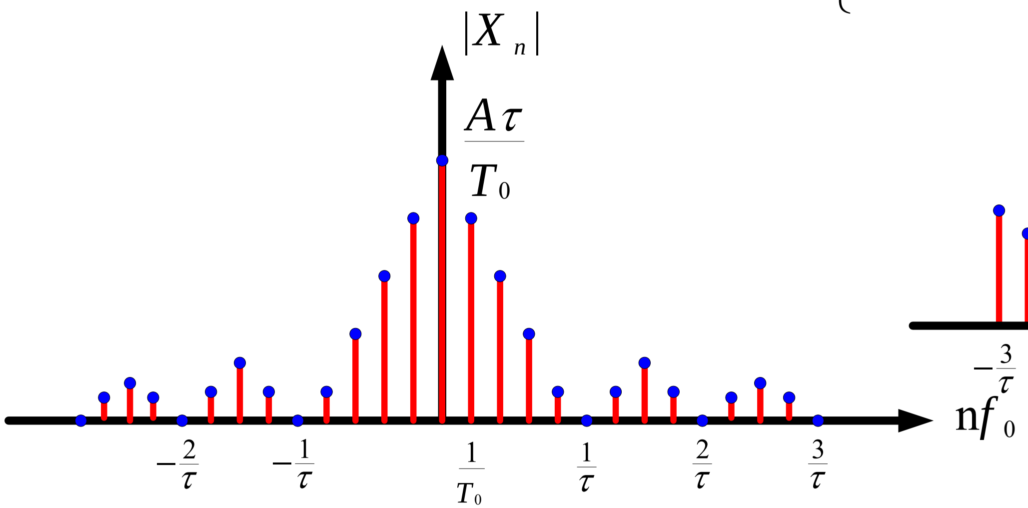


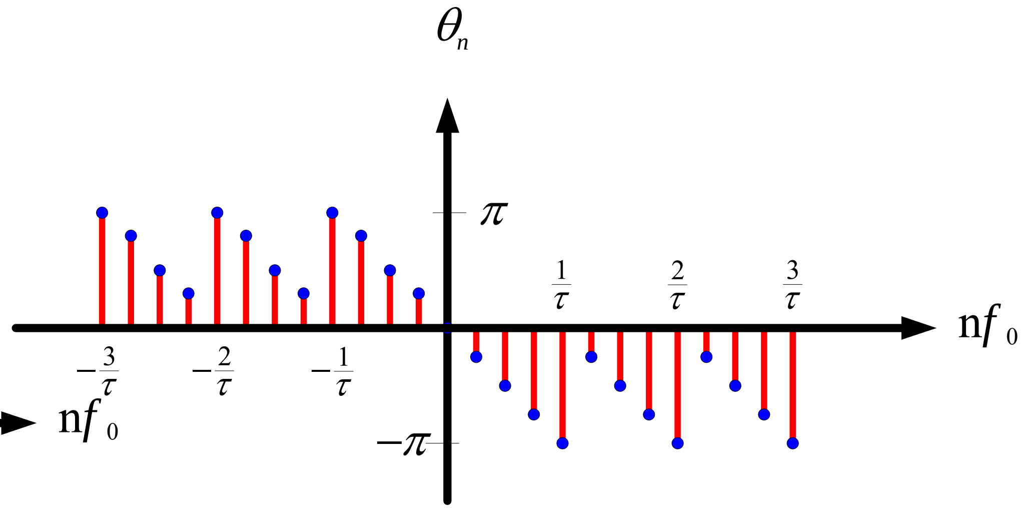
$$X_n = \frac{A\tau}{T_0}\,\mathrm{sinc}\left(nf_0\tau\right)e^{-j2\pi nf_0 t_0}$$

