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EE 207 Class Notes

Chapter 1

$$x(t) = \cos(18\pi t) + \cos(12\pi t)$$

$f_1 = 9$ Hz

$f_2 = 6$ Hz

Integer *frequency*

1 X 9

3 X 3

9 X 1

1 X 6

2 X 3

3 X 2

6 X 1

The common frequencies

3 X 3

9 X 1

2 X 3

6 X 1

f_0 = The maximum frequency → $f_0 = 3$ Hz

T_0 = The period = (1/3) second

You can show that the period T_0 of x(t) is (1/3) as follows

$$x(t) = \cos(18\pi t) + \cos(12\pi t)$$

$$x(t+(1/3)) = \cos(18\pi(t+(1/3))) + \cos(12\pi(t+(1/3)))$$

$$= \cos(18\pi t + 6\pi) + \cos(12\pi t + 4\pi)$$

Since

$$\cos(18\pi t + 6\pi) = \cos(18\pi t)\underbrace{\cos(6\pi)}_{1} - \sin(18\pi t)\underbrace{\sin(6\pi)}_{0}$$

$$= \cos(18\pi t)$$

Similarly

$$\cos(12\pi t + 4\pi) = \cos(12\pi t)\underbrace{\cos(4\pi)}_{1} - \sin(12\pi t)\underbrace{\sin(4\pi)}_{0}$$

$$= \cos(12\pi t)$$

$$\Rightarrow x(t+(1/3)) = \cos(18\pi t) + \cos(12\pi t) = x(t)$$

Phasor Signal and Spectra

From Circuit Theory

$$v(t) = 10\cos(10\pi t + 30^{o}) \xrightarrow{\text{phasor}} \boldsymbol{V} = 10\angle 30^{o}$$

In General

If $\boldsymbol{X} = A\angle\theta = Ae^{j\theta}$

ω_0

Then

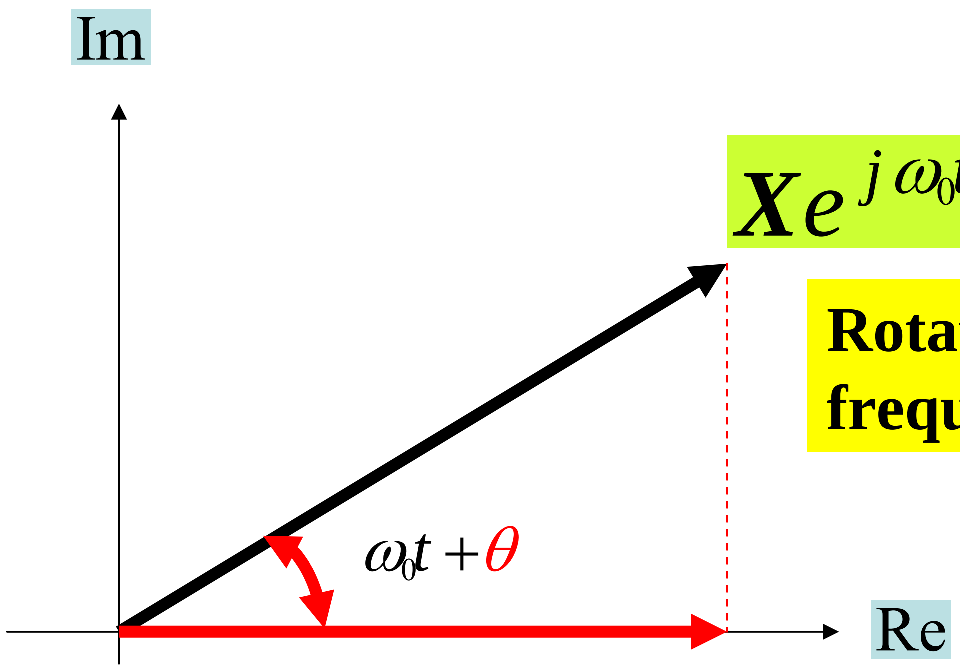
$$x(t) = \mathrm{Re}(\boldsymbol{X}e^{j\omega_0 t}) = \mathrm{Re}(Ae^{j\theta}e^{j\omega_0 t}) = \mathrm{Re}(Ae^{j(\omega_0 t + \theta)})$$

$$= \mathrm{Re}(A\cos(\omega_0 t + \theta) + \mathrm{j}\,\mathrm{A}\cos(\omega_0 t + \theta))$$

$$= A\cos(\omega_0 t + \theta)$$

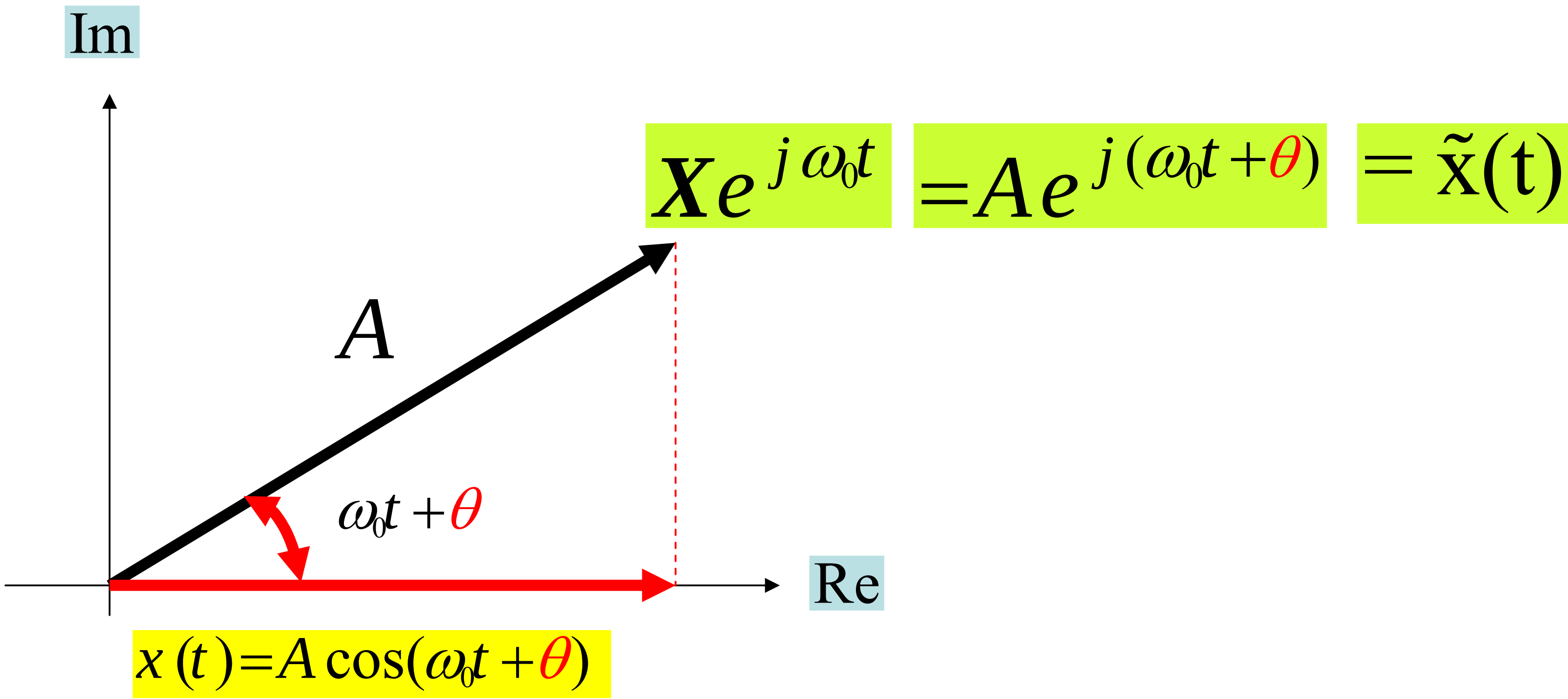
$$\boldsymbol{X}e^{j\omega_0 t} = Ae^{j(\omega_0 t+\theta)}$$

Rotating with angular frequency ω_o



$$x(t) = A\cos(\omega_0 t+\theta)$$

Projection of $Ae^{j(\omega_0 t+\theta)}$ **onto the real axis**



Let

$\tilde{x}(t) = Ae^{j(\omega_0 t+\theta)}$ **Be a rotating complex phasor function**

The Projection of $\tilde{x}(t)$ **onto the real axis gives** $x(t) = A\cos(\omega_0 t+\theta)$

