SE301:Numerical Methods
Topic 8
Solution of ODE

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Term 053

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(072)

Ordinary Differential Equations
Taylor Series Method

- Ordinary Differential Equations
- Taylor Series Method to solve ODE

Ordinary Differential Equations

Differential Equations involves one or more derivatives of unknown functions

$$\frac{dx(t)}{dt} - x(t) = e^t$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

A solution to a differential equations is a function that satisfies the equations.

Ordinary Differential Equations

$$x(t) = \cos(2t)$$

is a solution to the ODE Is it unique?

$$\frac{d^2 x(t)}{dt^2} + 4x(t) = 0$$

All functions of the form $x(t) = \cos(2t + c)$

(where c is a real constant) are solutions

Uniqueness of a solution

In order to uniquely specify a solution to an *n* th order differential equation we need *n* initial conditions.

$$\frac{d^2y(x)}{dt^2} + 4y(x) = 0$$

$$y(0) = a$$

$$\dot{y}(0) = b$$

Taylor Series Method

$$\text{Given}\ \ \frac{dy(x)}{dx} = f(y,x), \quad y(a) = y_a$$

Taylor Series expansion of *y(x)*

$$y(a+h) = y(a) + h\frac{dy}{dx} + h.o.t \quad \approx y(a) + h\ f(y(a),a)$$

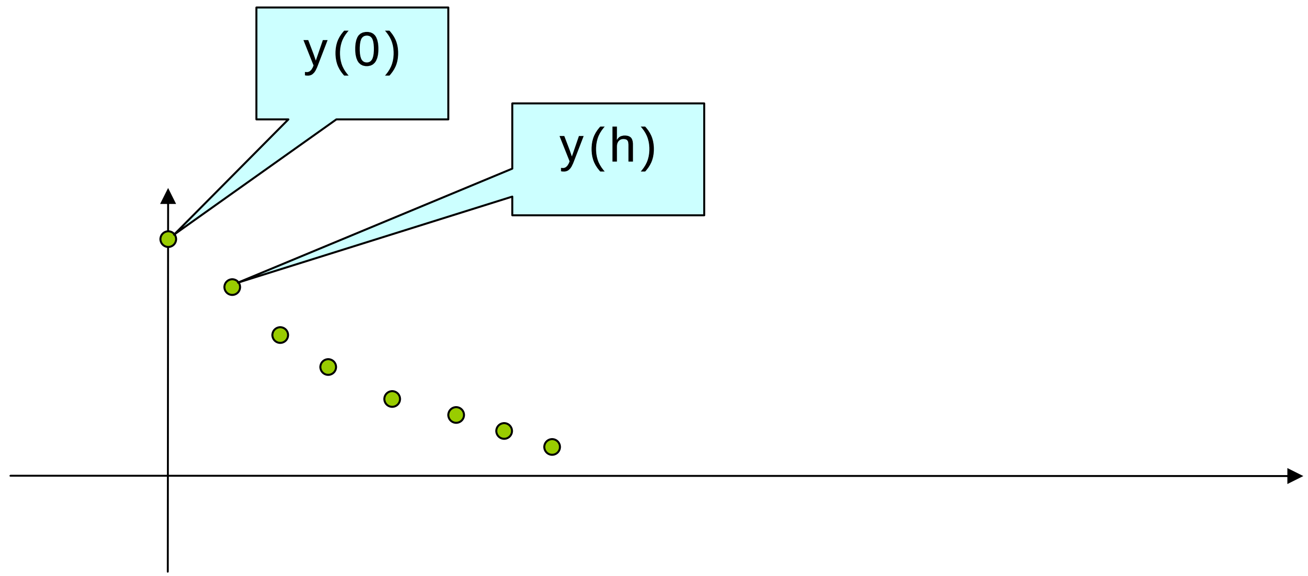
Similarly

$$y(a+2h) \approx y(a+h) + h\ f(y(a+h),a+h)$$

We use similar formulas to compute $y(a+3h),\ y(a+4h),....$

Example

$$Solve \quad \frac{dy}{dx} = f(y,x), \quad y(0) = y_a \quad use \quad h = 0.01$$