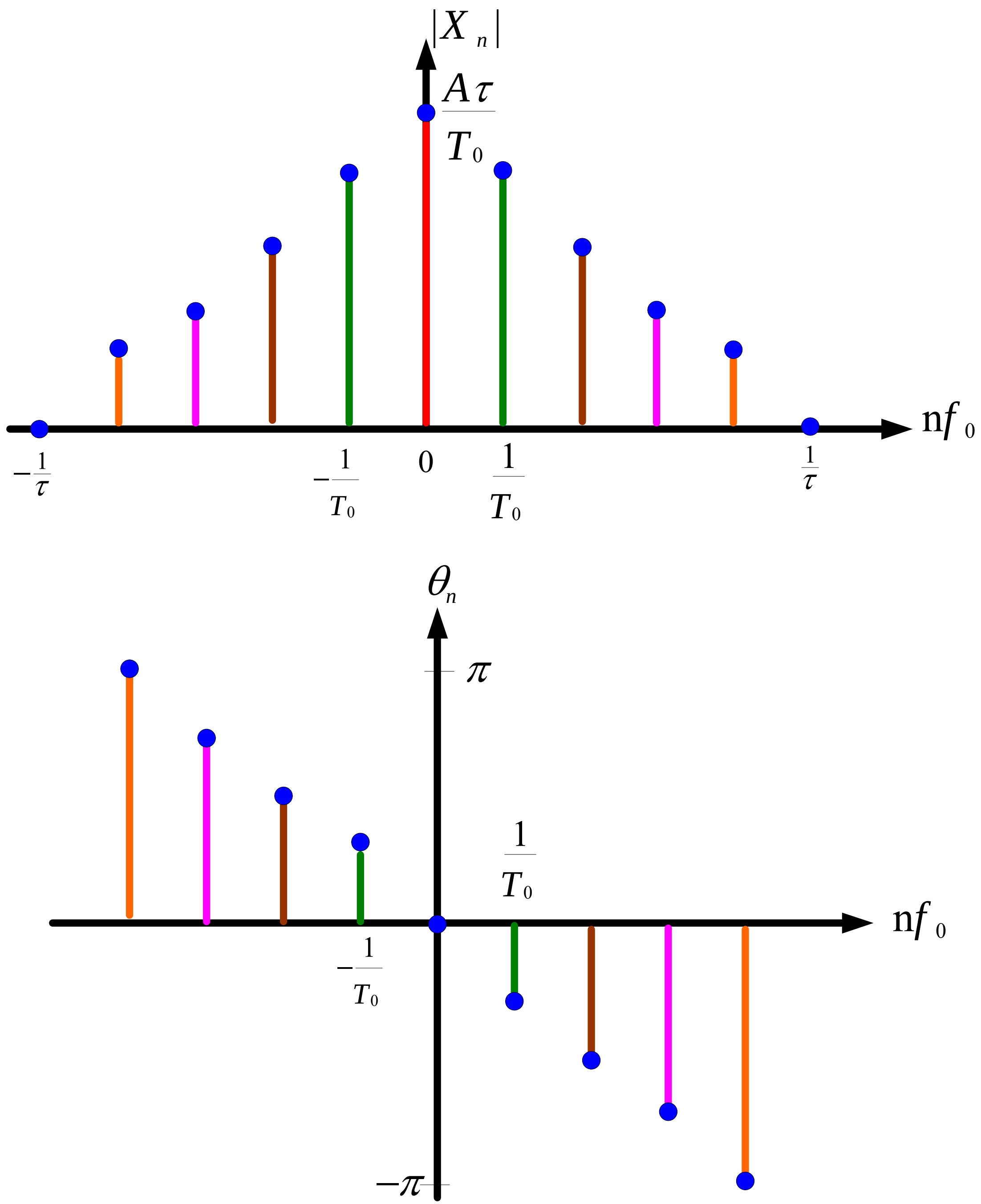
$$X_n = |X_n|\angle\theta_n$$

$$|X_n| = \frac{A\tau}{T_0}\left|\mathrm{sinc}\left(nf_0\tau\right)\right|$$

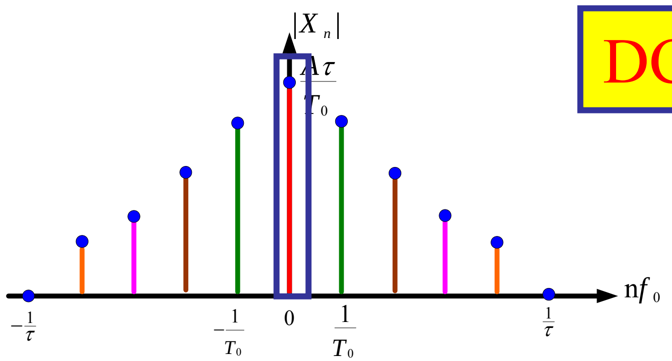
$$\theta_n = \begin{cases} -\pi nf_0\tau & \text{if } \mathrm{sinc}\left(nf_0\tau\right) > 0 \\ -\pi nf_0\tau + \pi & \text{if } nf_0 > 0 \text{ and } \mathrm{sinc}\left(nf_0\tau\right) < 0 \\ -\pi nf_0\tau - \pi & \text{if } nf_0 < 0 \text{ and } \mathrm{sinc}\left(nf_0\tau\right) < 0 \end{cases}$$