We can obtain $x(t) = A\cos(\omega_0 t + \theta)$ **from the complex signal**

$\tilde{x}(t) = Ae^{j(\omega_0 t + \theta)}$ **in two ways:**

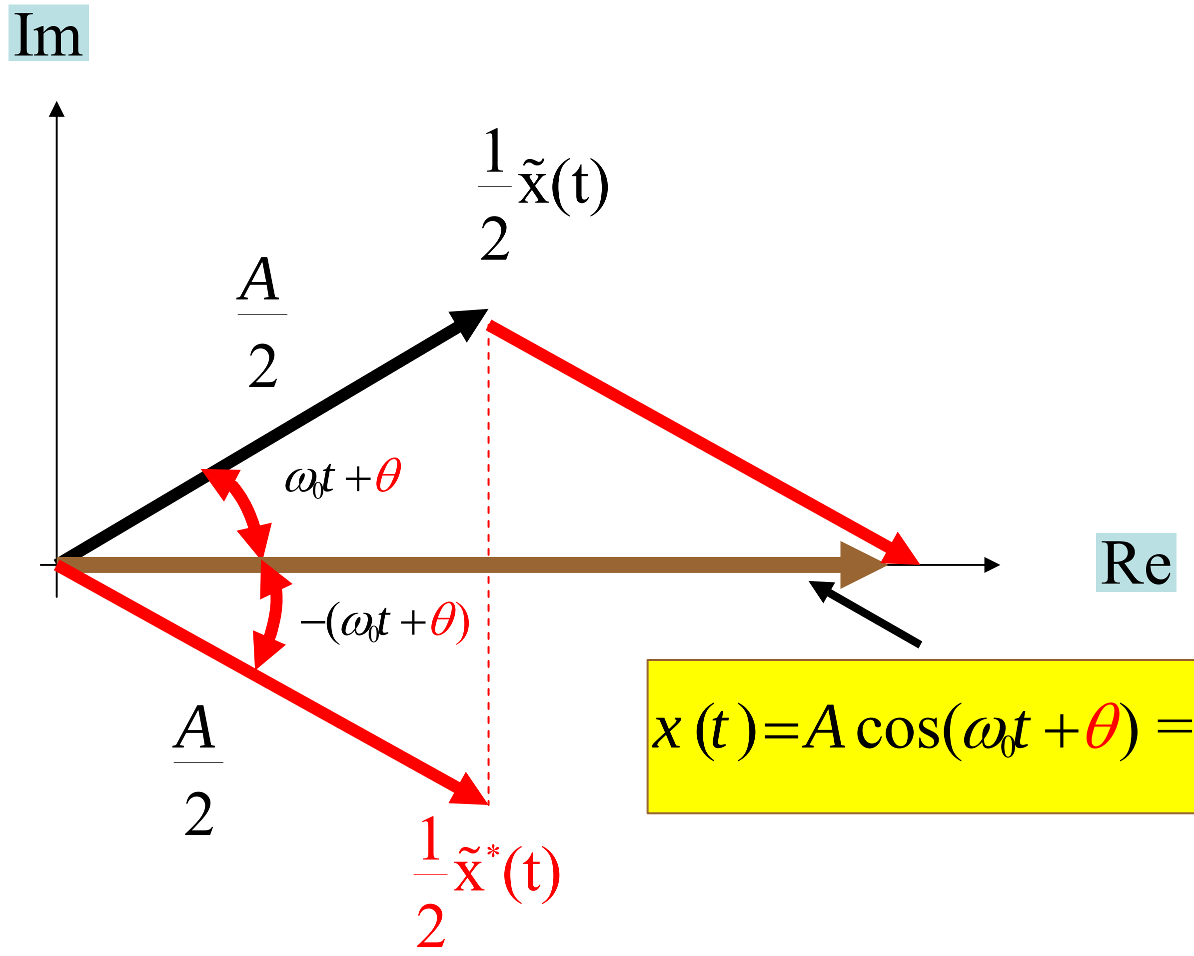
(I) $x(t) = A\cos(\omega_0 t + \theta) = \mathrm{Re}[Ae^{j(\omega_0 t + \theta)}] = \mathrm{Re}[\tilde{x}(t)]$

(II) $x(t) = A\cos(\omega_0 t + \theta) = \dfrac{Ae^{j(\omega_0 t + \theta)} + Ae^{-j(\omega_0 t + \theta)}}{2} = \dfrac{Ae^{j(\omega_0 t + \theta)}}{2} + \dfrac{Ae^{-j(\omega_0 t + \theta)}}{2}$

$$= \frac{1}{2}\tilde{x}(t) + \frac{1}{2}\tilde{x}^*(t)$$

conjugate of $\tilde{x}(t)$

Rotating phasors of opposite direction



$$x(t)=A\cos(\omega_0 t+\theta)=\frac{1}{2}\tilde{x}(t)+\frac{1}{2}\tilde{x}^*(t)$$

We note also that if $\tilde{x}(t)$ rotate with positive frequency

then $\tilde{x}^*(t)$ rotate with negative frequency

Note in reality there is no such thing as negative frequency.

It is only mathematical abstraction needed to get the real quantity

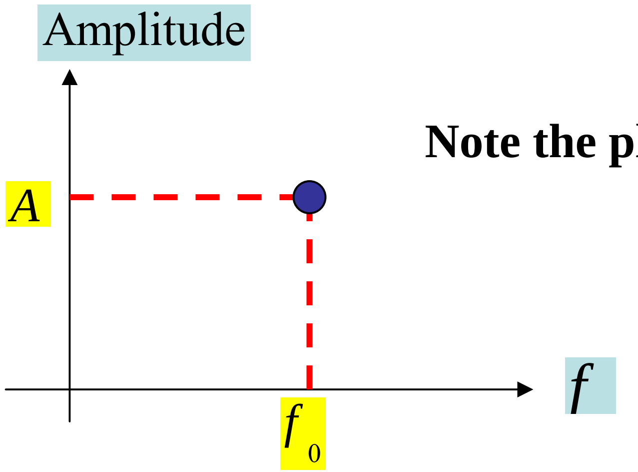
Frequency domain representation of x(t)

Since we have two representations of x(t) , then we will have two representations for the frequency representations as follows:

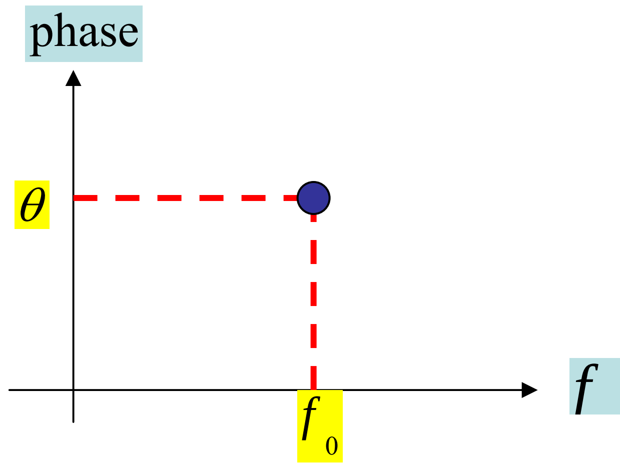
(I) $x(t) = A\cos(\omega_0 t + \theta) = \mathrm{Re}[Ae^{j(\omega_0 t + \theta)}] = \mathrm{Re}[\tilde{x}(t)]$

$\tilde{x}(t) = Ae^{j(\omega_0 t + \theta)}$ **is completely specified by** A and θ for a given ω_0 or f_0

Since we have a single frequency f_0 , then we have the following plot



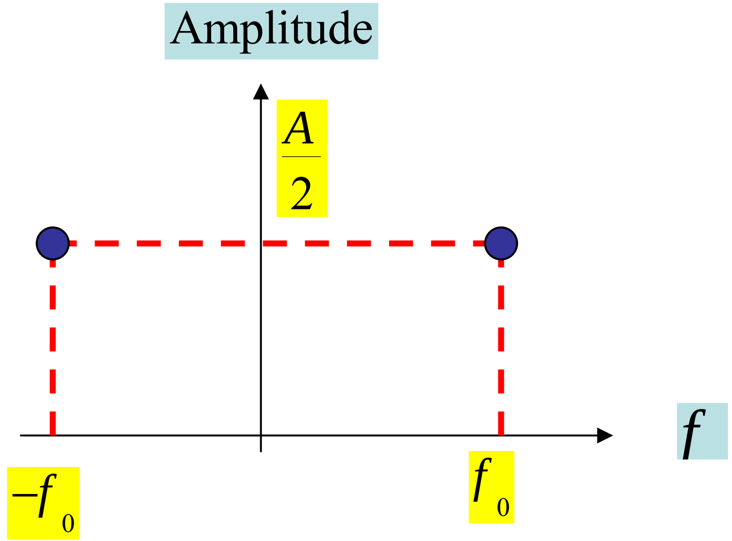
Note the plot consist of single points

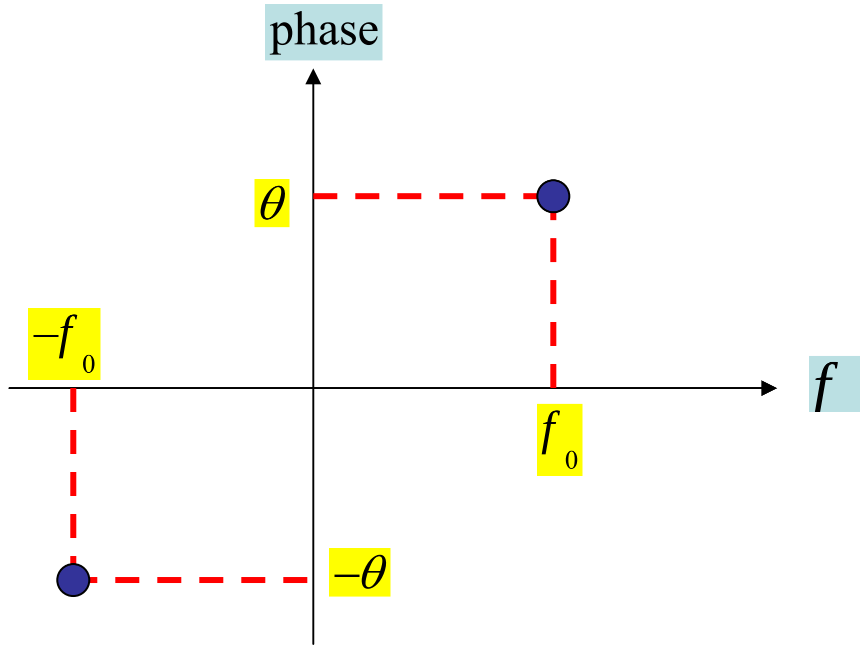


This plot is referred to as single-sided spectrum because it involves only positive frequencies , no negative frequencies

(II) $x(t) = A\cos(\omega_0 t + \theta) = \frac{1}{2}Ae^{j(\omega_0 t+\theta)} + \frac{1}{2}Ae^{-j(\omega_0 t+\theta)}$

Since we have two frequency f_0 and $-f_0$, then we have the following plot



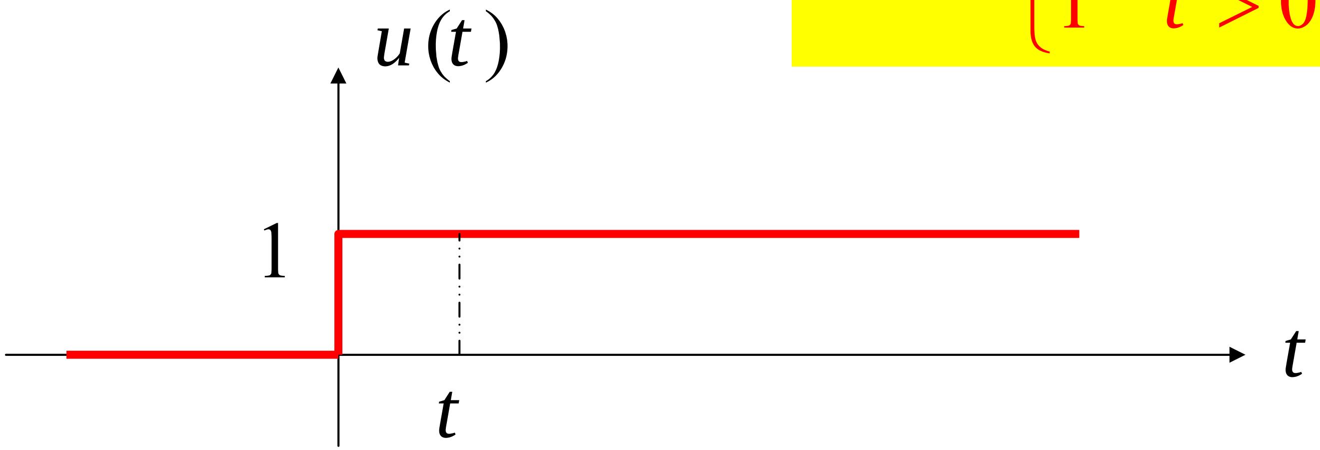


This plot is referred to as double-sided spectrum because it involves only positive and frequencies .

