SE301:Numerical Methods
Topic 9
Partial Differential Equations

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Term 053
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(072)

Lect 27: Partial Differential Equations

- Partial Differential Equations (PDE)
- What is a PDE
- Examples of Important PDE
- Classification of PDE

Partial Differential Equations

A partial differential equation (**PDE**) is an equation that involves an unknown function and its partial derivatives.

Examples:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

PDE involves two or more independent variables

(in the example x and t are independent variables)

Notation

$$u_{xx} = \frac{\partial^2 \ u(x,t)}{\partial x^2}$$

$$u_{xt} = \frac{\partial^2 \ u(x,t)}{\partial x \ \partial t}$$

order of the PDE = order of the highest order derivative

Linear PDE

Classification

A PDE is linear if it is linear in the unknown function and its derivatives

Example of linear PDE :

$$2\,u_{xx} + 1\,u_{xt} + 3\,u_{tt} + 4\,u_x + \cos(2t) = 0$$

$$2\,u_{xx} + -3\,u_t + 4\,u_x = 0$$

Examples of Nonlinear PDE

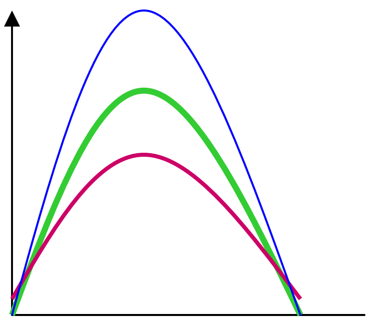
$$2\,u_{xx} + \left(u_{xt}\right)^2 + 3\,u_{tt} = 0$$

$$\sqrt{u_{xx}} + 2\,u_{xt} + 3u_t = 0$$

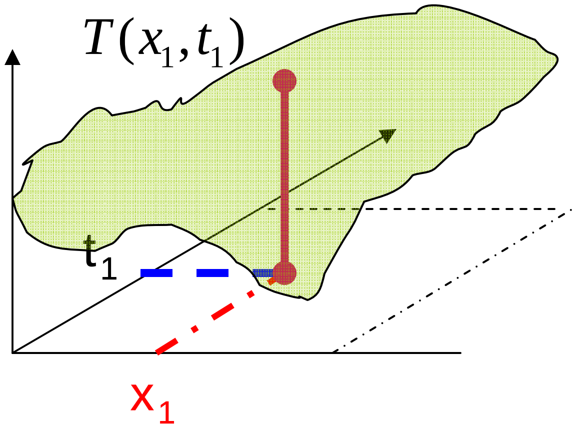
$$2\,u_{xx} + 2\,u_{xt}\,u_t + 3u_t = 0$$

Representing the solution of PDE (two independent variables)

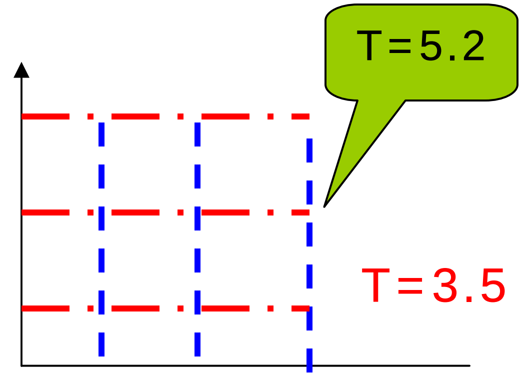
- Three main ways to represent the solution



Different curves are used for different values of one of the independent variable



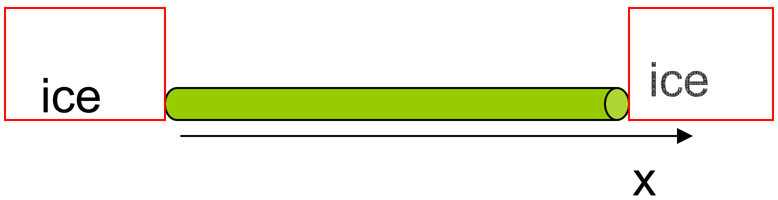
Three dimensional plot of the function T(x,t)



The axis represent the independent variables. The value of the function is displayed at grid points

Heat Equation

Different curve is used for each value of t

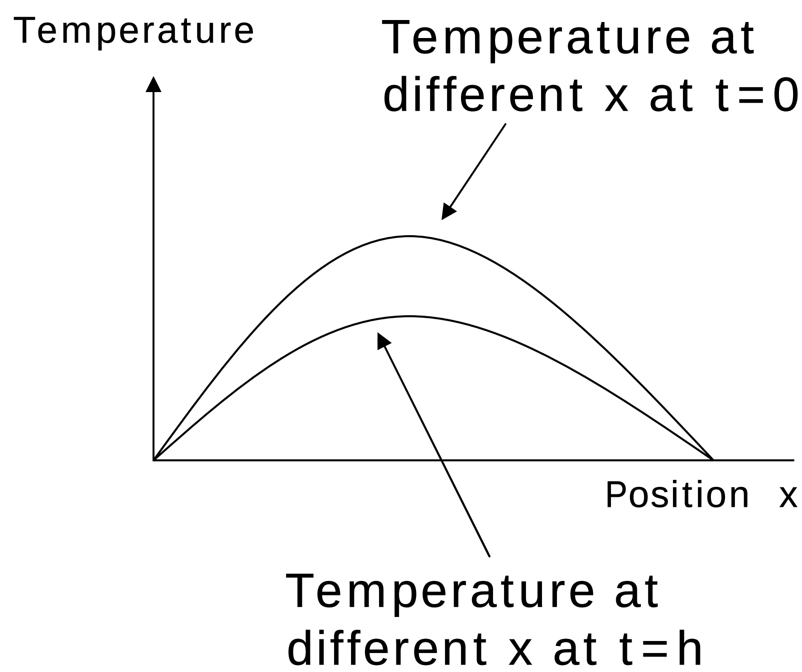


Thin metal rod insulated everywhere except at the edges. At t =0 the rod is placed in ice

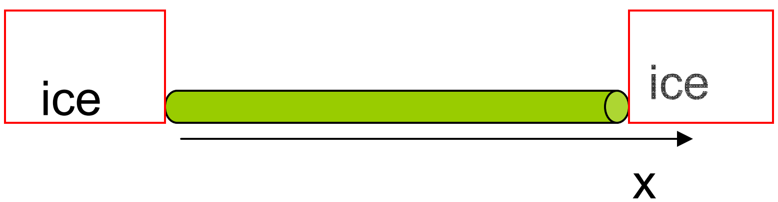
$$\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0$$

$$T(0,t) = T(1,t) = 0$$

$$T(x,o) = \sin(\pi x)$$



Heat Equation



$$\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0$$

$$T(0,t) = T(1,t) = 0$$

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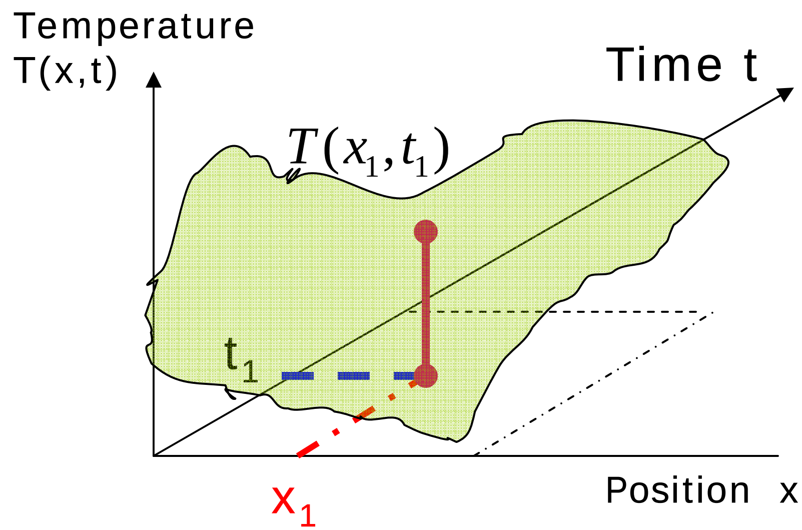


Linear Second Order PDE

Classification

A second order linear PDE (2-indepent variables)

$$A\ u_{xx} + B\ u_{xy} + C\ u_{yy} + D = 0,$$

D is a function of x, y, u_x, u_y

is classfied based on $(B^2 - 4AC)$ as follows:

$B^2 - 4AC = 0$	Parabolic
$B^2 - 4AC > 0$	Hyperbolic
$B^2 - 4AC < 0$	Elliptic

Linear Second Order PDE

Examples (Classification)

Heat Equation $$k\frac{\partial^2 T(x,t)}{\partial x^2}-\frac{\partial T(x,t)}{\partial t}=0$$

$$A=1, B=0, C=0 \Rightarrow B^2-4AC=0$$

$\Rightarrow$ Heat Equation *is Parabolic*

Wave Equation $$\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial^2 u(x,t)}{\partial t^2}=0$$

$$A=1, B=0, C=-1 \Rightarrow B^2-4AC>0$$

$\Rightarrow$ Wave Equation *is Hyperbolic*

Classification of PDE

Linear Second order PDE are important set of equations that are used to model many systems in many different fields of science and engineering.

Classification is important because

- Each category relates to specific engineering problems
- Different approaches are used to solve these categories

Examples of PDE

PDE are used to model many systems in many different fields of science and engineering.

Important Examples:

- Wave Equation
- Heat Equation
- Laplace Equation
- Biharmonic Equation

Heat Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} = \frac{\partial u(x,y,z,t)}{\partial t}$$

The function *u(x,y,z,t)* is used to represent the temperature at time *t* in a physical body at a point with coordinates *(x,y,z)* .

Simpler Heat Equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

u(x,t) is used to represent the temperature at time *t* at the point *x* of the thin rod.

Wave Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} = \frac{\partial^2 u(x,y,z,t)}{\partial t^2}$$

The function *u(x,y,z,t)* is used to represent the displacement at time t of a particle whose position at rest is *(x,y,z)* .

Used to model movement of 3D elastic body

Laplace Equation

$$\frac{\partial^2 u(x,y,z,t)}{\partial x^2}+\frac{\partial^2 u(x,y,z,t)}{\partial y^2}+\frac{\partial^2 u(x,y,z,t)}{\partial z^2}=0$$

Used to describe the steady state distribution of heat in a body.

Also used to describe the steady state distribution of electrical charge in a body.

Biharmonic Equation

$$\frac{\partial^4 u(x,y,t)}{\partial x^4} + 2\frac{\partial^4 u(x,y,t)}{\partial x^2\ \partial y^2} + \frac{\partial^4 u(x,y,t)}{\partial y^4} = 0$$

Used in the study of elastic stress.