Example

$$Solve \quad \frac{dy}{dx} = 1 + x^2, \; y(1) = -4 \qquad use \quad h = 0.01$$

$$y(a+h) = y(a) + h\, f\,(y(a),\; a)$$

$$y(1) = -4$$

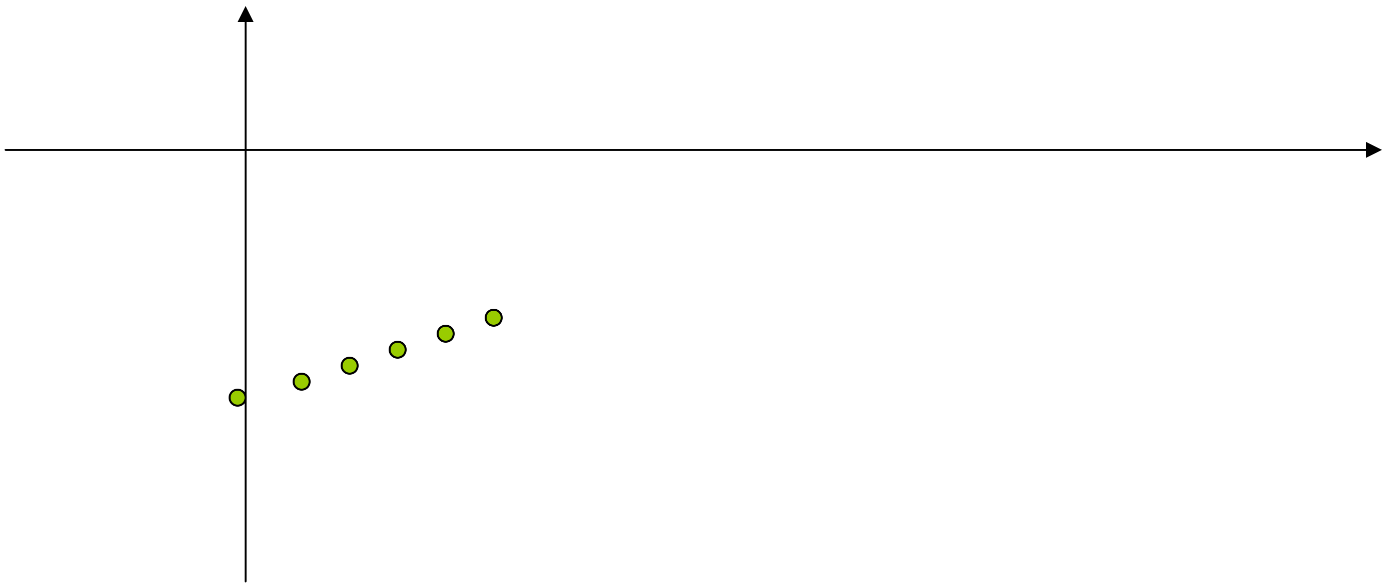
$$y(1.01) = -4 + 0.01(1 + (1)^2) = -3.98$$

$$y(1.02) = -3.98 + 0.01\left(1 + (1.01)^2\right) = -3.9598$$

$$y(1.03) = -3.9598 + 0.01\left(1 + (1.02)^2\right) = -3.9394$$

Example

$$Solve \quad \frac{dx(t)}{dt} = 1 + t^2, x(1) = -4 \quad use \quad h = 0.01$$



Euler Method

The Euler Method == First order Taylor series method

$$Given \quad \frac{dy(x)}{dx} = f(y,x), \quad y(a) = y_a$$

$$Solution:$$

$$y(a+h) = y(a) + h\, f(y(a), a)$$

$$y(a+2h) = y(a+h) + h\, f(y(a+h), a+h)$$

$$y(a+3h) = y(a+2h) + h\, f(y(a+h), a+2h)$$

Types of Errors

Types of Errors:

- Local truncation error:

  error due to the use of truncated Taylor series to compute x(t+h).

- Round off error:

  error due to finite number of bits used in representation of numbers. This error could be accumulated and magnified in succeeding steps.

Example 2

$$\text{Solve} \quad \frac{dx(t)}{dt} + 2x(t) = 1, \; x(0) = 1 \qquad \text{use} \quad h = 0.01$$