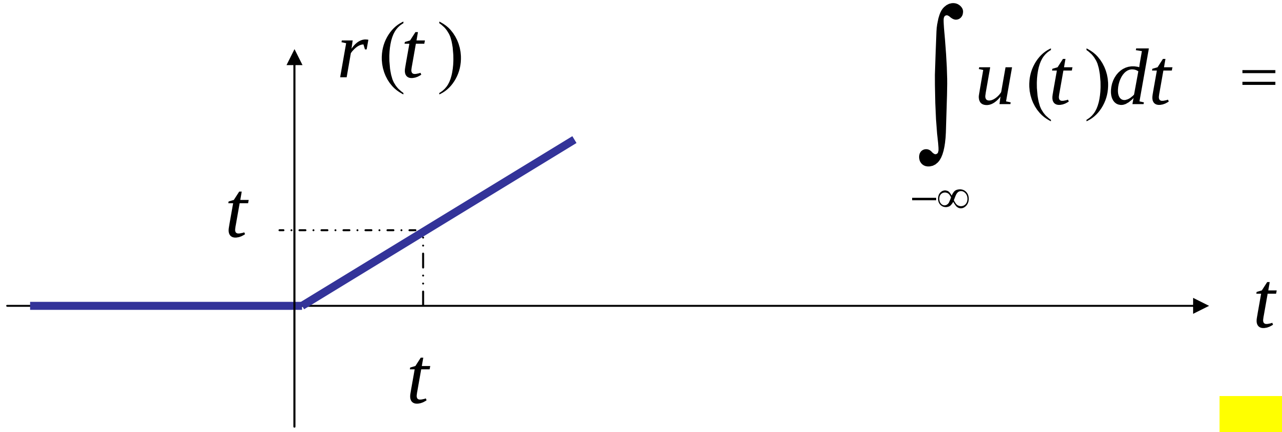
If x(t) is given as sin we write it as cosine using $\sin(\omega_0 t + \theta) = \cos(\omega_0 t + \theta - \frac{\pi}{2})$

Step Function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



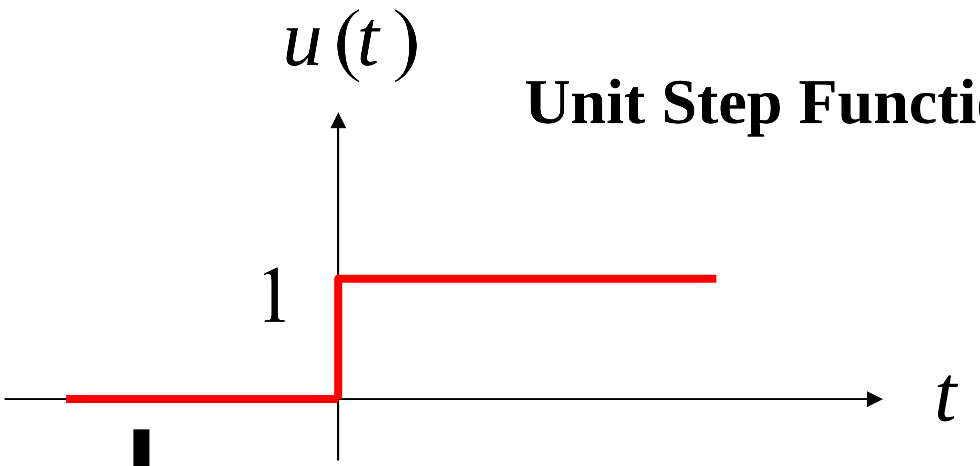
$$\int_{-\infty}^{t} u(t)dt = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$



Ramp Function

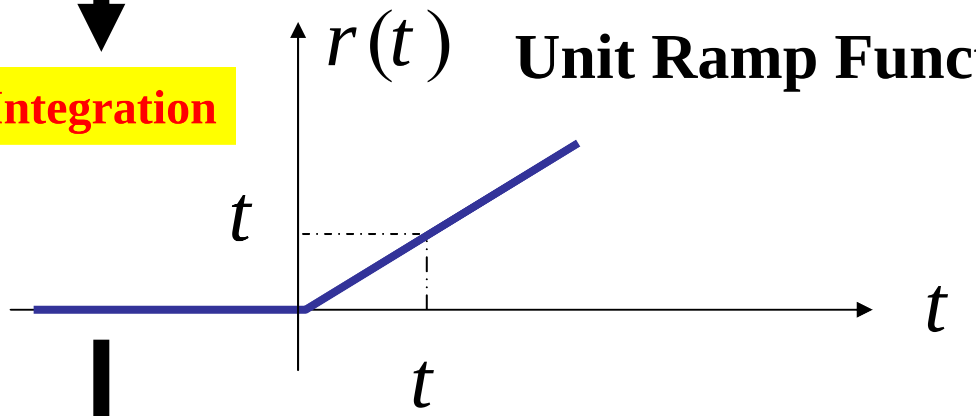
$$r(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