Boundary conditions for PDE

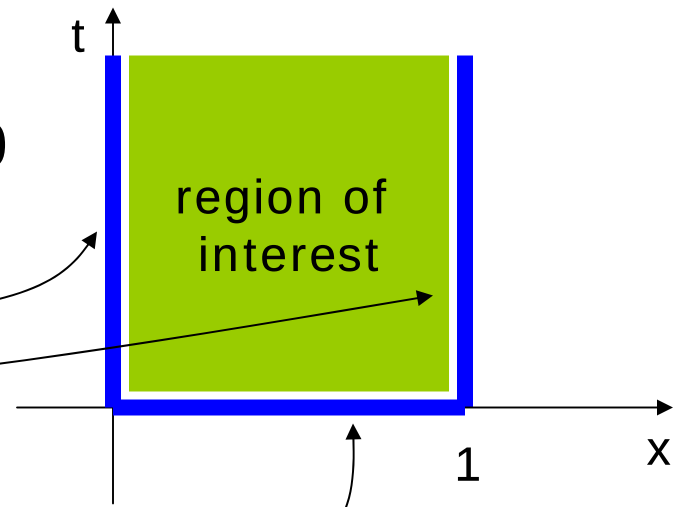
- To uniquely specify a solution to the PDE, a set of boundary conditions are needed.
- Both regular and irregular boundaries are possible

Heat Equation $$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$



The solution Methods for PDE

- Analytic solutions are possible for simple and special (idealized) cases only.
- To make use of the nature of the equations, different methods are used to solve different classes of PDE.
- The methods discussed here are based on finite difference technique

Elliptic Equations

- Elliptic Equations
- Laplace Equation
- Solution

Elliptic Equations

A second order linear PDE (2 - indepent variables x , y)

$$A\ u_{xx} + B\ u_{xy} + C\ u_{yy} + D = 0,$$

where D is a function of x, y, u_x, u_y

is Elliptic if

$$B^2 - 4AC < 0$$

Laplace Equation

Laplace equation appears is several engineering problems such as

- Studying the steady state distribution of heat in a body
- Studying the steady state distribution of electrical in a body

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = f(x,y)$$

T : steady state temperature at point (x, y)

f(x, y) : heat source (or heat sink)

Laplace Equation

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = f(x,y)$$

$$A = 1, B = 0, C = 1$$

$$B^2 - 4AC = -4 \quad < 0 \quad Elliptic$$

- Temperature is function of the position (x and y)
- When no heat source is available → $f(x,y) = 0$

Solution Technique

- A grid is used to divide region of interest
- Since the PDE is satisfied at each point in the area, it must be satisfied at each point of the grid.
- A finite difference approximation is obtained at each grid point.

$$\frac{\partial^2 \, T(x,y)}{\partial x^2} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}, \quad \frac{\partial^2 \, T(x,y)}{\partial y^2} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

Solution Technique

$$\frac{\partial^2 T(x,y)}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2},$$

$$\frac{\partial^2 T(x,y)}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

$$\Rightarrow \frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0$$

is approximated by

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

Solution Technique

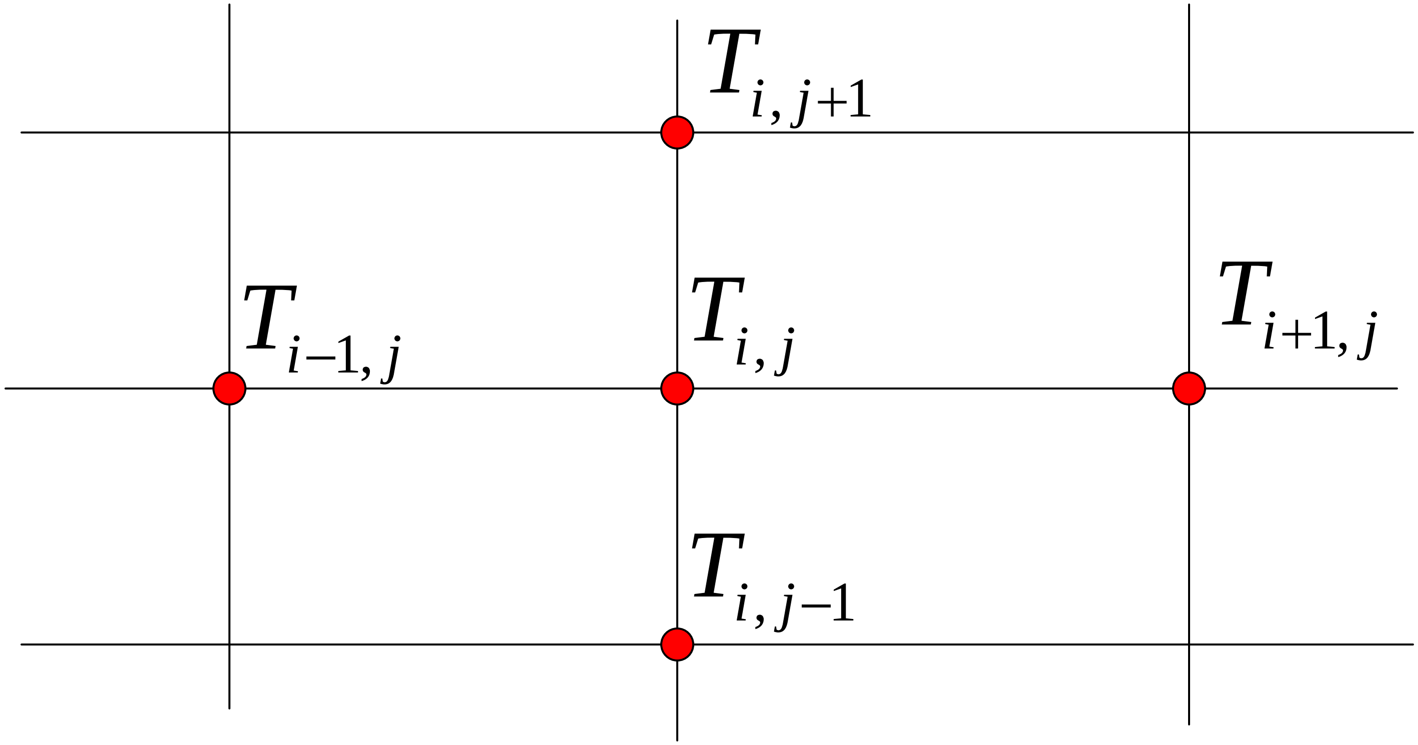
$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

$$(Laplacian\ Difference Equation)$$

$$assume\ \Delta x = \Delta y = h$$

$$\Rightarrow T_{i+1,j} - 4T_{i,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} = 0$$

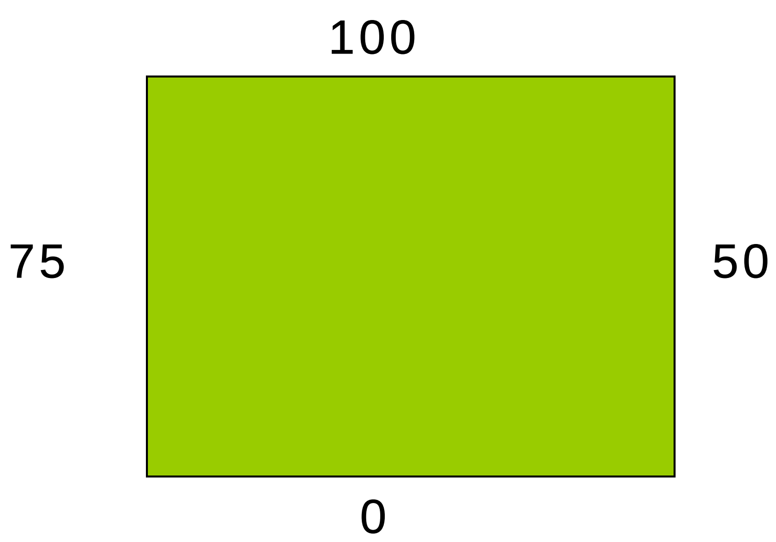
Solution Technique



$$T_{i+1,j} - 4T_{i,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} = 0$$

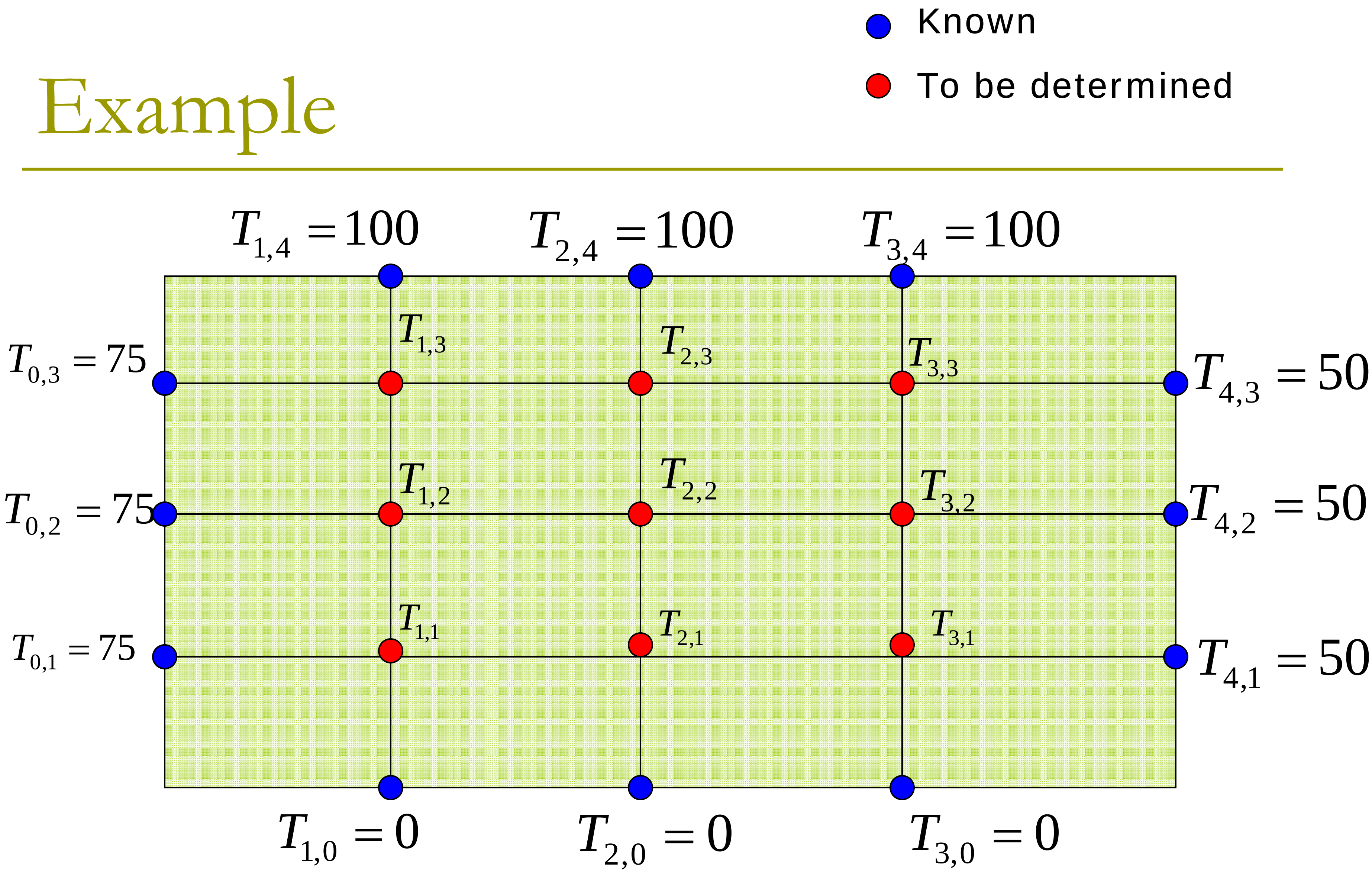
Example

It is required to determine the steady state temperature at all points of a heated sheet of metal. The edges of the sheet are kept at constant temperature 100,50, 0 and 75 degrees.