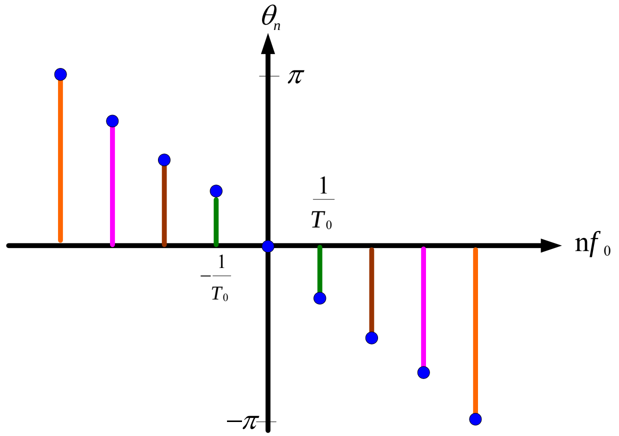




$|X_n|$
$\frac{A\tau}{T_0}$
nf_0
$-\frac{1}{\tau}$
$-\frac{1}{T_0}$
0
$\frac{1}{T_0}$
$\frac{1}{\tau}$
θ_n
π
nf_0
$\frac{1}{T_0}$
$-\frac{1}{T_0}$
$-\pi$



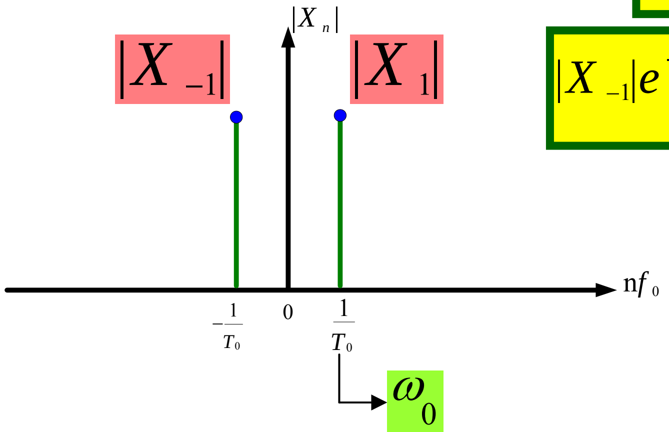
DC

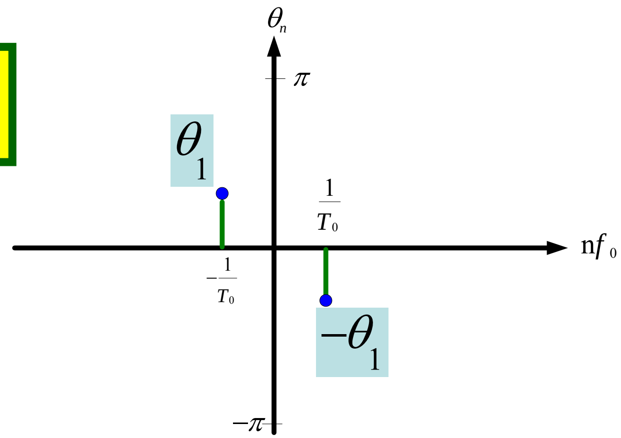


$$X_{-1}\, e^{-j\omega_0 t} + X_1\, e^{j\omega_0 t}$$

$$|X_{-1}|e^{-j\theta_1}\, e^{-j\omega_0 t} + |X_1|e^{j\theta_1}\, e^{j\omega_0 t}$$