Unit Step Function



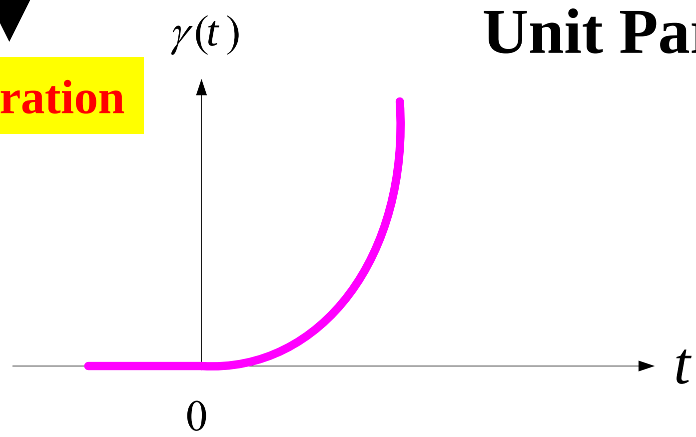
Integration

Unit Ramp Function



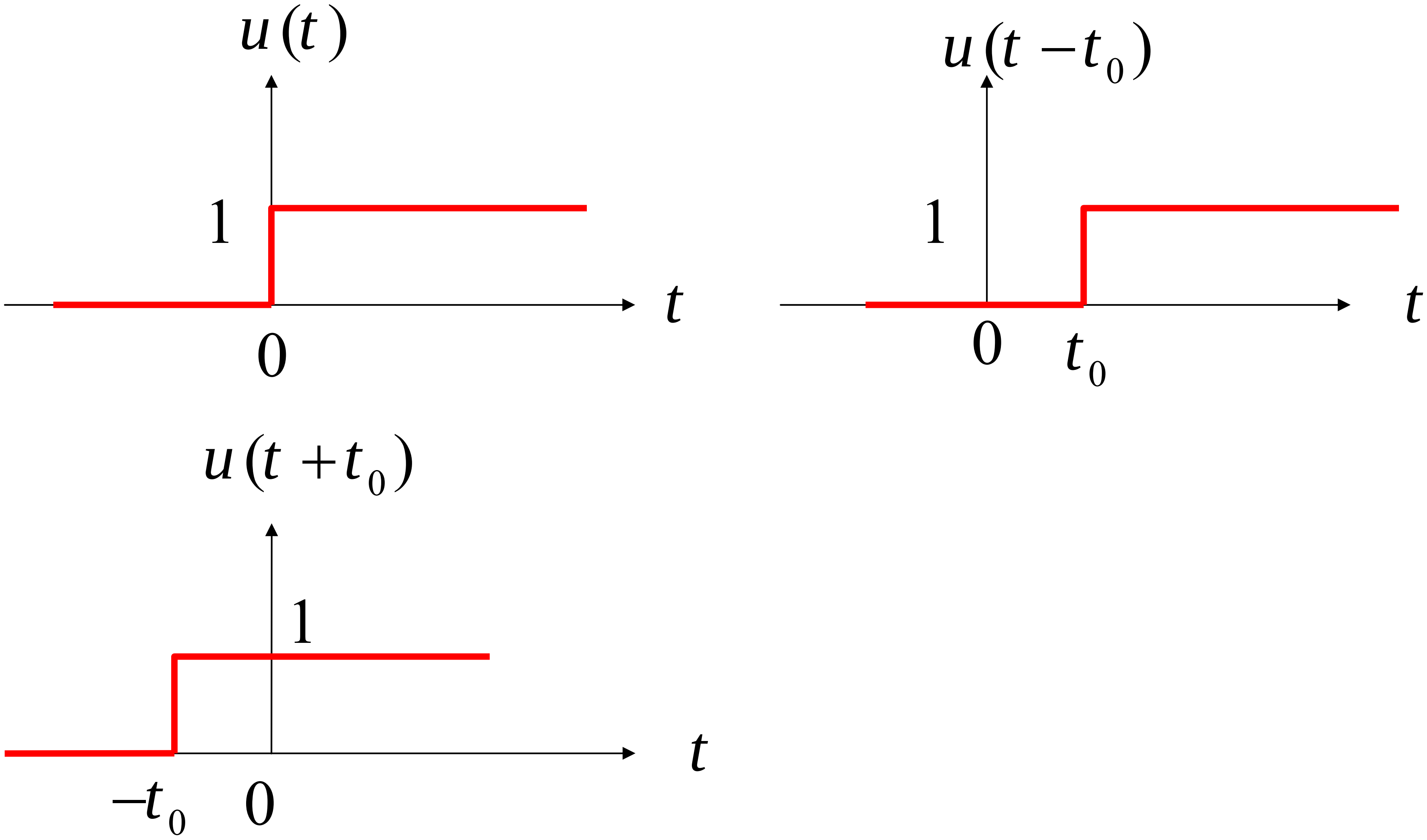
Integration

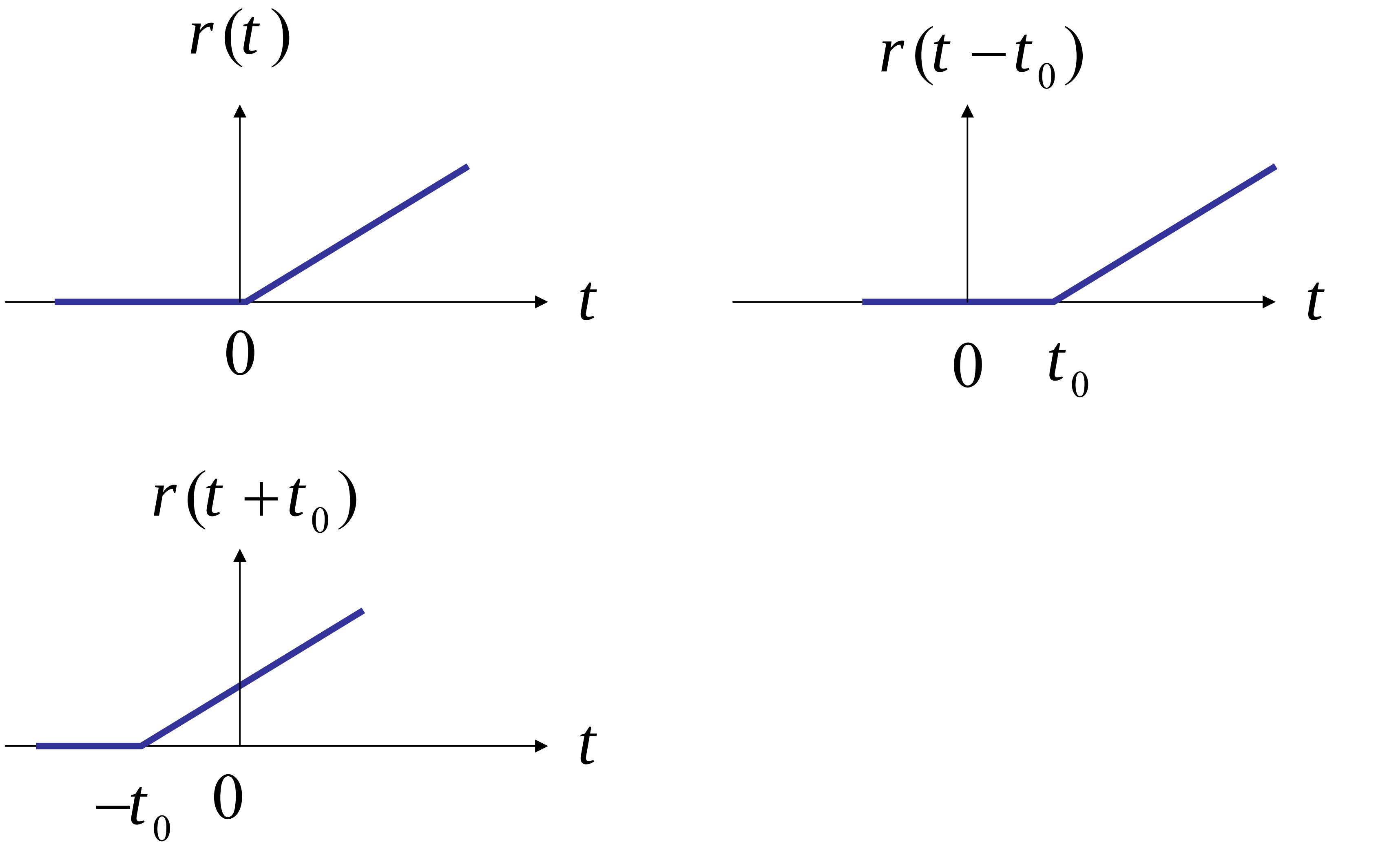
Unit Parabolic Function



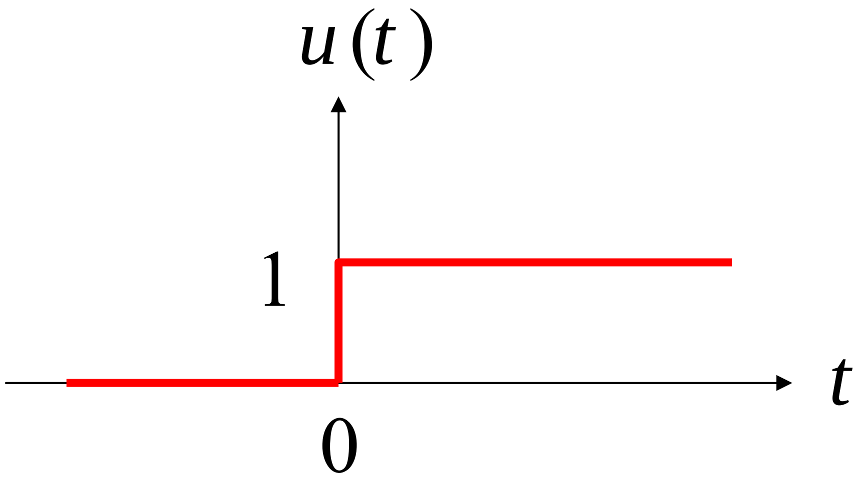
$$\gamma(t) = \int_{-\infty}^{t} r(t')\,dt' = \begin{cases} 0 & t < 0 \\ \dfrac{t^2}{2} & t \geq 0 \end{cases}$$