$$x(a+h) = x(a) + h\, f(a, x(a))$$

$$f(x) = \quad ?$$

$$x(0.0) \quad = ?$$

$$x(0.01) = ?$$

$$x(0.02) = ?$$

$$x(0.03) = ?$$

Example 2

$$Solve \quad \frac{dx(t)}{dt} + 2x(t) = 1, x(0) = 1 \qquad use \quad h = 0.01$$

$$x(a+h) = x(a) + h\ f(a, x(a))$$

$$f(t,x) = \quad 1 - 2x(t)$$

$$x(0.01) \quad = 1 + .01(1 - 2(1)) = 1 - .01 = .99$$

$$x(0.02) = 0.99 + 0.01(1 - 2(0.99)) = 0.9802$$

$$x(0.03) = 0.9706$$

$$x(0.04) = 0.9612$$

Lecture 29
Modified Euler Methods

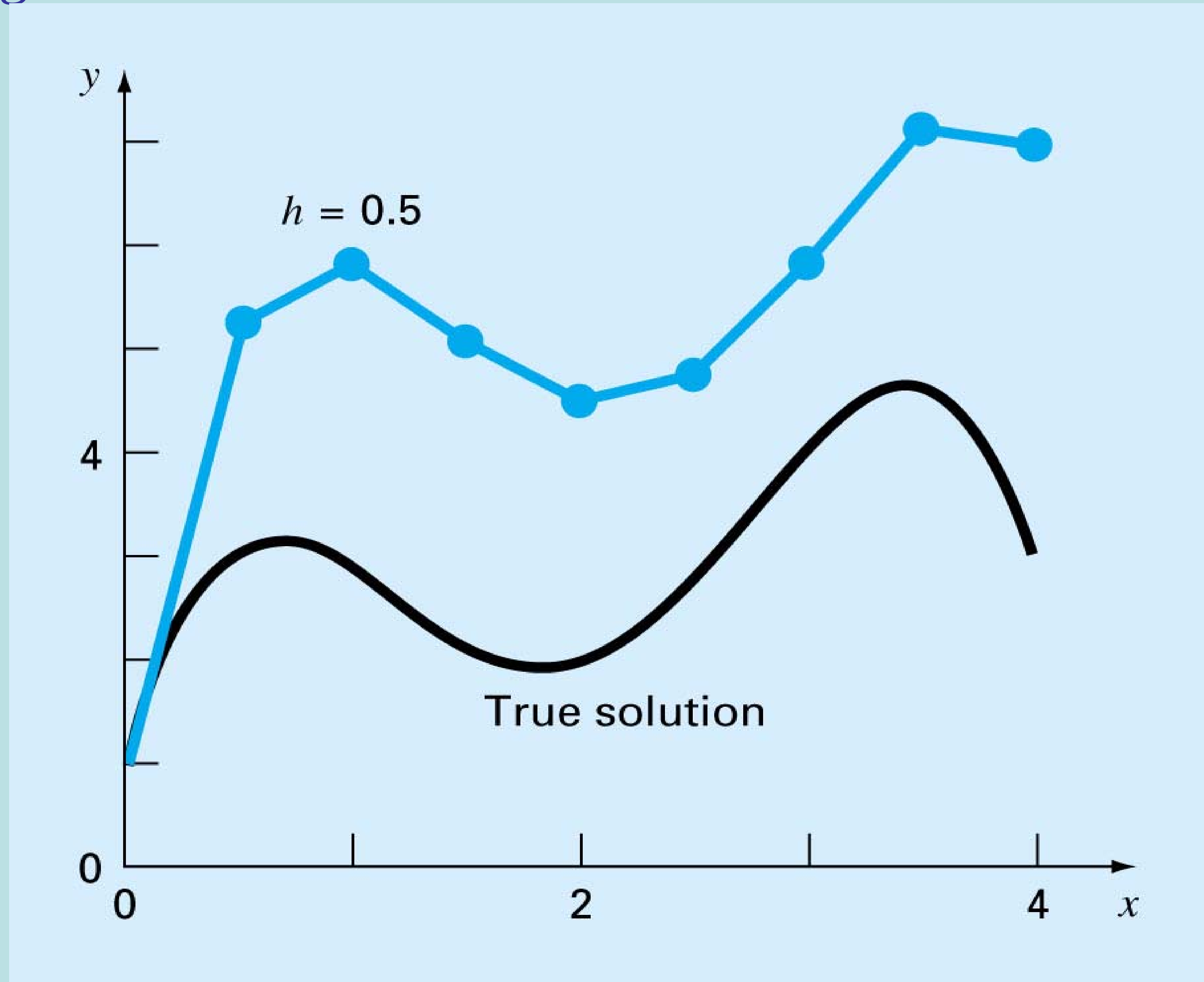
Review Euler Method

Heun's Method

Midpoint method

Runge-Kutta method

Figure 25.3



Error Analysis for Euler's Method/

- Numerical solutions of ODEs involves two types of error:
  - *Truncation* error
    - *Local truncation error*