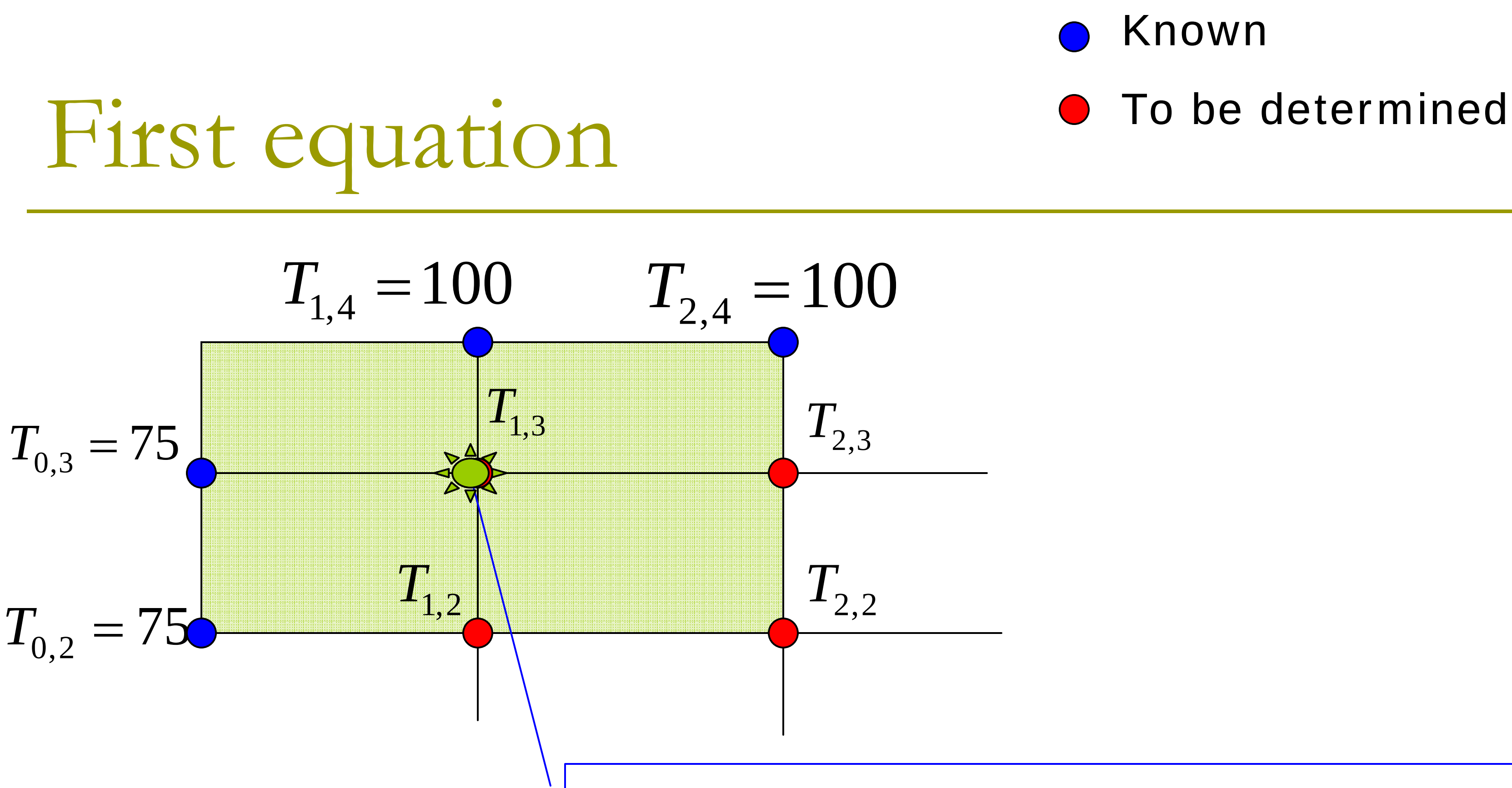
The sheet is divided by 5X5 grids

Example



Known
To be determined
$T_{1,4} = 100$
$T_{2,4} = 100$
$T_{3,4} = 100$
$T_{0,3} = 75$
$T_{0,2} = 75$
$T_{0,1} = 75$
$T_{1,3}$
$T_{2,3}$
$T_{3,3}$
$T_{1,2}$
$T_{2,2}$
$T_{3,2}$
$T_{1,1}$
$T_{2,1}$
$T_{3,1}$
$T_{4,3} = 50$
$T_{4,2} = 50$
$T_{4,1} = 50$
$T_{1,0} = 0$
$T_{2,0} = 0$
$T_{3,0} = 0$

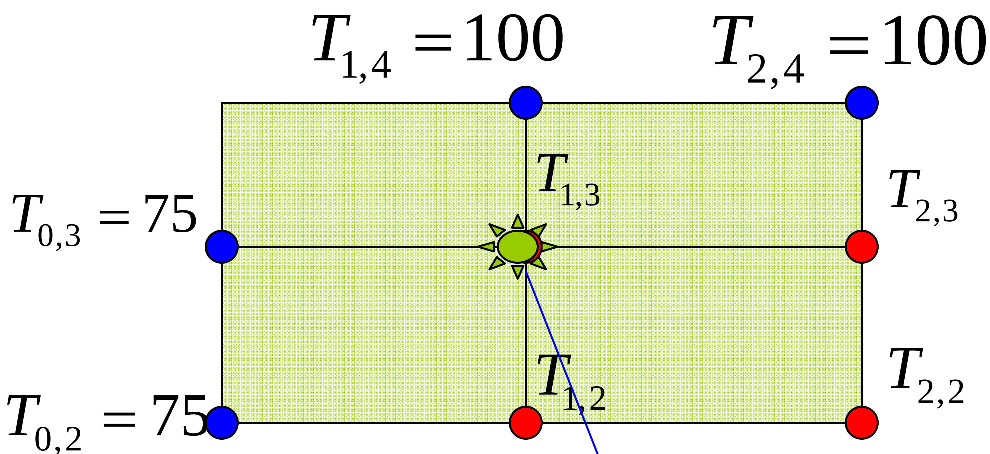
First equation



$$T_{0,3} + T_{1,4} + T_{1,2} + T_{2,3} + -4T_{1,3} = 0$$

$$75 + 100 + T_{1,2} + T_{2,3} + -4T_{1,3} = 0$$

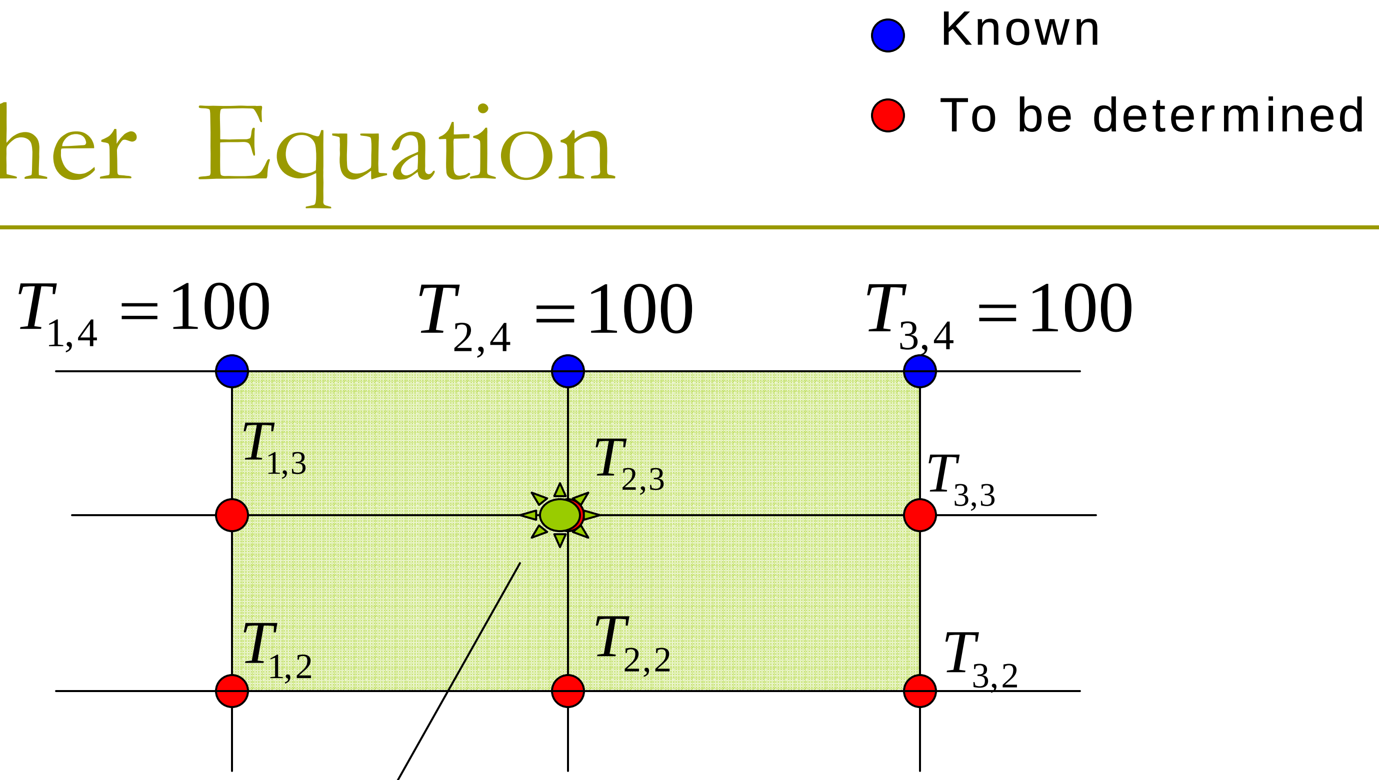
Example



$$T_{0,3} + T_{1,4} + T_{1,2} + T_{2,3} + -4T_{1,3} = 0$$

$$75 + 100 + T_{1,2} + T_{2,3} + -4T_{1,3} = 0$$

Another Equation



$$T_{1,3} + T_{2,4} + T_{3,3} + T_{2,2} + -4T_{2,3} = 0$$

$$T_{1,3} + 100 + T_{3,3} + T_{2,2} + -4T_{2,3} = 0$$

Solution

The rest of the equations

$$\begin{pmatrix} 4 & -1 & 0 & -1 & & & & & \\ -1 & 4 & -1 & 0 & -1 & & & & \\ 0 & -1 & 4 & 0 & 0 & -1 & & & \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & & \\ & -1 & 0 & -1 & 4 & -1 & 0 & -1 & \\ & & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ & & & -1 & 0 & 0 & 4 & -1 & 0 \\ & & & & -1 & 0 & -1 & 4 & -1 \\ & & & & & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{pmatrix} = \begin{pmatrix} 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{pmatrix}$$

Convergence and stability of solution

- Convergence

  The solutions converge means that the solution obtained using finite difference method approaches the true solution as the steps $\Delta x \ and \ \Delta t$ approaches zero.

- Stability:

  An algorithm is stable if the errors at each stage of the computation are not magnified as the computation progresses.

The Liebmann Method/

- Most numerical solutions of Laplace equation involve systems that are very large.
- For larger size grids, a significant number of terms will b e zero.
- For such sparse systems, most commonly employed approach is *Gauss-Seidel*, which when applied to PDEs is also referred as *Liebmann's method*.

Iterative method

- Solve the finite difference equation iteratively for j=1 to n and i=1 to m; Unknown values assume zeros!

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

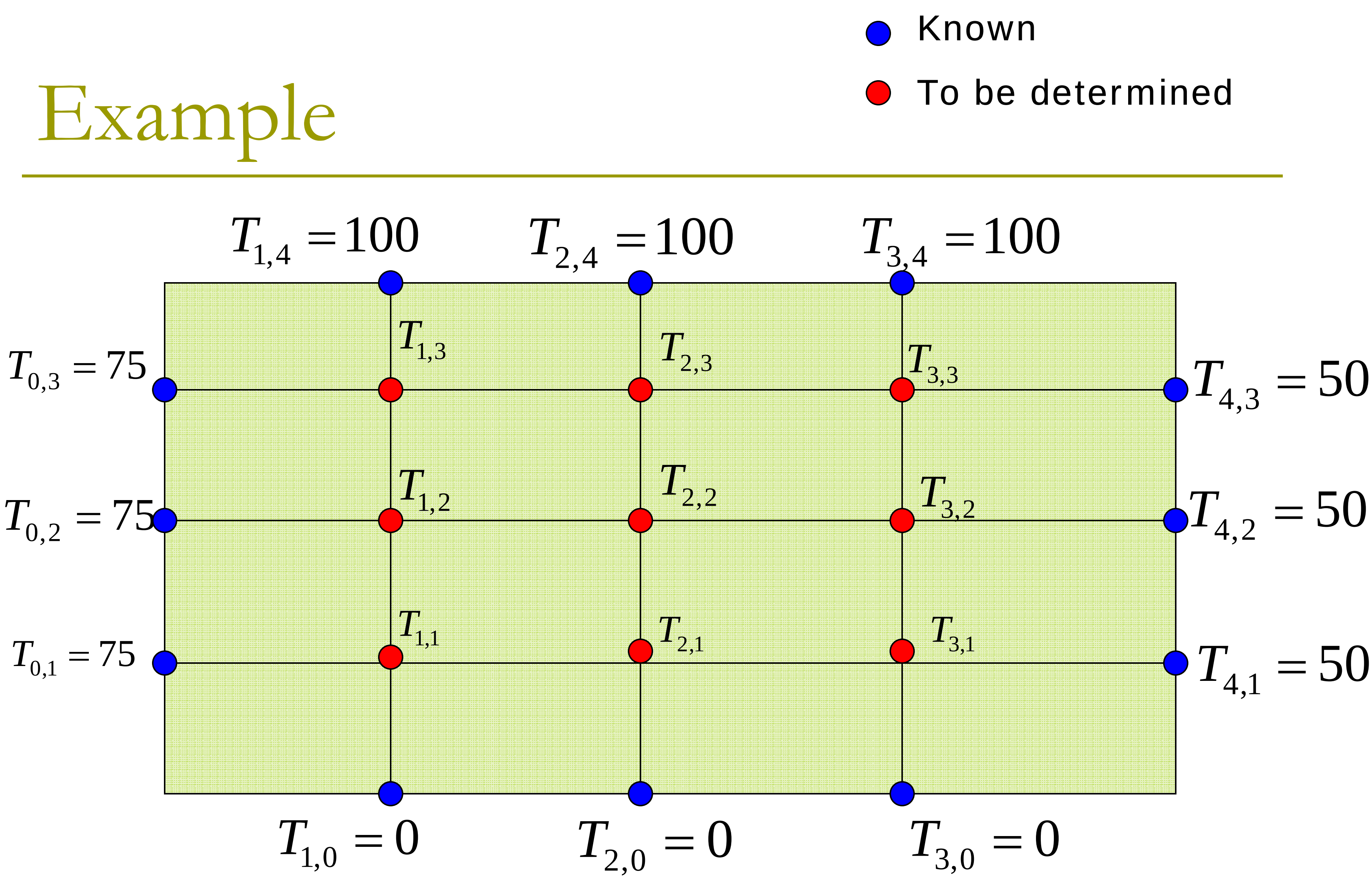
- To accelerate the rate of convergence, sometimes overrelaxation is used

$$T_{i,j}^{new} = \lambda T_{i,j}^{new} + (1-\lambda) T_{i,j}^{old}$$

- Percent relative error is

$$(|\varepsilon_a)_{i,j}| = \left| \frac{T_{i,j}^{new} - T_{i,j}^{old}}{T_{i,j}^{new}} \right| 100\ \%$$