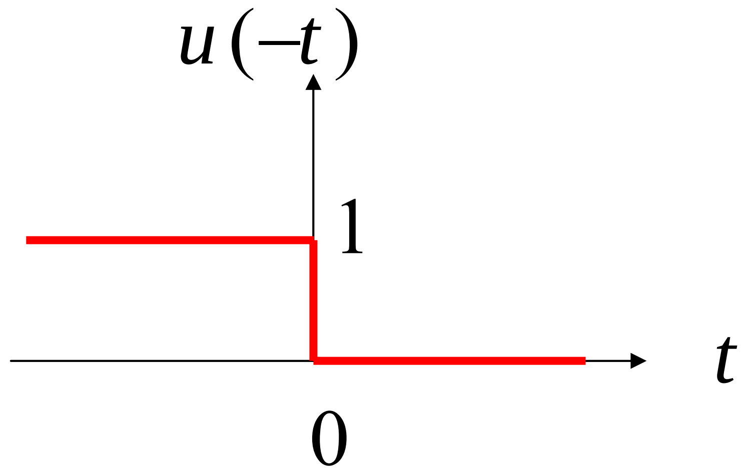
Time Shift

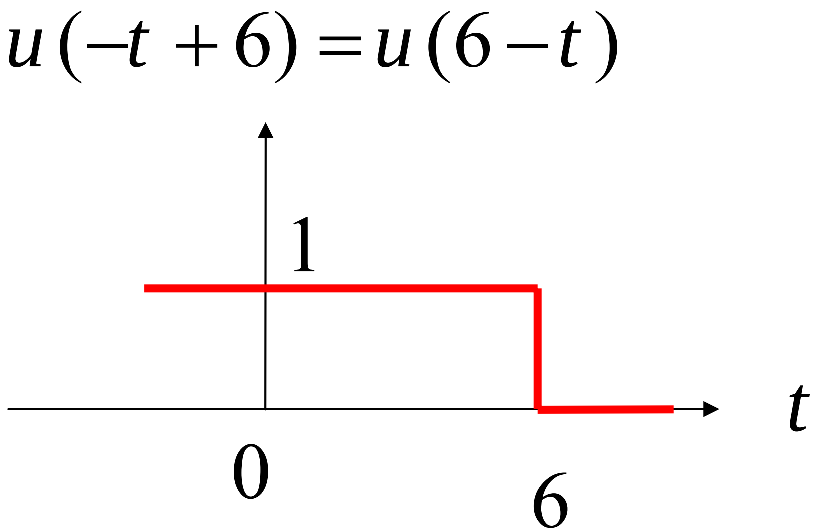


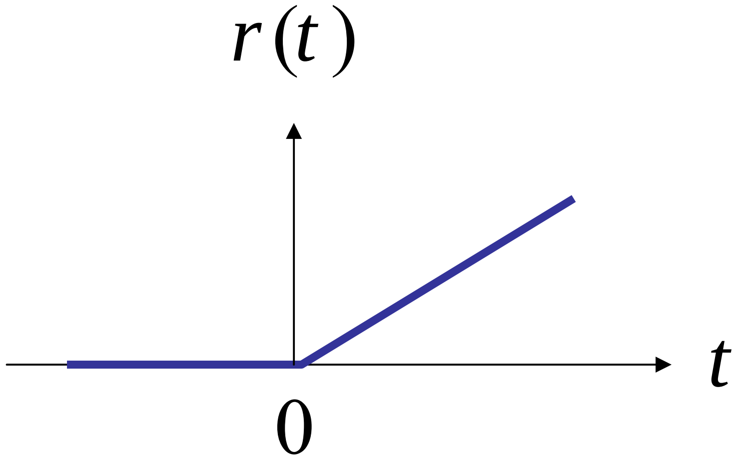

$r(t)$
t
0
$r(t-t_0)$
t
0
t_0
$r(t+t_0)$
t
$-t_0$
0

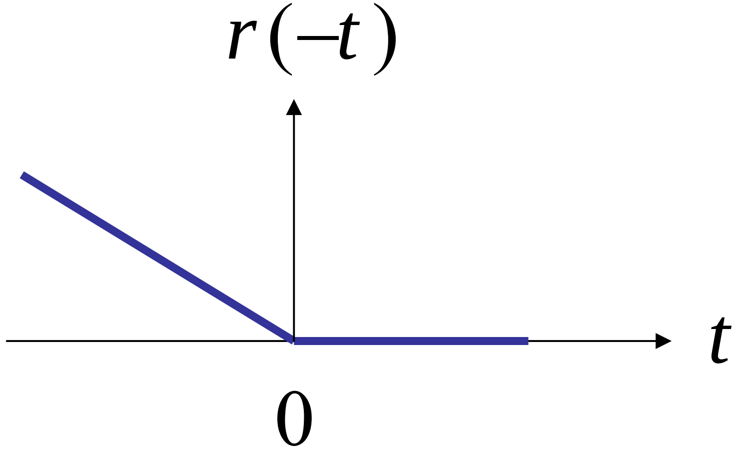
Time folding



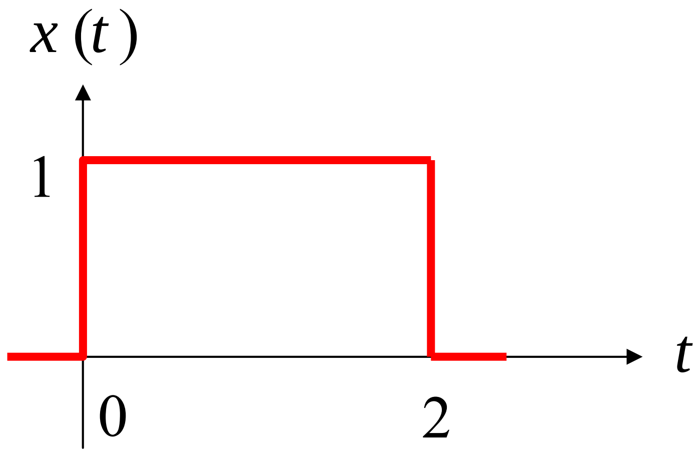


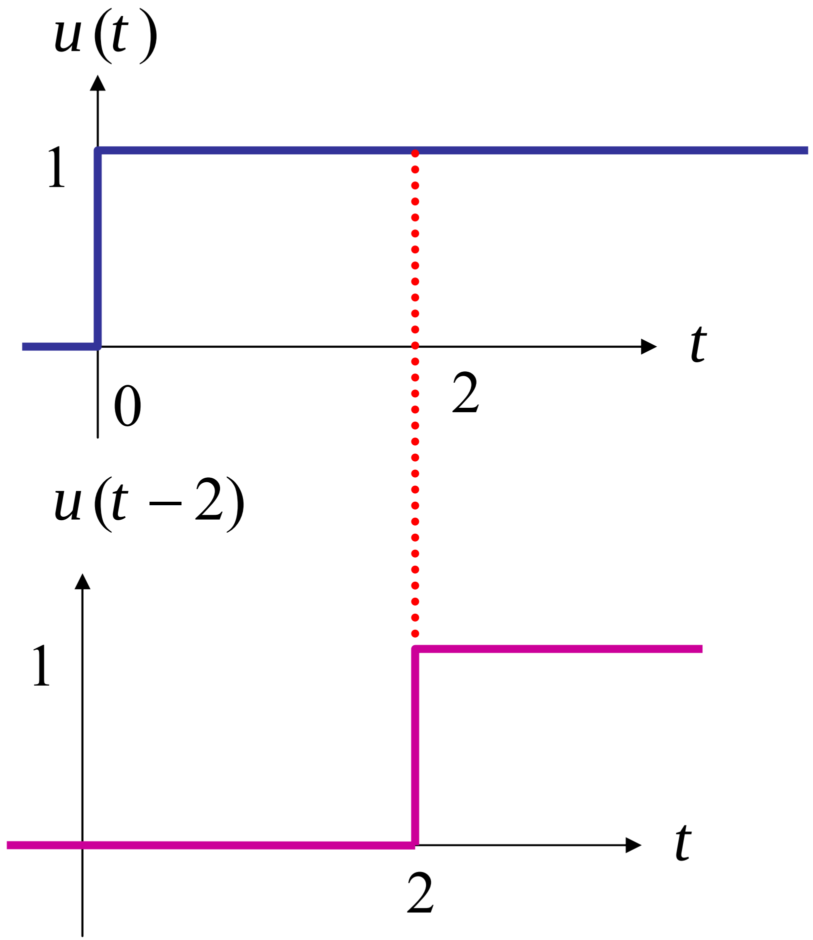






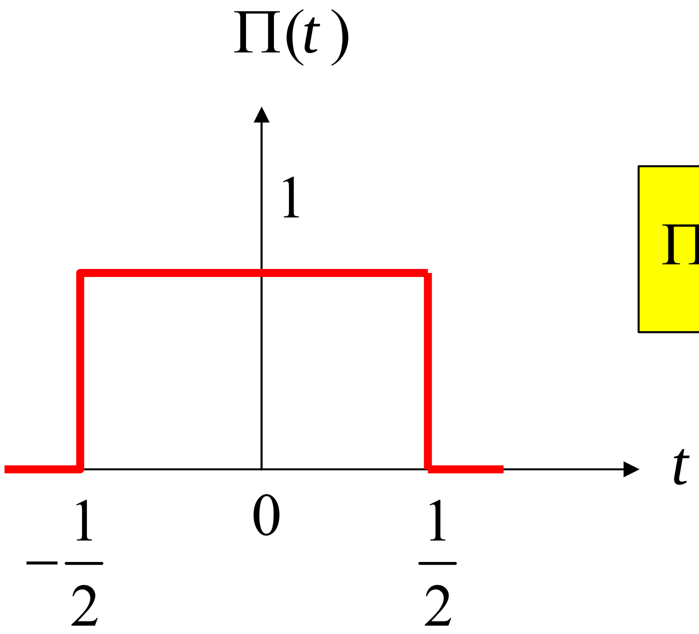
$$r(t) = \begin{cases} 0 & t > 0 \\ -t & t \le 0 \end{cases}$$