$$2|X_1|\cos(\omega_0 t + \theta_1)$$

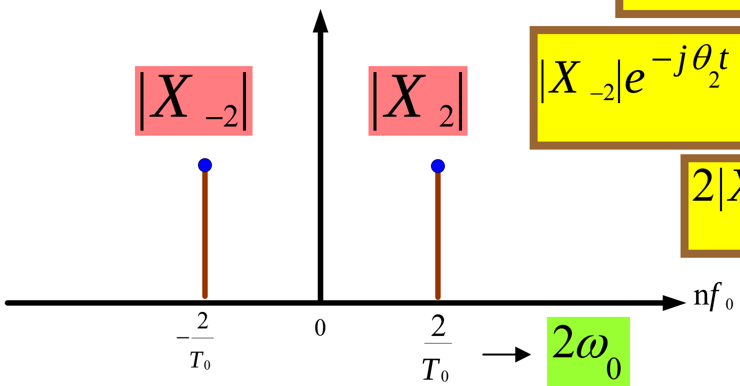


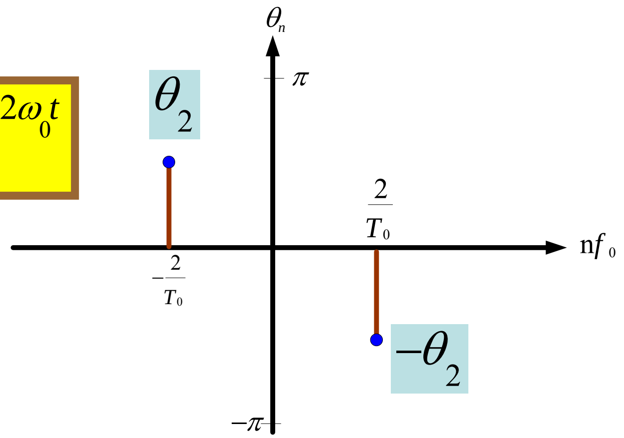


$$X_{-2}\, e^{-j2\omega_0 t} + X_2\, e^{j2\omega_0 t}$$

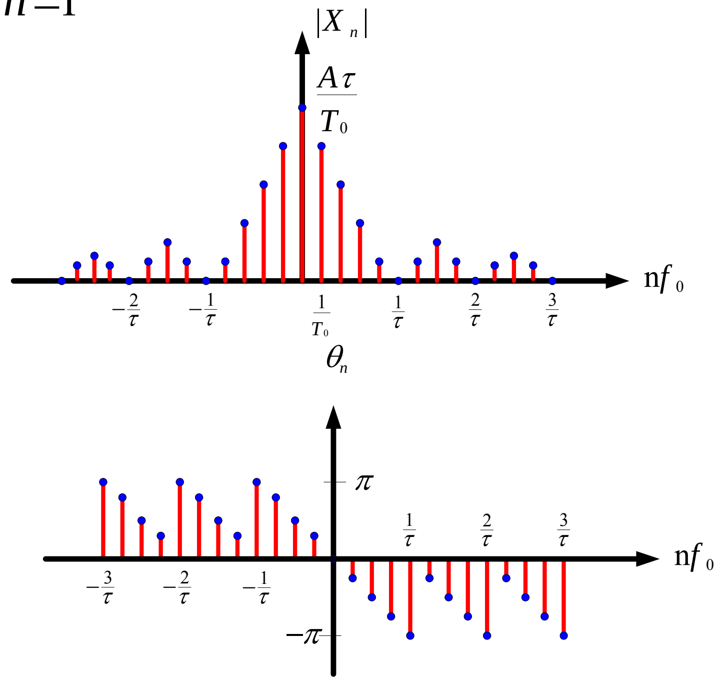
$$|X_{-2}|e^{-j\theta_2 t}\, e^{-j2\omega_0 t} + |X_2|e^{j\theta_2 t}\, e^{j2\omega_0 t}$$