Example



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$T_{4,3} = 50$
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$T_{2,2}$
$T_{3,2}$
$T_{0,2} = 75$
$T_{4,2} = 50$
$T_{1,1}$
$T_{2,1}$
$T_{3,1}$
$T_{0,1} = 75$
$T_{4,1} = 50$
$T_{1,0} = 0$
$T_{2,0} = 0$
$T_{3,0} = 0$

Parabolic Equations Chapter 30

- Parabolic Equations
- Heat Conduction Equation
- Explicit Method
- Implicit Method
- Cranks Nicolson Method

Parabolic Equations

A second order linear PDE (2 - indepent variables x , y)

$$A\ u_{xx} + B\ u_{xy} + C\ u_{yy} + D = 0,$$

where D is a function of x, y, u_x, u_y

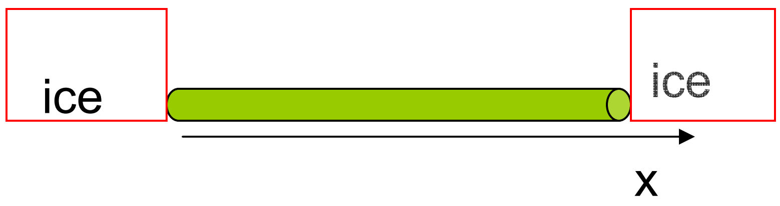
is parabolic if $\boxed{B^2 - 4AC = 0}$

Parabolic Problems

Heat Equation $$\frac{\partial^2 T(x,t)}{\partial x^2} - \frac{\partial T(x,t)}{\partial t} = 0$$

$$T(0,t) = T(1,t) = 0$$

$$T(x,o) = \sin(\pi x)$$



* Parabolic problem $(B^2 - 4AC = 0)$
* Boundary conditions are needed to uniquely specify a solution

Parabolic Equations

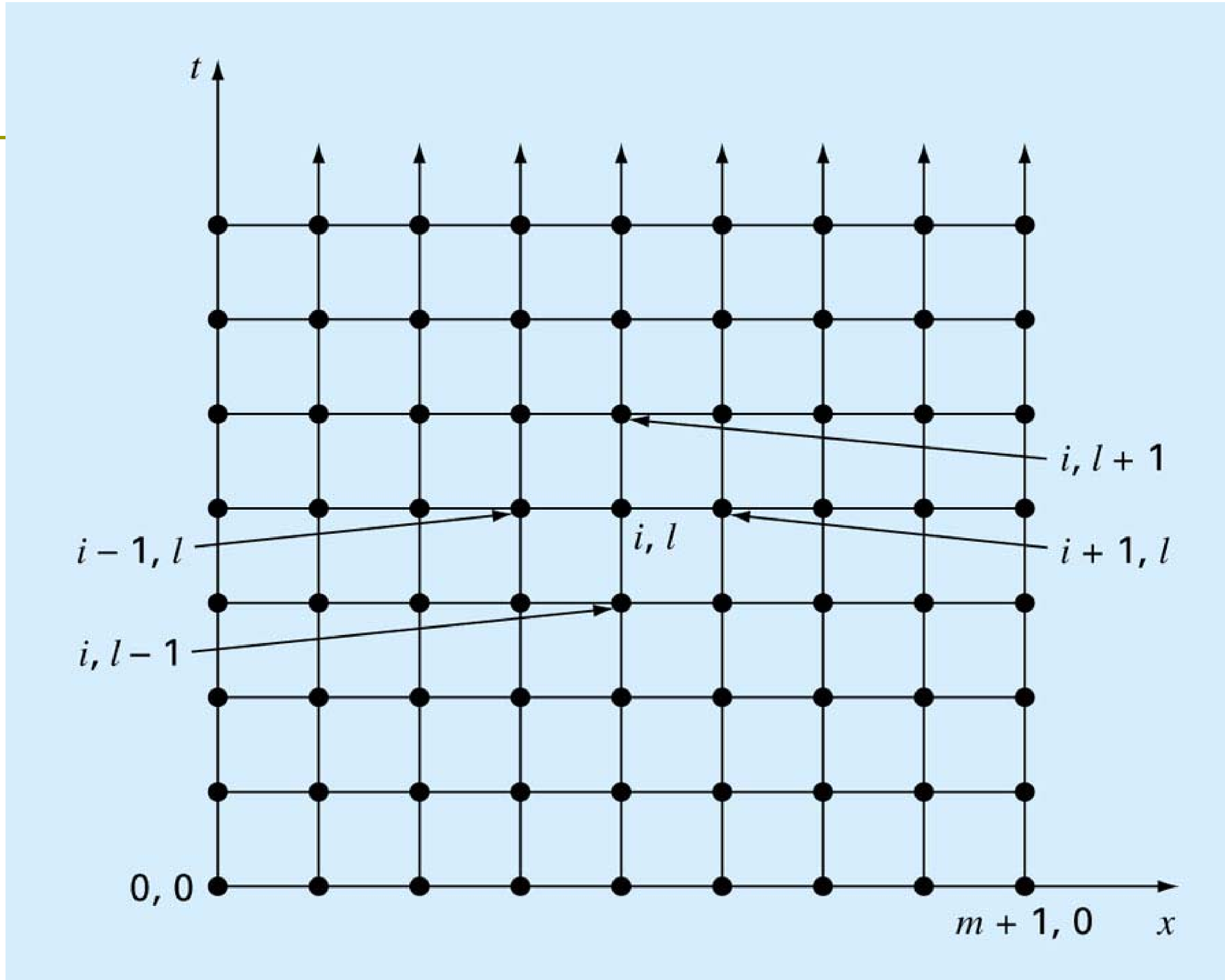
- Parabolic equations are employed to characterize time-variable (*unsteady-state*) problems.
- Conservation of energy can be used to develop an *unsteady-state* energy balance for the differential element in a long, thin insulated rod.

- Energy balance together with Fourier's law of heat conduction yields *heat-conduction equation*:

$$k\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t}$$

- Just as elliptic PDEs, parabolic equations can be solved by substituting finite divided differences for the partial derivatives.
- In contrast to elliptic PDEs, we must now consider changes in time as well as in space.
- Parabolic PDEs are temporally open-ended and involve new issues such as stability.

Figure 30.2



Solution of the Heat Equation

Two solutions to the Parabolic Equation (Heat Equation) will be presented

1. Explicit Method:

Simple, Stability Problems

2. Crank-Nicolson Method:

involves solution of Tridiagonal system of equations, stable.

Finite Difference Methods

New Notation

Central Difference Formulas :

$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T_{i+1}^{l} - T_{i-1}^{l}}{2\,\Delta x}$$

$$\frac{\partial^2\ T(x,t)}{\partial x^2} \approx \frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{(\Delta x)^2}$$

$$\frac{\partial^2\ T(x,t)}{\partial t^2} \approx \frac{T_i^{l+1} - 2T_i^{l} + T_i^{l-1}}{(\Delta t)^2}$$

Superscript for t-axis
And
Subscript for x-axis
$T_i^{l-1} = T(x, t-\Delta t)$

Forward Difference Formula :

$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T_{i+1}^{l} - T_i^{l}}{\Delta x}, \quad \frac{\partial T(x,t)}{\partial t} \approx \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

Explicit Methods

- The heat conduction equation requires approximations for the second derivative in space and the first derivative in time:

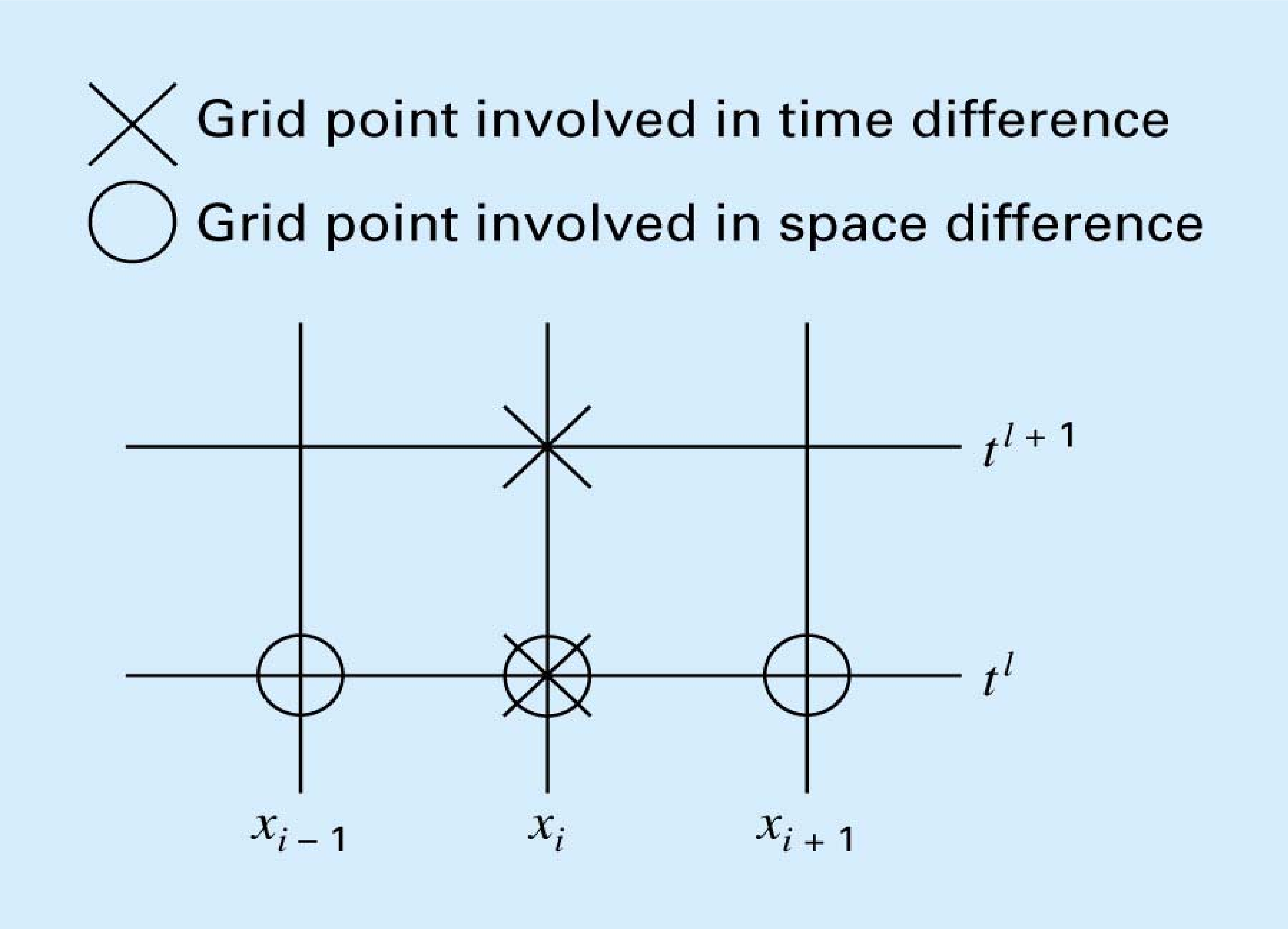
$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2}$$

$$\frac{\partial T}{\partial t} = \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

$$k\,\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{(\Delta x)^2} = \frac{T_i^{l+1} - T_i^{l}}{\Delta t}$$

$$T_i^{l+1} = T_i^{l} + \lambda\left(T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}\right) \qquad \lambda = \frac{k\Delta t}{(\Delta x)^2}$$

Figure 30.3


Grid point involved in time difference
Grid point involved in space difference
t^{l+1}
t^{l}
x_{i-1}
x_i
x_{i+1}

- This equation can be written for all interior nodes on the rod.
- It provides an explicit means to compute values at each node for a future time based on the present values at the node and its neighbors.