$$x(t) = u(t) - u(t-2)$$

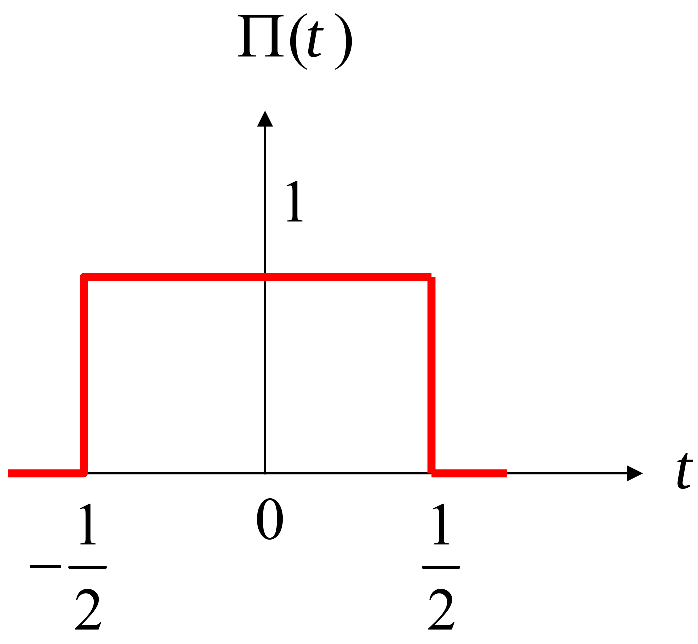
Pulse Function



$$\Pi(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right)$$

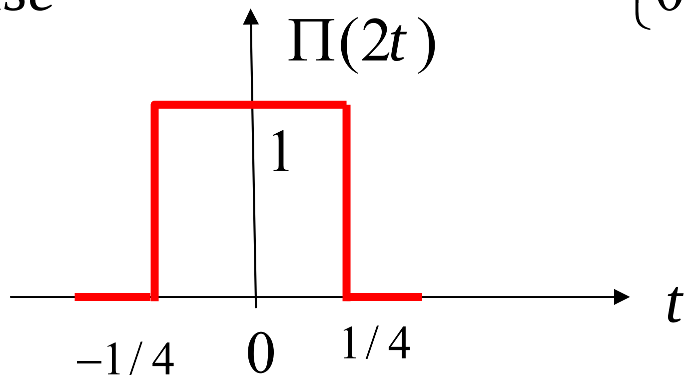
Chang of Scale

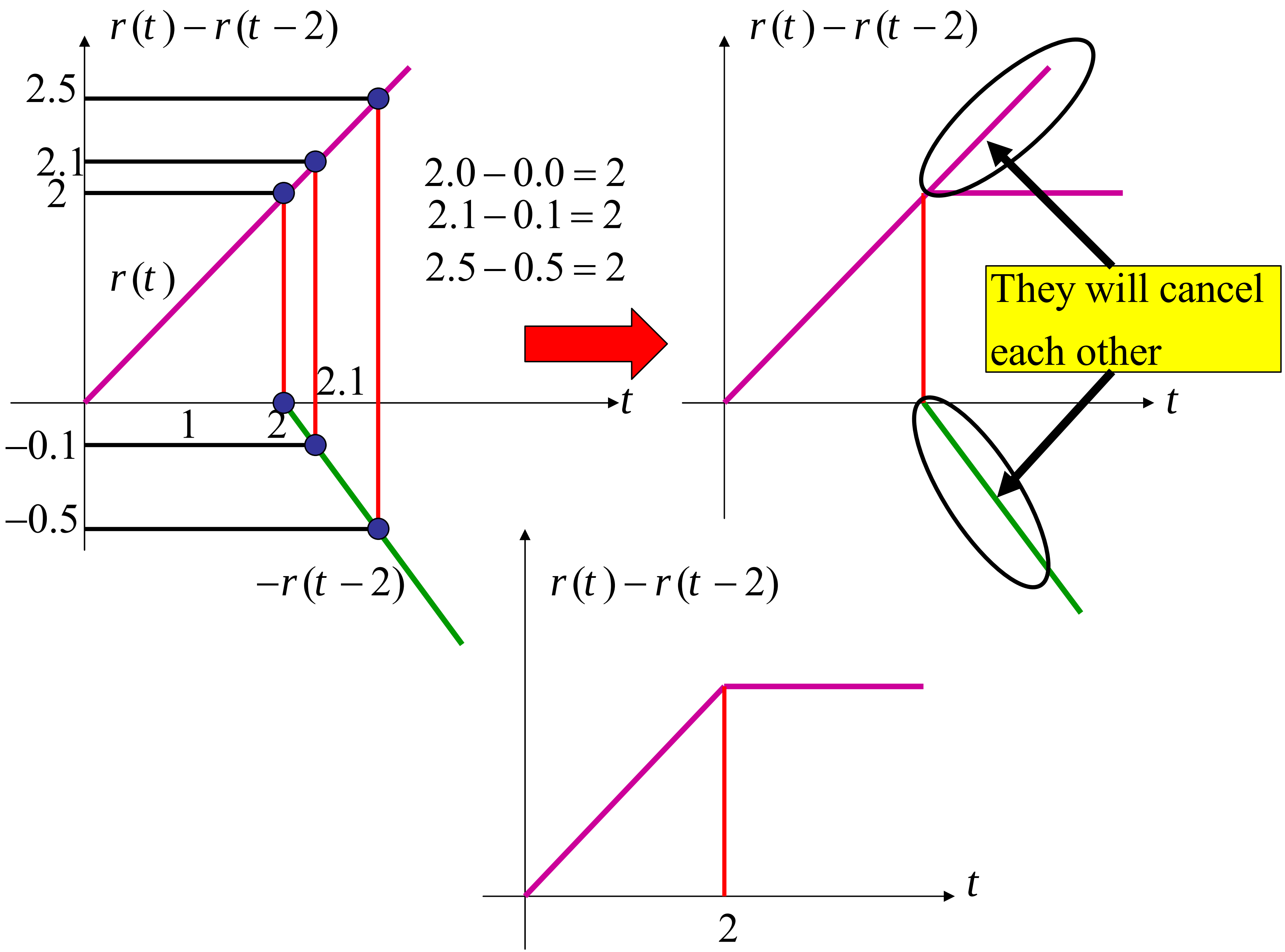
Consider the pulse function $\Pi(t)$



Plot $\Pi(2t)$?

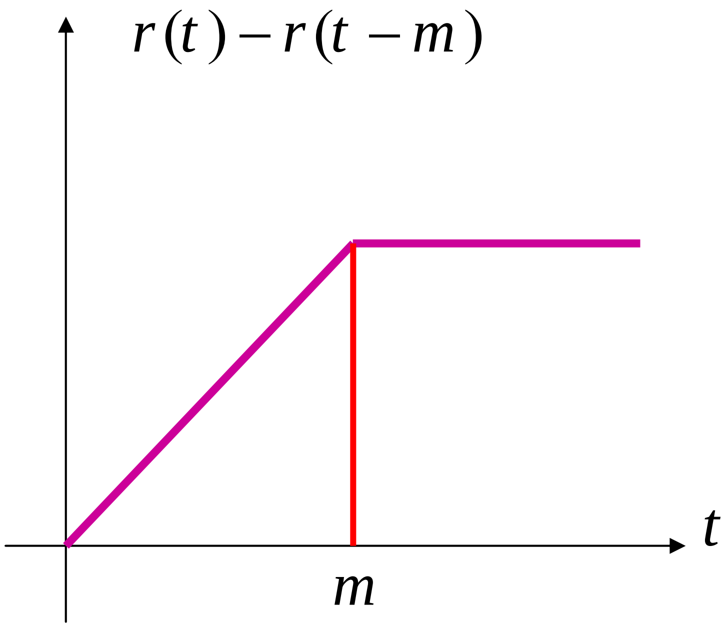
Since $\Pi(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & else \end{cases}$ Since $\Pi(2t) = \begin{cases} 1 & -\frac{1}{2} \le 2t \le \frac{1}{2} \Rightarrow -\frac{1}{4} \le t \le \frac{1}{4} \\ 0 & else \end{cases}$



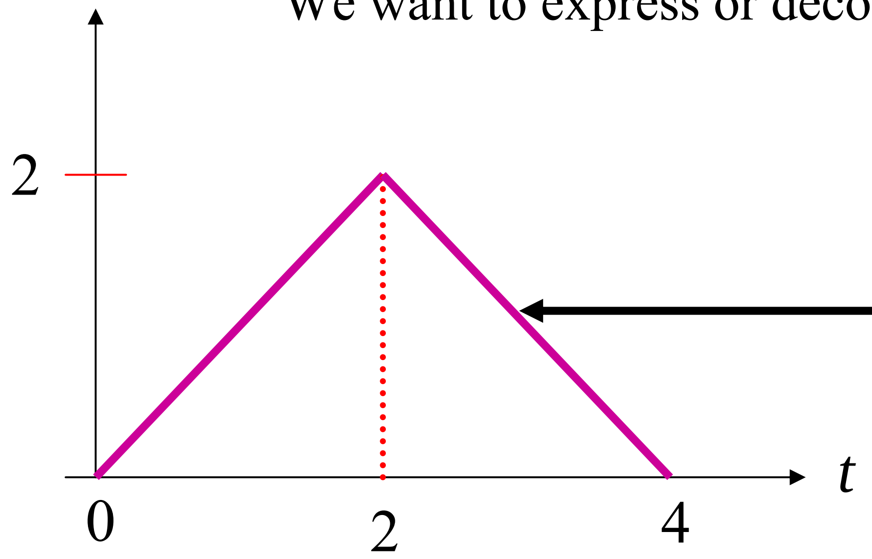


r(t) − r(t − 2)
2.5
2.1
2
r(t)
1
2
2.1
t
−0.1
−0.5
−r(t − 2)
2.0 − 0.0 = 2
2.1 − 0.1 = 2
2.5 − 0.5 = 2
r(t) − r(t − 2)
t
They will cancel each other
r(t) − r(t − 2)
t
2

In general



We want to express or decompose this function to singularities functions

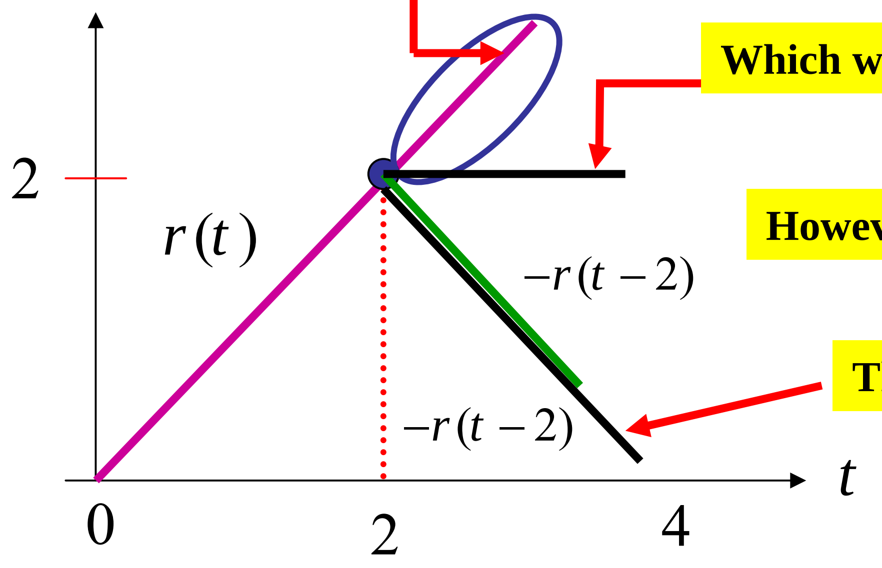


First we cancel the uprising ramp after t =2 with subtracting downward ramp

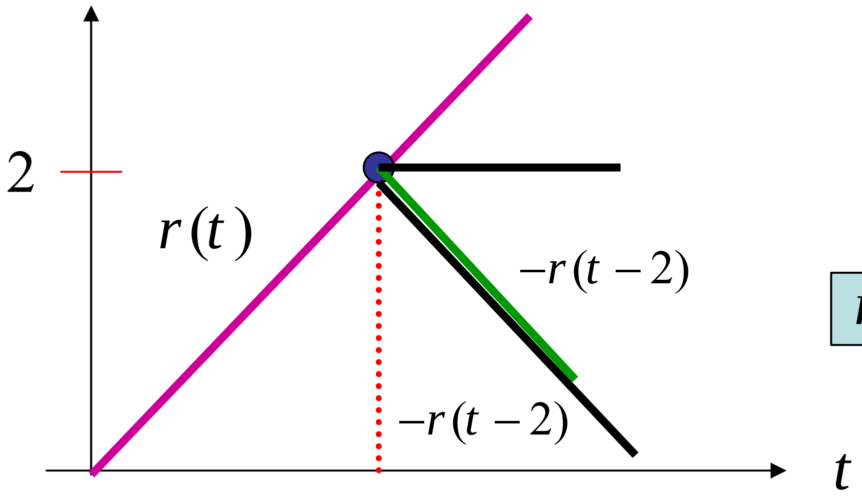
Which will result in horizontal line segment