$$2|X_2|\cos(2\omega_0 t + \theta_2)$$



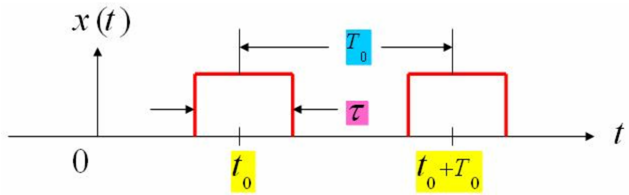


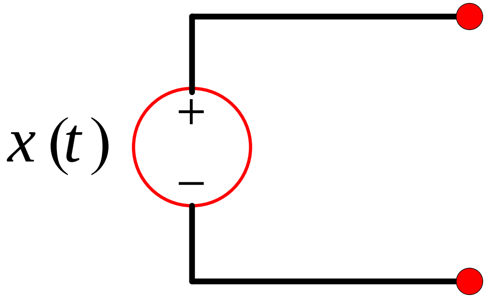
$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t$$

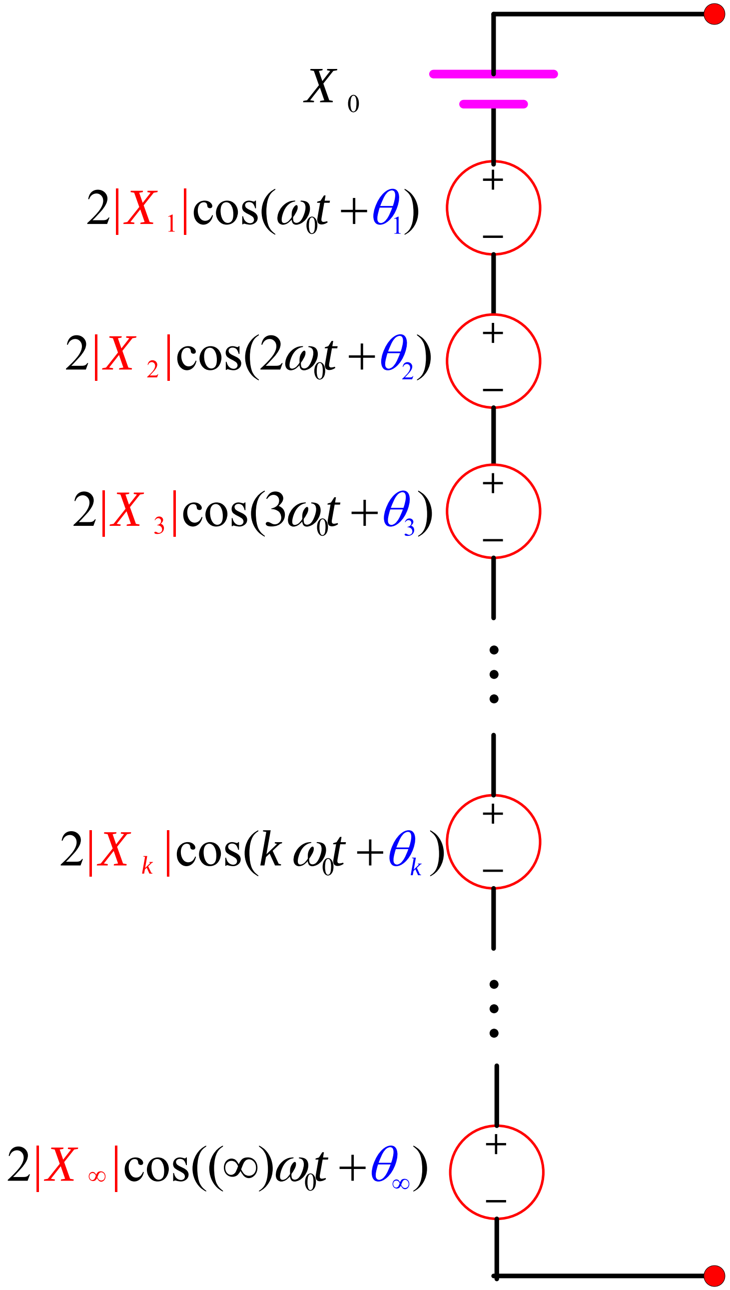


$$=\sum_{n=-\infty}^{\infty}X_n e^{jn\omega_0 t}$$

$$=X_0+2\sum_{n=1}^{\infty}|X_n|\cos(n\omega_0 t+\theta_n)$$







$$x(t)=X_0+2\sum_{n=1}^{\infty}|X_n|\cos(n\omega_0 t+\theta_n)$$