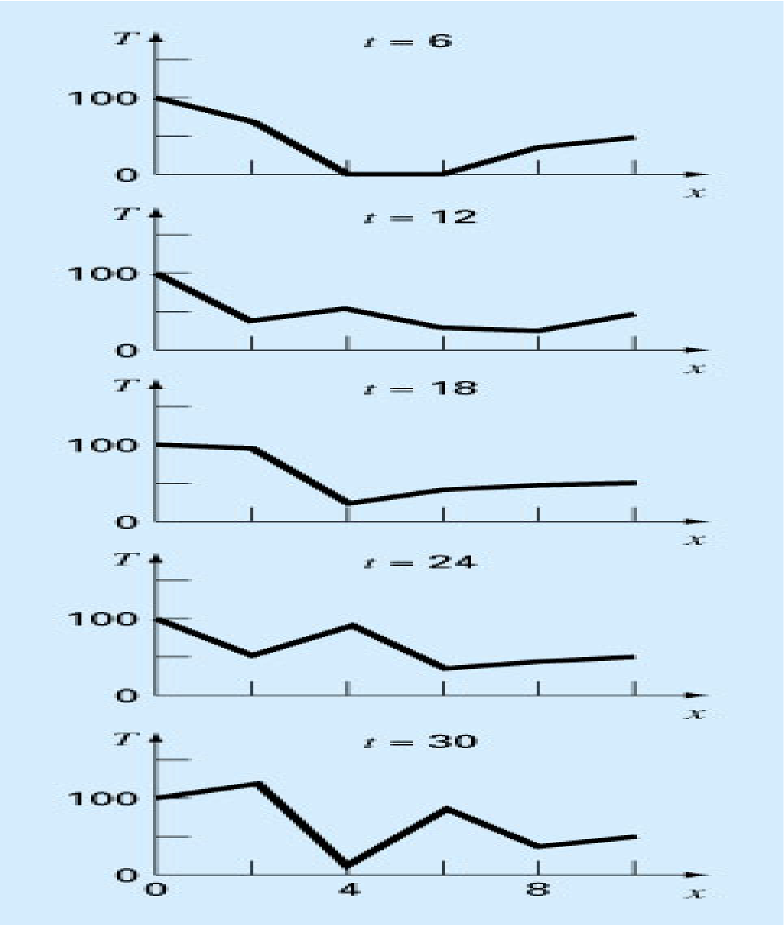
Convergence and Stability

- Convergence means that as Dx and Dt approach zero, the results of the finite difference method approach the true solution.
- Stability means that errors at any stage of the computation are not amplified but are attenuated as the computation progresses.
- The explicit method is both convergent and stable if

$$\lambda \le 1/2 \quad or \qquad \Delta t \le \frac{1}{2}\frac{\Delta x^2}{k}$$

Convergence and Stability

- Slow
- May oscillate



Derivative Boundary Conditions

- As was the case for elliptic PDEs, derivative boundary conditions can be readily incorporated into parabolic equations.

$$T_0^{l+1} = T_0^l + \lambda\left(T_1^l - 2T_0^l + T_{-1}^l\right)$$

- Thus an imaginary point is introduced at *i*= -*1*, providing a vehicle for incorporating the derivative boundary condition into the analysis.

A simple Implicit Method

- Implicit methods overcome difficulties associated with explicit methods at the expense of somewhat more complicated algorithms.
- In implicit methods, the spatial derivative is approximated at an advanced time interval *l+1*:

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}$$

rate.

$$k\frac{T_{i+1}^{l+1}-2T_i^{l+1}+T_{i-1}^{l+1}}{(\Delta x)^2}=\frac{T_i^{l+1}-T_i^l}{\Delta t}$$

$$-\lambda T_{i-1}^{l+1}+(1+2\lambda)T_i^{l+1}-\lambda T_{i+1}^{l+1}=T_i^l$$

This eqn. applies to all but the first and the last interior nodes, which must be modified to reflect the boundary conditions:

$$T_0^{l+1}=f_0\left(t^{l+1}\right)$$

$$(1+2\lambda)T_1^{l+1}-\lambda T_2^{l+1}=T_1^l+f_0\left(t^{l+1}\right)$$

$$i=m$$

$$(1+2\lambda)T_m^{l+1}-\lambda T_{m-1}^{l+1}=T_m^l+f_{m+1}\left(t^{l+1}\right)$$

Resulting *m* unknowns and *m* linear algebraic equations

Figures 30.6

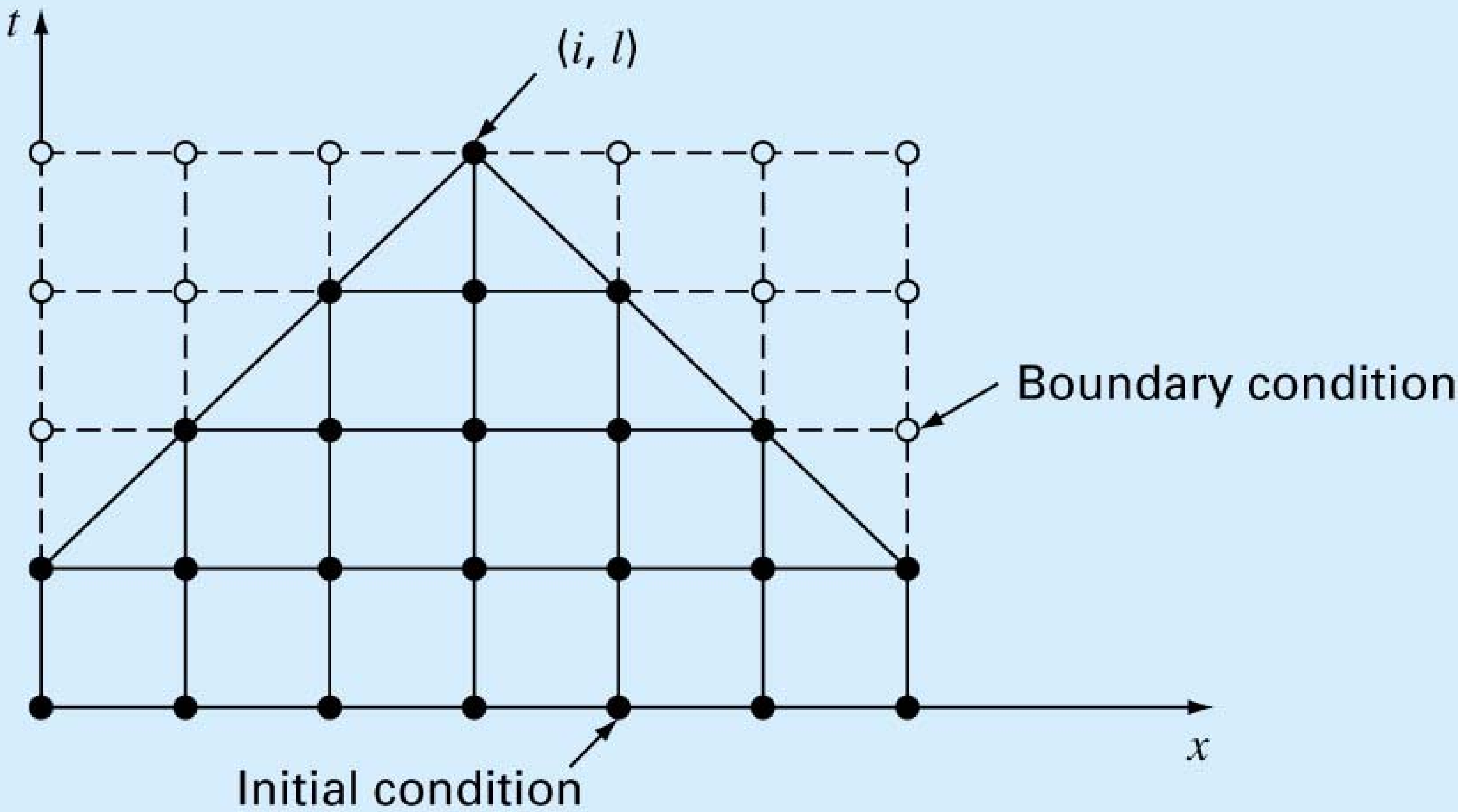
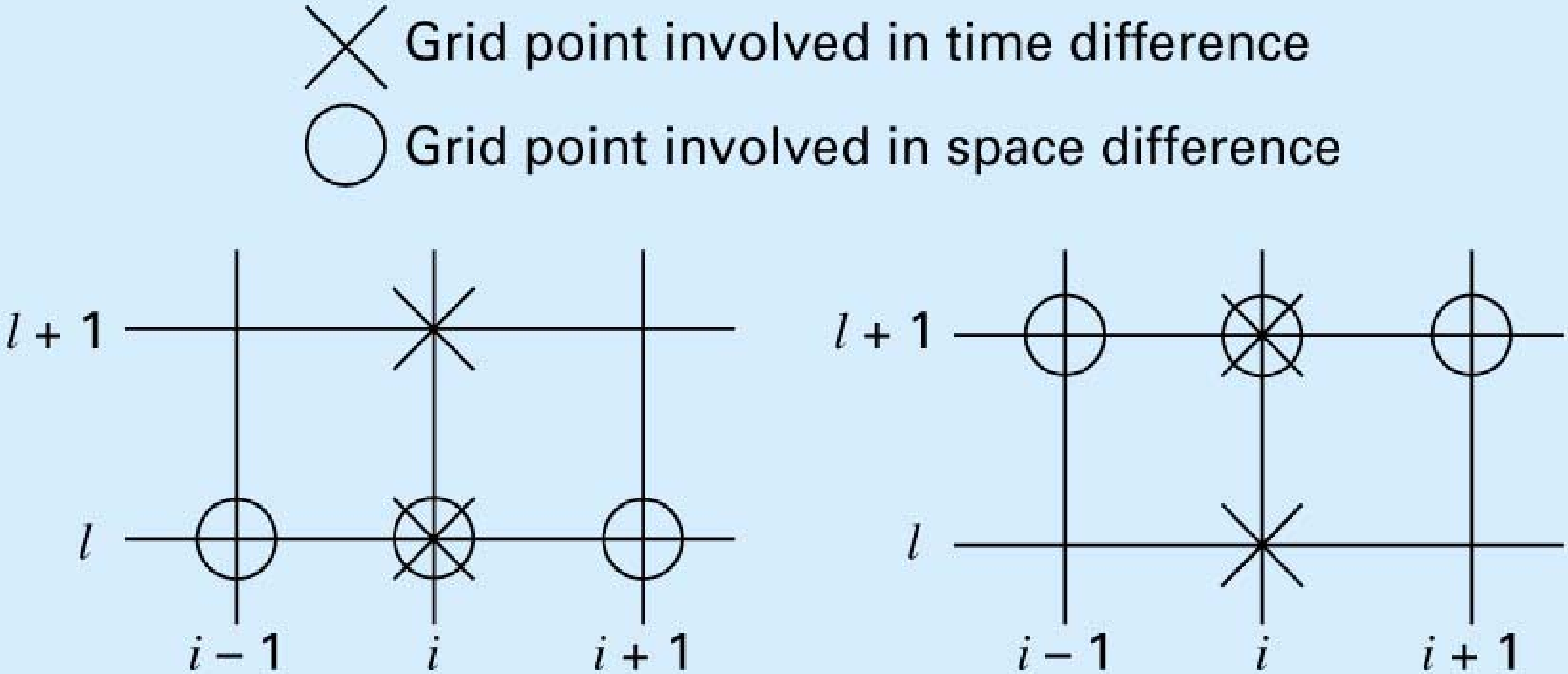