However what we need is a downward ramp

Therefore we need to subtract another ramp



$$r(t)-2r(t-2)$$

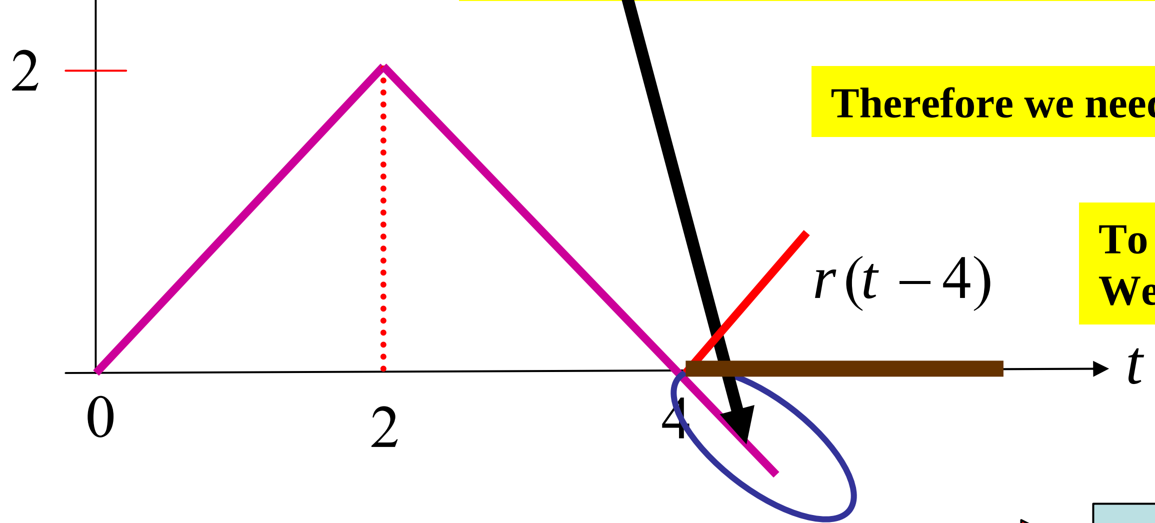


$$r(t)-2r(t-2)$$

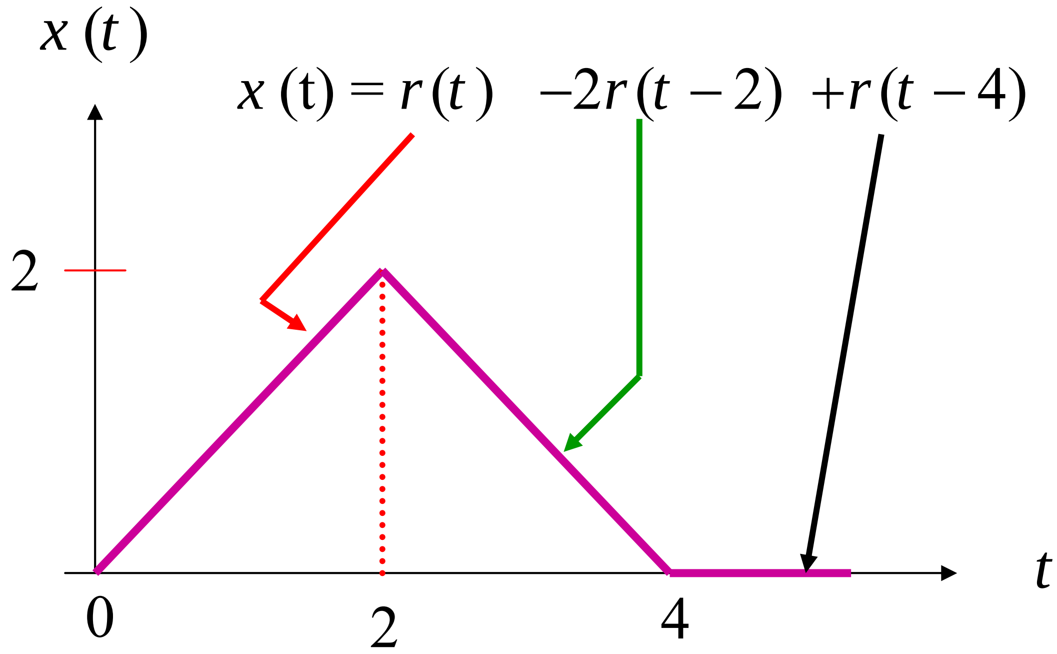
However the downward ramp will continue indefinitely

Therefore we need to cancel this part of downward ramp

To cancel this part of downward ramp
We add an upward ramp at t = 4



$$x(t)=r(t)-2r(t-2)+r(t-4)$$

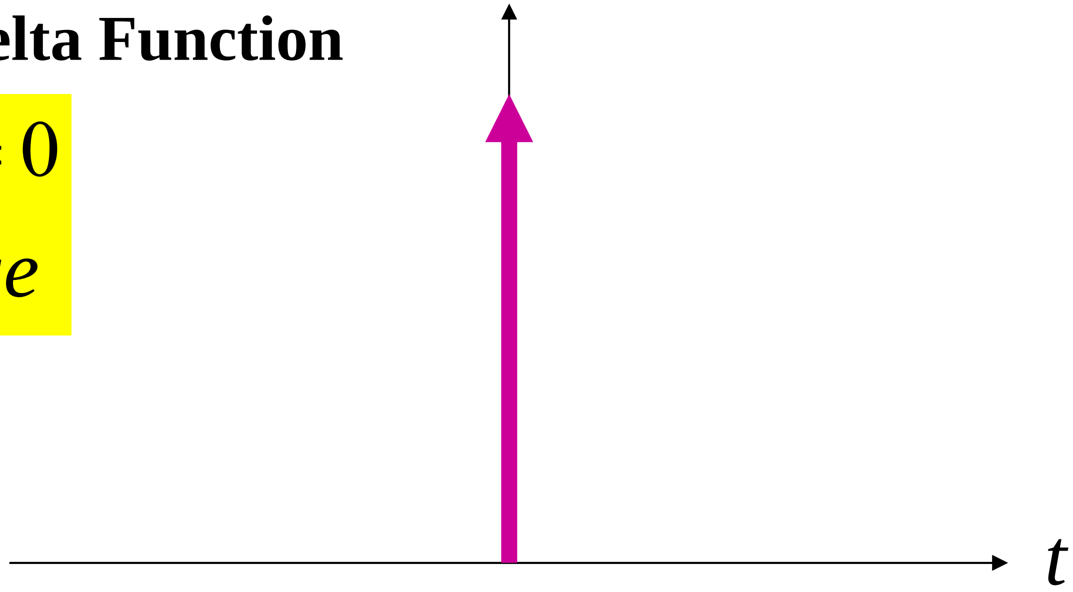


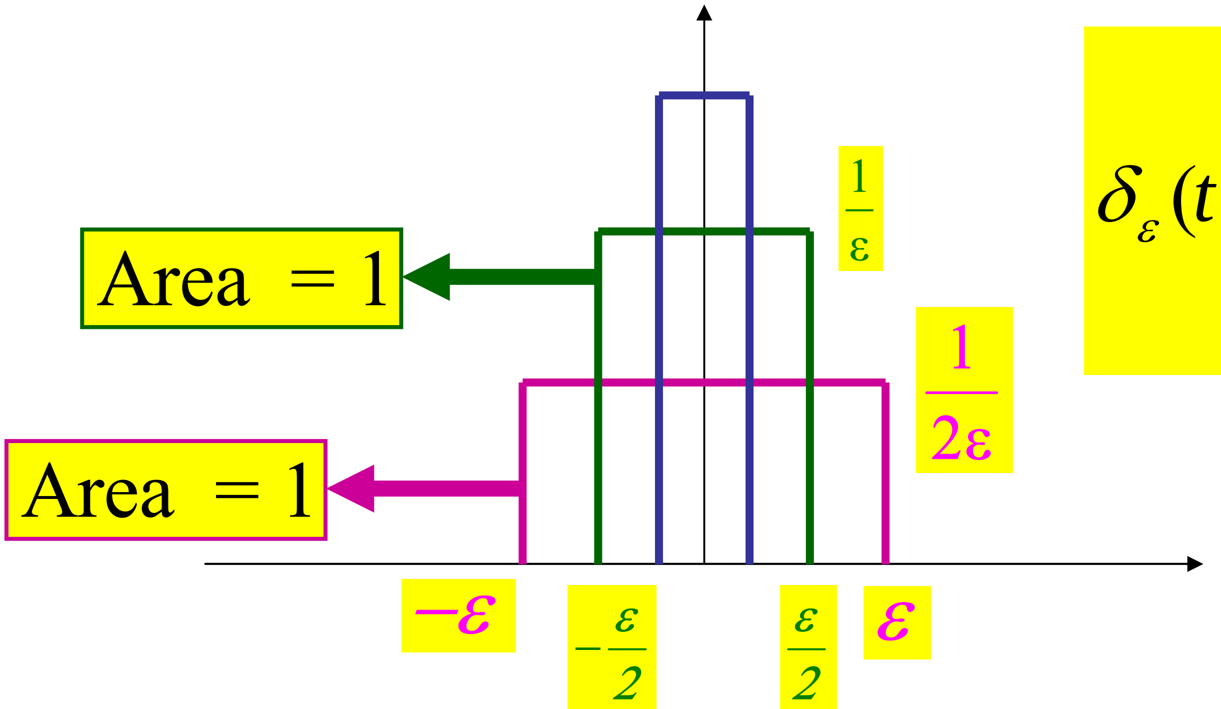
$x(t)$
$x(t) = r(t) \; -2r(t-2) \; +r(t-4)$
2
0
2
4
t

Impulse or Dirac Delta Function

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & else \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