Figure 30.7



(a) Explicit (b) Implicit

The Crank-Nicolson Method

- Provides an alternative implicit scheme that is second order accurate both in space and time.
- To provide this accuracy, difference approximations are developed at the midpoint of the time increment:

$$\frac{\partial T}{\partial t} \cong \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

$$\frac{\partial^2 T}{\partial x^2} \cong \frac{1}{2}\left[\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}\right]$$

Crank-Nicolson Method

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{u(x,t) - u(x,t-k)}{k}$$

$$\frac{1}{h^2}\big(u(x+h,t) - 2u(x,t) + u(x-h,t)\big) - \frac{1}{k}\big(u(x,t) + u(x,t-k)\big) = 0$$

$$Define \quad s = \frac{h^2}{k}, \quad r = 2 + s$$

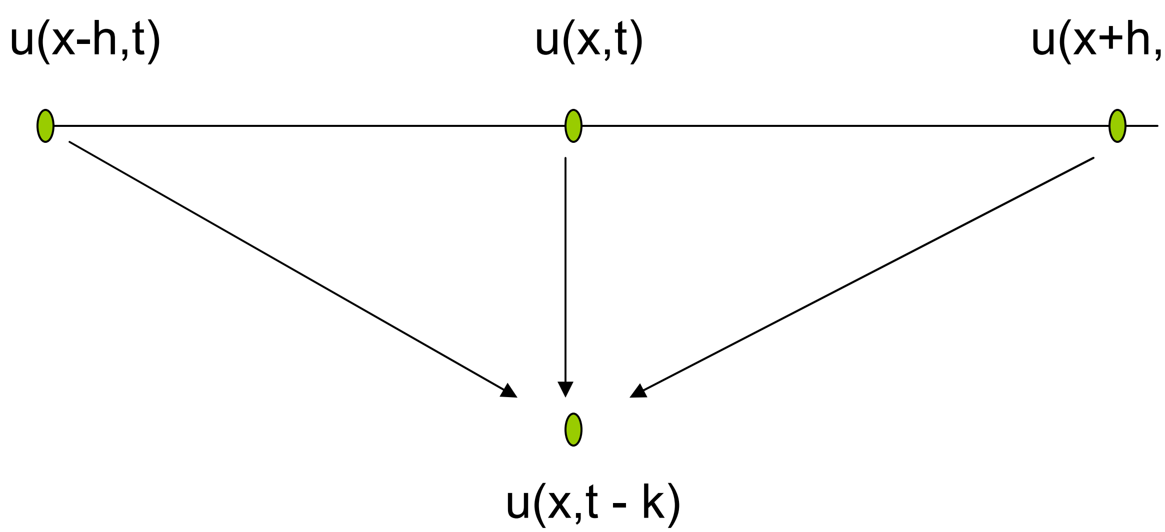
$$s\ u(x,t-k) = -\ u(x-h,t) + r\ u(x,t) - u(x+h,t)$$

Explicit Method

How do we compute

$$s u(x, t-k) = u(x+h, t) + r\ u(x, t) + \ u(x-h, t)$$

means



Crank-Nicolson Method

The equation

$$s\ u(x,t-k) = -\ u(x-h,t) + r\ u(x,t) - u(x+h,t)$$

can be expressed as a Tridiagonal system of equations

$$\begin{bmatrix} r & -1 & & \\ -1 & r & -1 & \\ & -1 & r & -1 \\ & & -1 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Crank-Nicolson Method

The method involves solving a Tridiagonal system of linear equations

The method is stable (No magnification of error)

$\rightarrow$ We can use larger h, k (compared to the Explicit Method)

Outlines

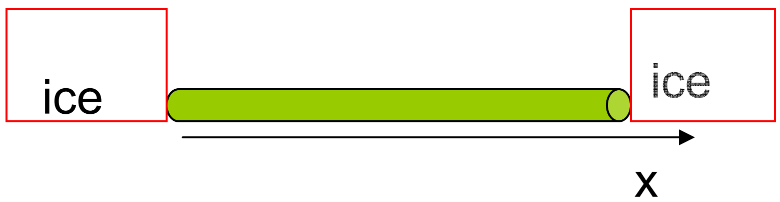
- Examples
  - Explicit method to solve Parabolic PDE
  - Cranks-Nicholson Method

Heat Equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,o) = \sin(\pi x)$$



* Parabolic problem $(B^2 - 4AC = 0)$

* Auxiliary conditions are needed to uniquely specify a solution

Example 1

Solve the PDE

$$\frac{\partial^2 u(x,t)}{\partial^2 x} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$

$Use \quad h = 0.25, \quad k = 0.25$ to find u(t, x) for $x \in [0,1], t \in [0,1]$

$$\sigma = \frac{k}{h^2} = 4$$

Example 1 (cont.)

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

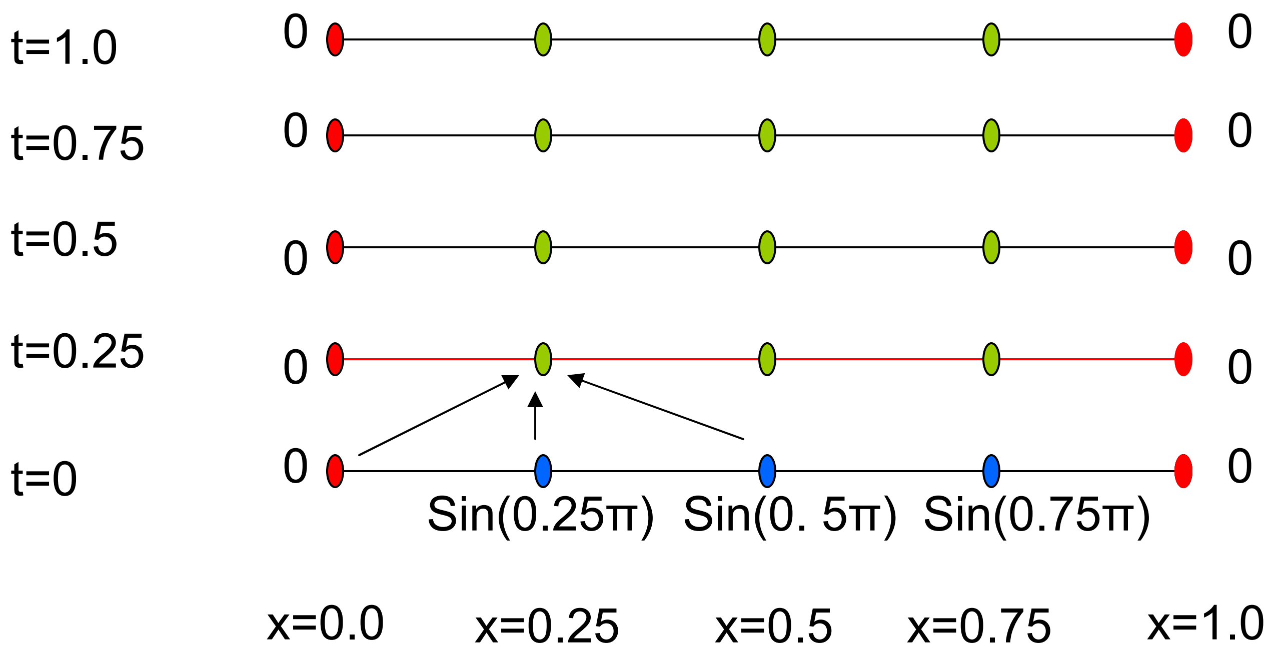
$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} - \frac{u(x,t+k) - u(x,t)}{k} = 0$$

$$16\big(u(x+h,t) - 2u(x,t) + u(x-h,t)\big) - 4\big(u(x,t+k) + u(x,t)\big) = 0$$

$$u(x,t+k) = 4\ u(x+h,t) - 7\ u(x,t) + 4\ u(x-h,t)$$

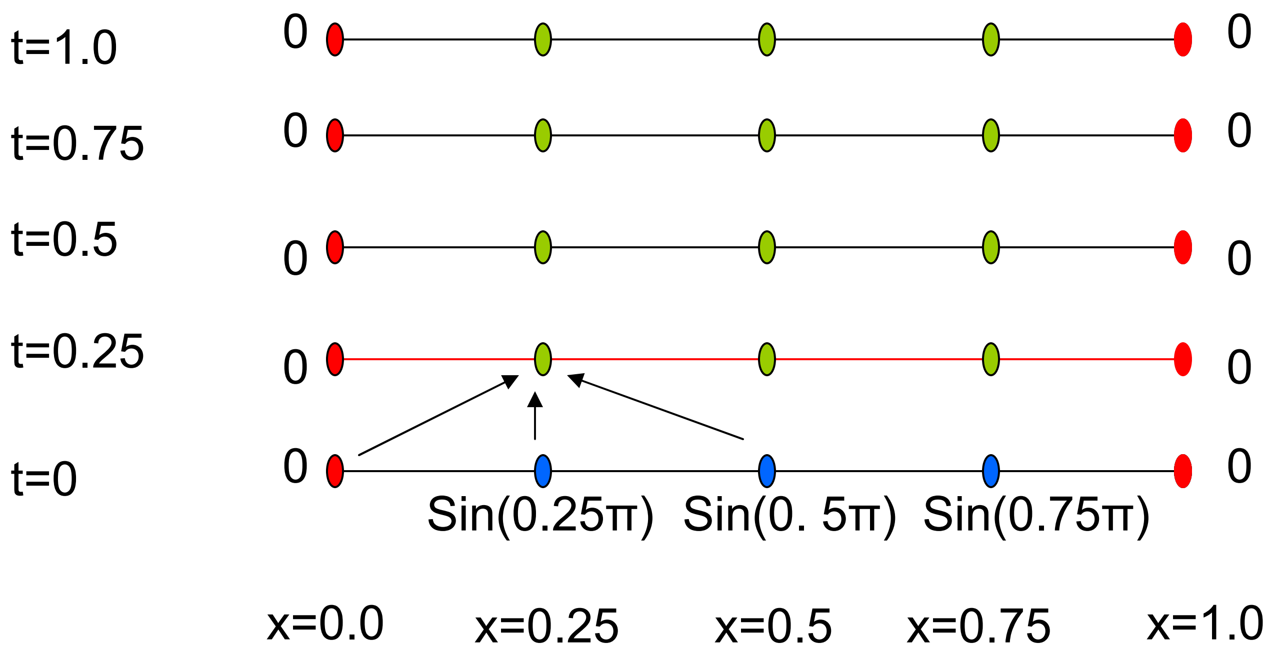
Example 1

$$u(x,t+k) = 4\ u(x+h,t) - 7\ u(x,t) + 4\ u(x-h,t)$$



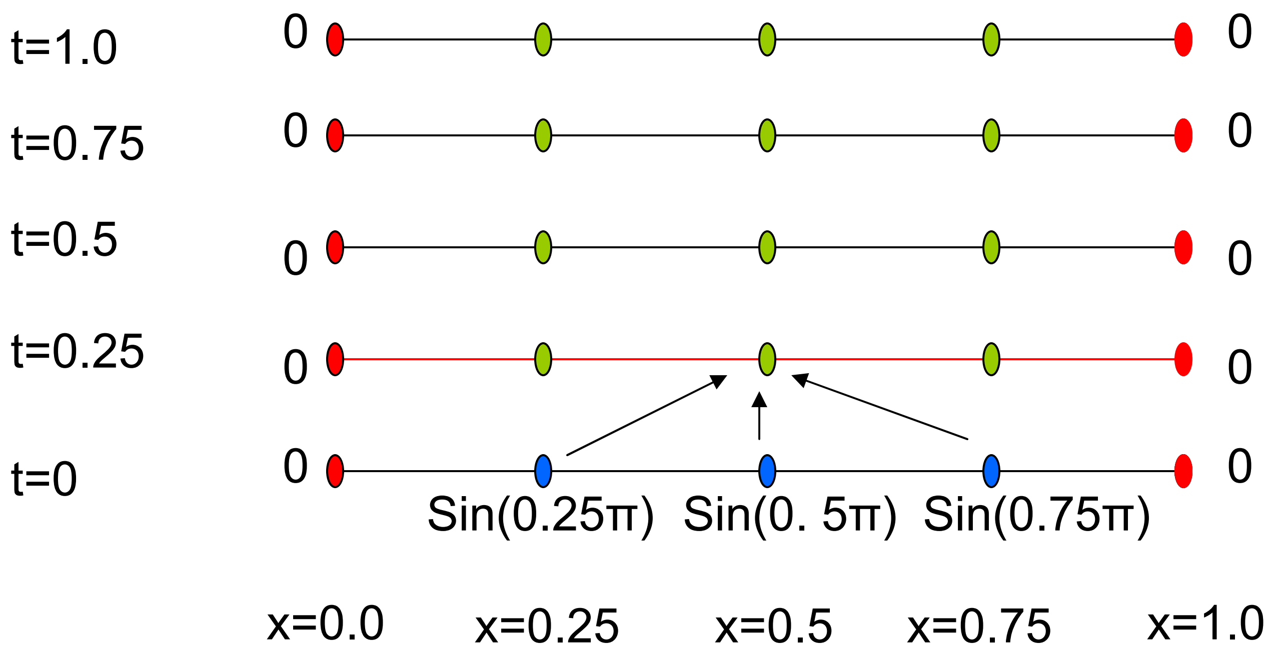
Example 1

$$u(0.25,0.25) = 4\ u(.5,0) - 7\ u(.25,0) + 4\ u(0,0)$$
$$= 4\sin(\pi/2) - 7\sin(\pi/4) + 0 = -0.9497$$



Example 1

$$u(0.5,0.25) = 4\,u(0.75,0) - 7\,u(0.5,0) + 4\,u(0.25,0)$$
$$= 4\sin(3\pi/4) - 7\sin(\pi/2) + 4\sin(\pi/2) = -0.1716$$



Remarks on Example 1

The obtained results are probably not accurate

because $1-2\sigma=-7$

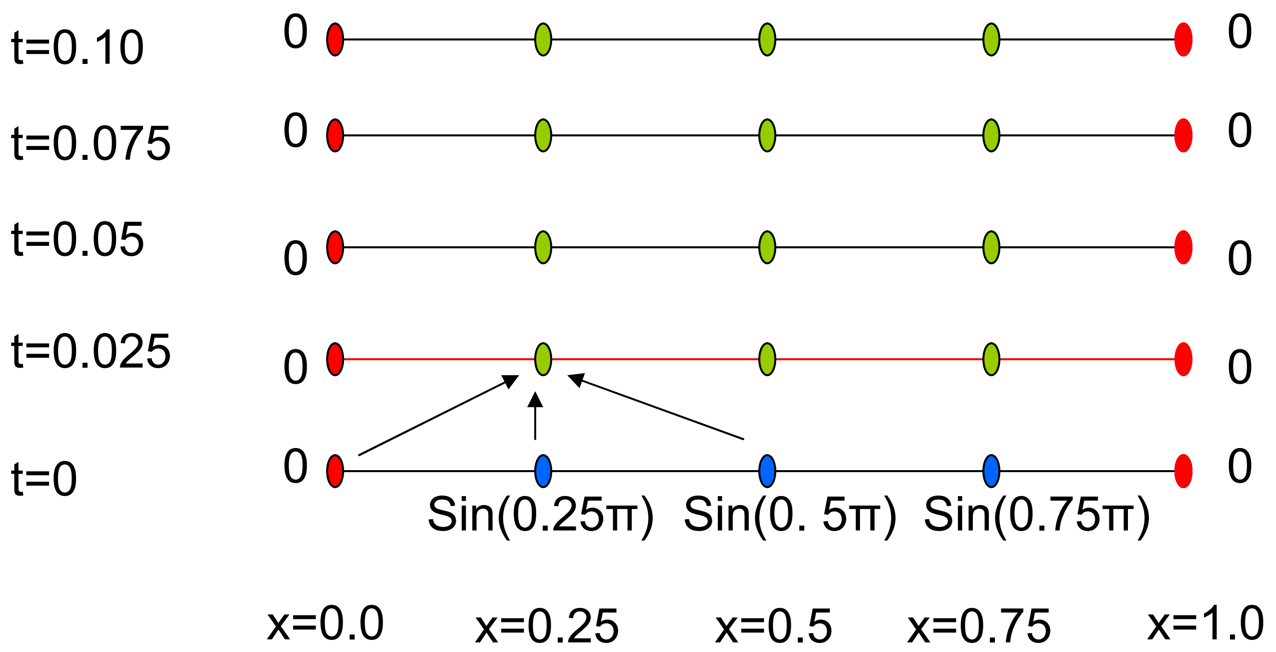
For accurate results $1-2\sigma>0$

One need to select $k<0.03125$

Let $k=0.025$

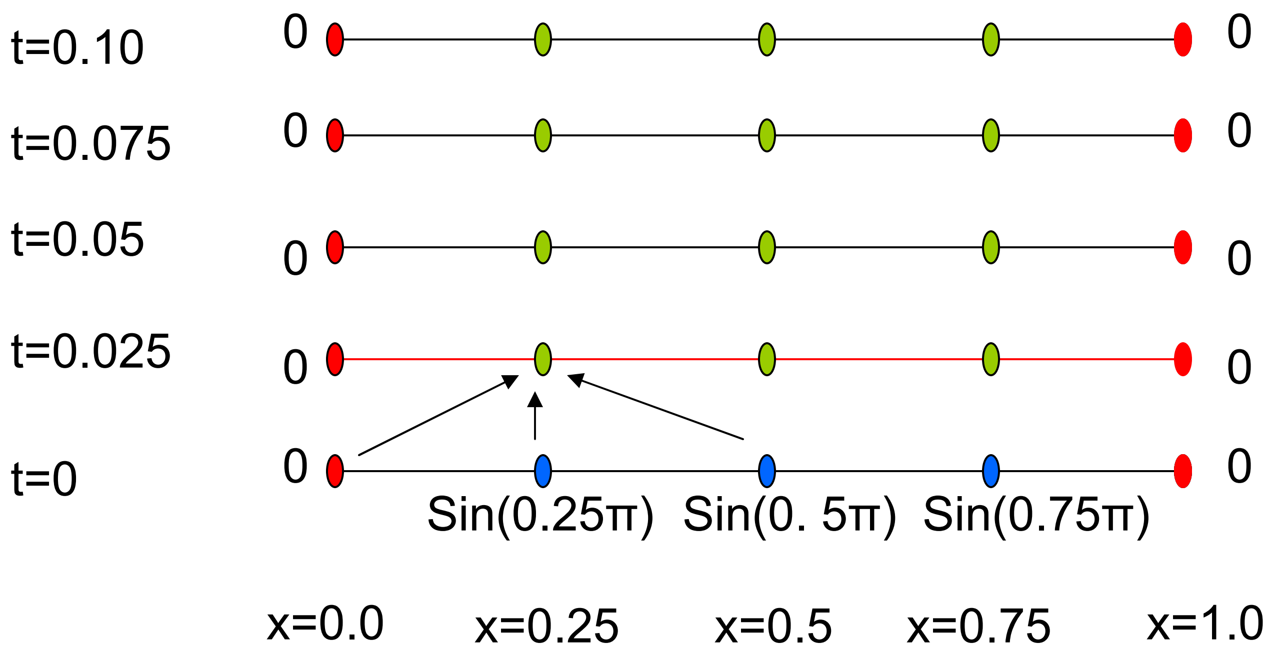
Example 1

$$u(x,t+k)=0.4\ u(x+h,t)+0.2\ u(x,t)+0.4\ u(x-h,t)$$



Example 1

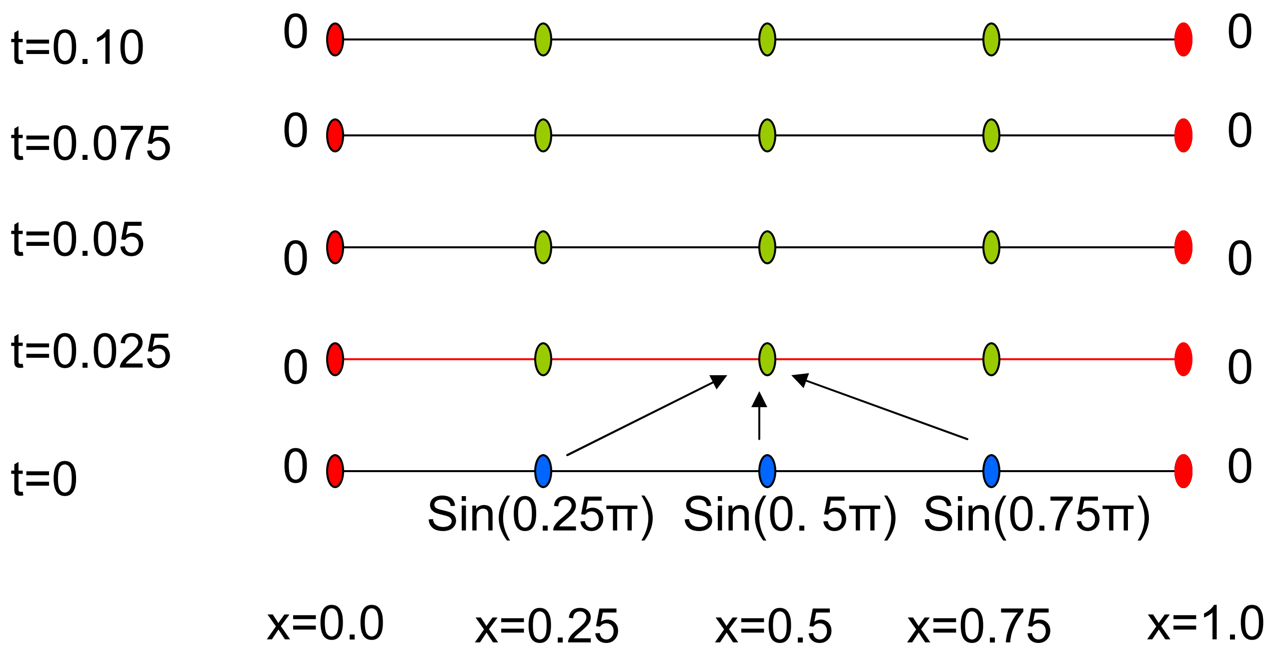
$$u(0.25,0.025) = 0.4\ u(0.5,0) + 0.2\ u(0.25,0) + 0.4\ u(0,0)$$
$$= 0.4\sin(\pi/2) + .2\sin(\pi/4) + 0 = 0.5414$$



Example 1

$$u(0.5,0.025) = 0.4\ u(0.75,0) + 0.2\ u(0.5,0) + 0.4\ u(0.25,0)$$

$$= 0.4\sin(3\pi/4) + .2\sin(\pi/2) + 0.4\sin(\pi/4) = 0.7657$$



Example 2

Solve the PDE

$$\frac{\partial^2 u(x,t)}{\partial^2 x} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(\pi\, x)$$

Solve using Crank - Nicolson method

$Use \quad h = 0.25, \quad k = 0.25$ to find u(t, x) for $x \in [0,1], t \in [0,1]$