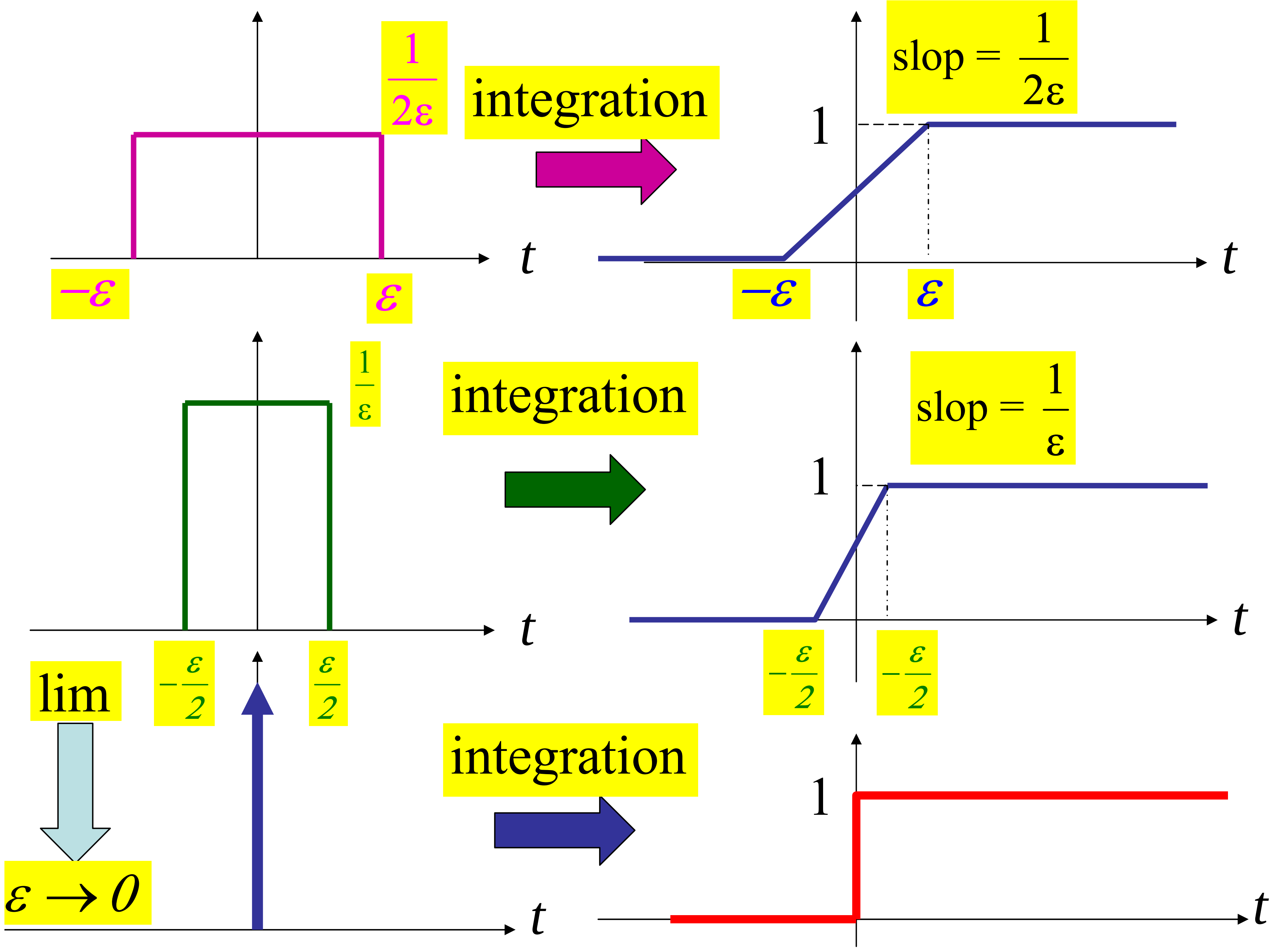




t

$$\delta_\varepsilon(t) = \begin{cases} \dfrac{1}{2\varepsilon} & |t| \le 0 \\ 0 & else \end{cases}$$

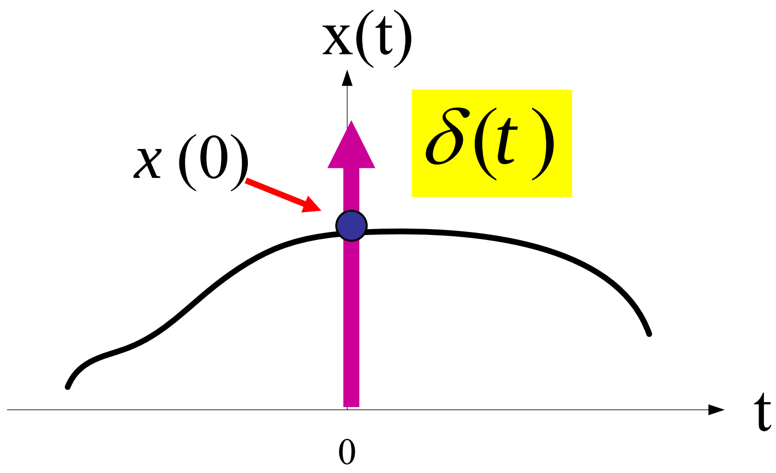
$$\lim_{\varepsilon \to 0} \delta_\varepsilon(t) = \delta(t)$$

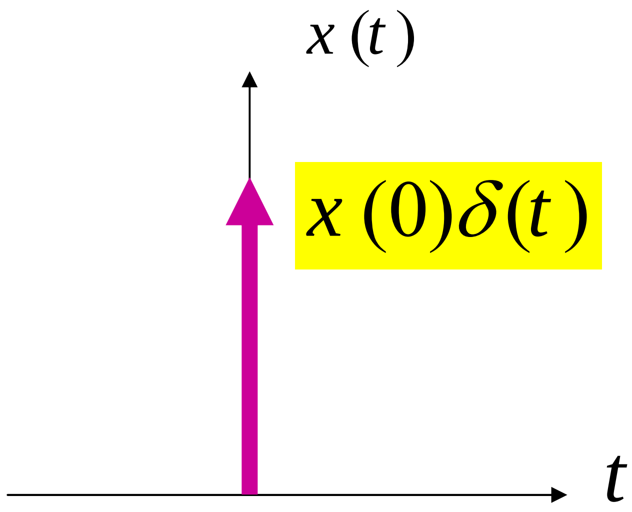


$\frac{1}{2\varepsilon}$
integration
slop = $\frac{1}{2\varepsilon}$
1
t
$-\varepsilon$
ε
t
$-\varepsilon$
ε
$\frac{1}{\varepsilon}$
integration
slop = $\frac{1}{\varepsilon}$
1
t
t
$-\frac{\varepsilon}{2}$
$\frac{\varepsilon}{2}$
$-\frac{\varepsilon}{2}$
$-\frac{\varepsilon}{2}$
lim
$\varepsilon \to 0$
integration
1
t
t

Properties of Delta Function

(I)

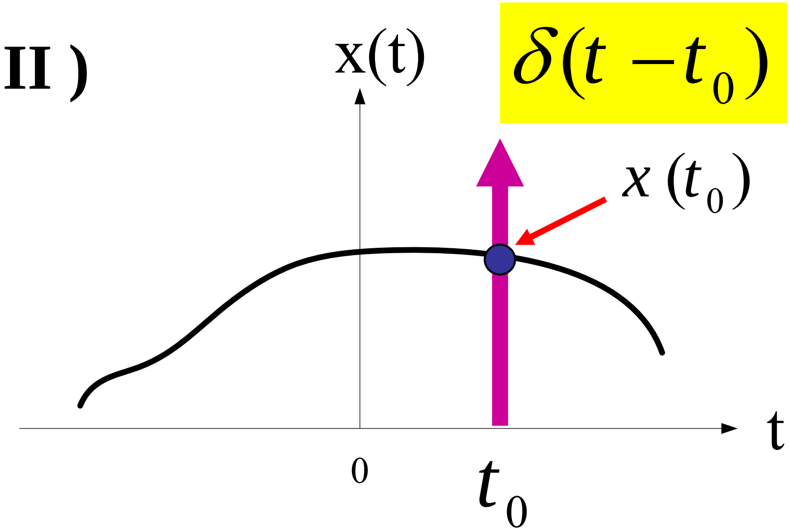


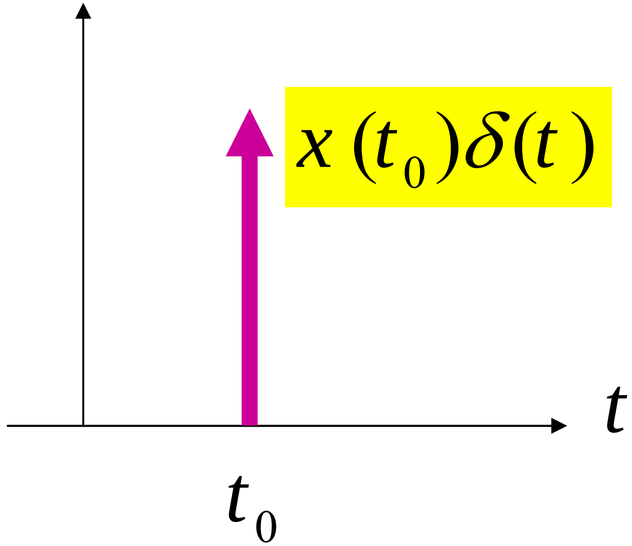


$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

Note $0\delta(t) = 0$

(II)





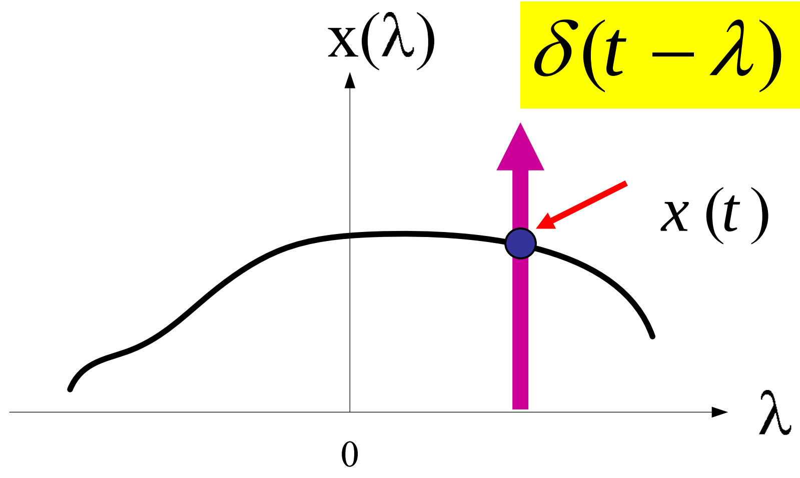
$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$$

Exercise

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} \delta(t - 10)dt = e^{-\alpha (10)^2} = e^{-100\alpha}$$

(II)

$$\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)dt = x(t)$$



This property is known as **"convolution"** which will be useful in chapter 2

Note $\delta(-t) = \delta(t)$ "Even Function"