Example 2

Crank-Nicolson Method

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{u(x,t) - u(x,t-k)}{k}$$

$$16\big(u(x+h,t) - 2u(x,t) + u(x-h,t)\big) - 4\big(u(x,t) - u(x,t-k)\big) = 0$$

$$Define \quad s = \frac{h^2}{k} = 0.25, \quad r = 2 + s = 2.5$$

$$0.25u(x,t-k) = -u(x-h,t) + 2.5\,u(x,t) - (x+h,t)$$

$$u(x,t-k) = -4u(x-h,t) + 10\,u(x,t) - 4(x+h,t)$$

Example 2

Crank-Nicolson Method

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} = \frac{u(x,t) - u(x,t-k)}{k}$$

$$16\big(u(x+h,t) - 2u(x,t) + u(x-h,t)\big) - 4\big(u(x,t) - u(x,t-k)\big) = 0$$

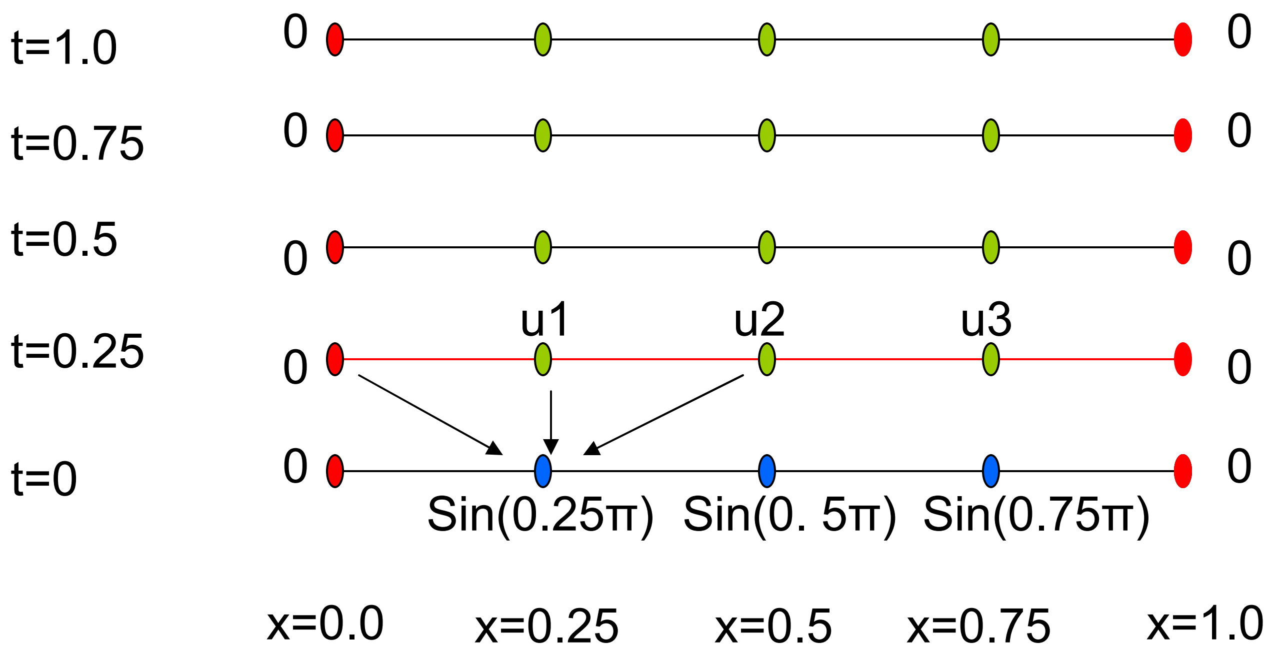
$$Define\ s = \frac{h^2}{k} = 0.25, \quad r = 2 + s = 2.25$$

$$0.25u(x,t-k) = -u(x-h,t) + 2.25\,u(x,t) - (x+h,t)$$

Example 2

$$0.25u(0.0,0.25) = -\,u(0,0.25) + 2.25\,u(0.25,0.25) - u(0.5,0.25)$$

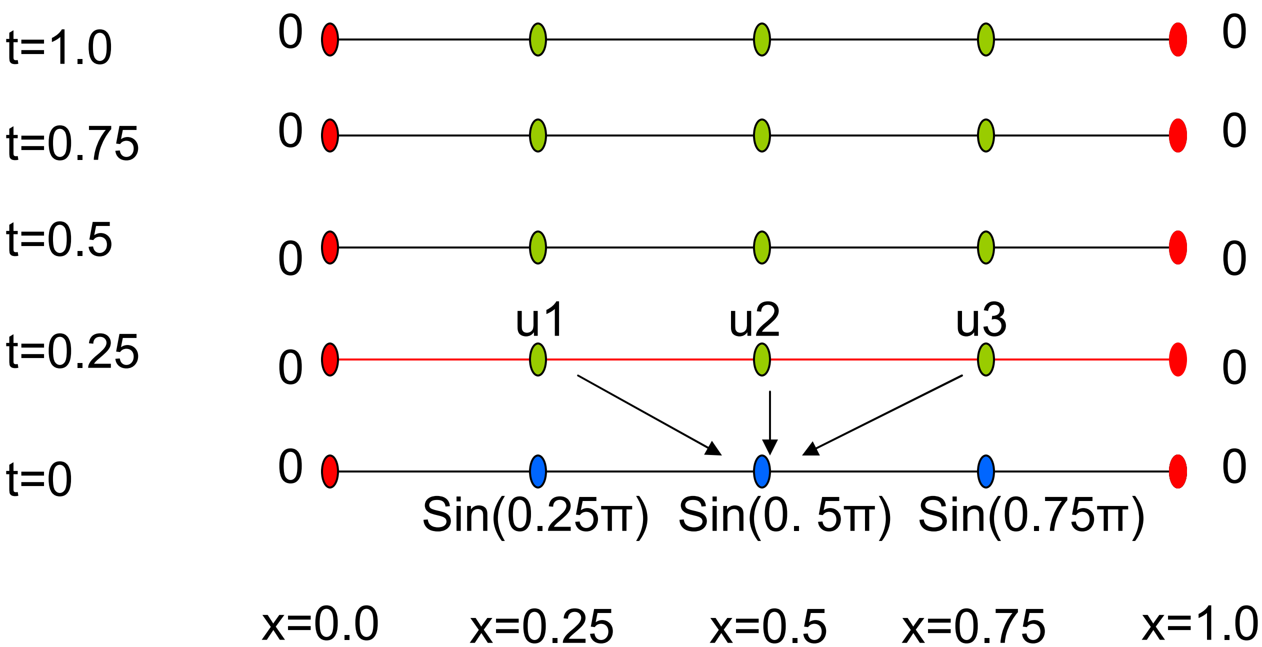
$$0.25\sin(0.25\pi) = 0 + 2.25u_1 - u_2$$



Example 2

$$0.25u(0,0.5) = -u(0.25,0.25) + 2.25\,u(.5,0.25) - u(0.75,0.25)$$

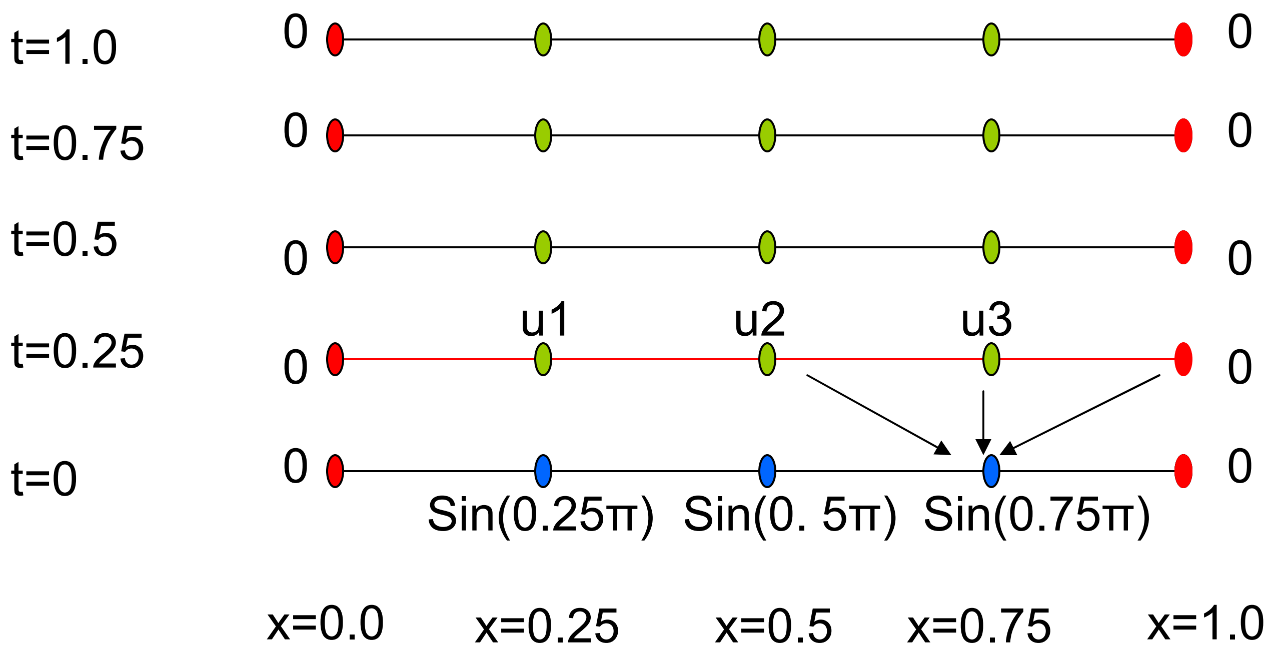
$$0.25\sin(0.5\pi) = -u_1 + 2.25u_2 - u_3$$



Example 2

$$0.25u(0.75,0) = -u(0.5,0.25) + 2.25\,u(.75,0.25) - u(1,0.25)$$

$$0.25\sin(0.75\pi) = -u_2 + 2.25u_3 - 0$$



Example 2
Crank-Nicolson Method

The solution of the PDE is converted to solution of the following tridiagonal system

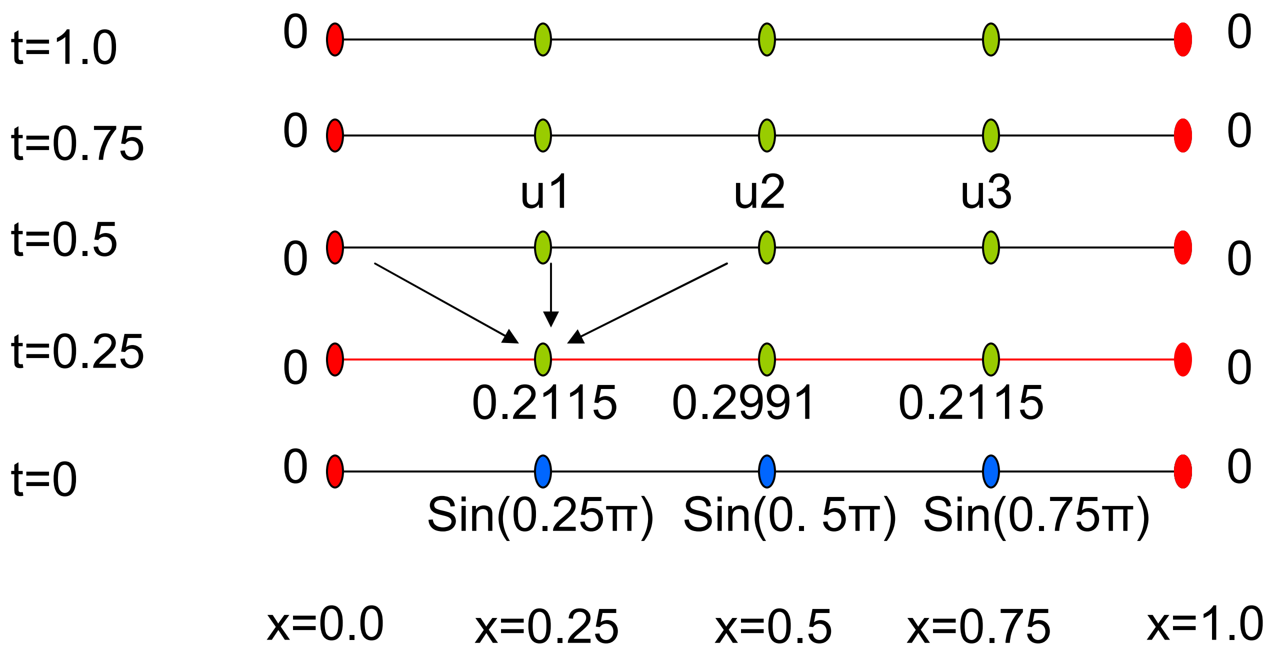
$$\begin{bmatrix} 2.25 & -1 & \\ -1 & 2.25 & -1 \\ & -1 & 2.25 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.25\sin(0.25\pi) \\ 0.25\sin(0.5\pi) \\ 0.25\sin(0.75\pi) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.21152 \\ 0.2912 \\ 0.21151 \end{bmatrix}$$

Example 2
Second Row

$$0.25u(0.25,0.25) = -\,u(0,0.5) + 2.25\,u(0.25,0.5) - u(0.5,0.5)$$

$$0.2115 = 0 + 2.25u_1 - u_2$$



Example 2

The process is continued until the values of u(x,t) on the desired grid are computed.

Remarks

The explicit metod:

- one need to select small k to ensure **stability**
- This requires a lot of computation.

Cranks Nicolson

- Requires solution of **Tridiagonal** system
- Stable (larger k can be used).