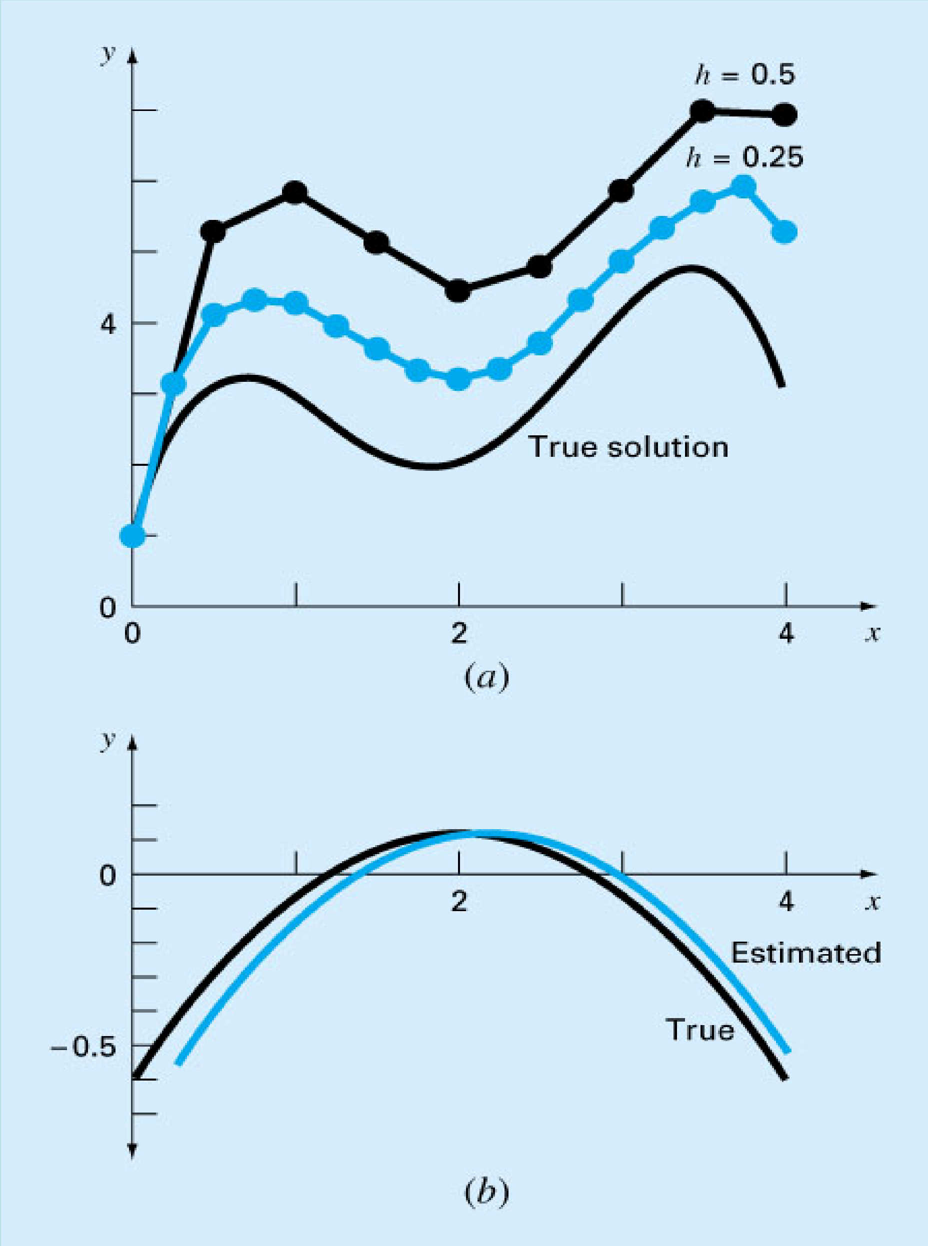
$$E_a = \frac{f'(x_i, y_i)}{2!} h^2$$

$$E_a = O(h^2)$$

    - *Propagated truncation error*
  - The sum of the two is the *total or global truncation error*
  - *Round-off* errors

- The Taylor series provides a means of quantifying the error in Euler’s method. However;
  - The Taylor series provides only an estimate of the local truncation error-that is, the error created during a single step of the method.
  - In actual problems, the functions are more complicated than simple polynomials. Consequently, the derivatives needed to evaluate the Taylor series expansion would not always be easy to obtain.
- In conclusion,
  - the error can be reduced by reducing the step size
  - If the solution to the differential equation is linear,

Figure 25.4



Improvements of Euler's method

- A fundamental source of error in Euler's method is that the derivative at the beginning of the interval is assumed to apply across the entire interval.
- Two simple modifications are available to circumvent this shortcoming:
  - Heun's Method
  - The Midpoint (or Improved Polygon) Method

Outlines

- Euler Method
- Heun's Predictor Corrector
- Midpoint method
- Comparison

Euler Method

Problem

$$\dot{y}(x) = f(x, y)$$

$$y(x_0) = y_0$$

Euler Method

$$y_0 = y(x_0)$$

$$y_{i+1} = y_i + h\ f(x_i, y_i)$$

$$for\ i = 1,2,...$$

Local Truncation Error $O(h^2)$

Global Truncation Error $O(h)$

Heun's Predictor Corrector Method

Problem

$\dot{y}(x) = f(x, y)$

$y(x_0) = y_0$

Heun's Method

$$y_0 = y(x_0)$$

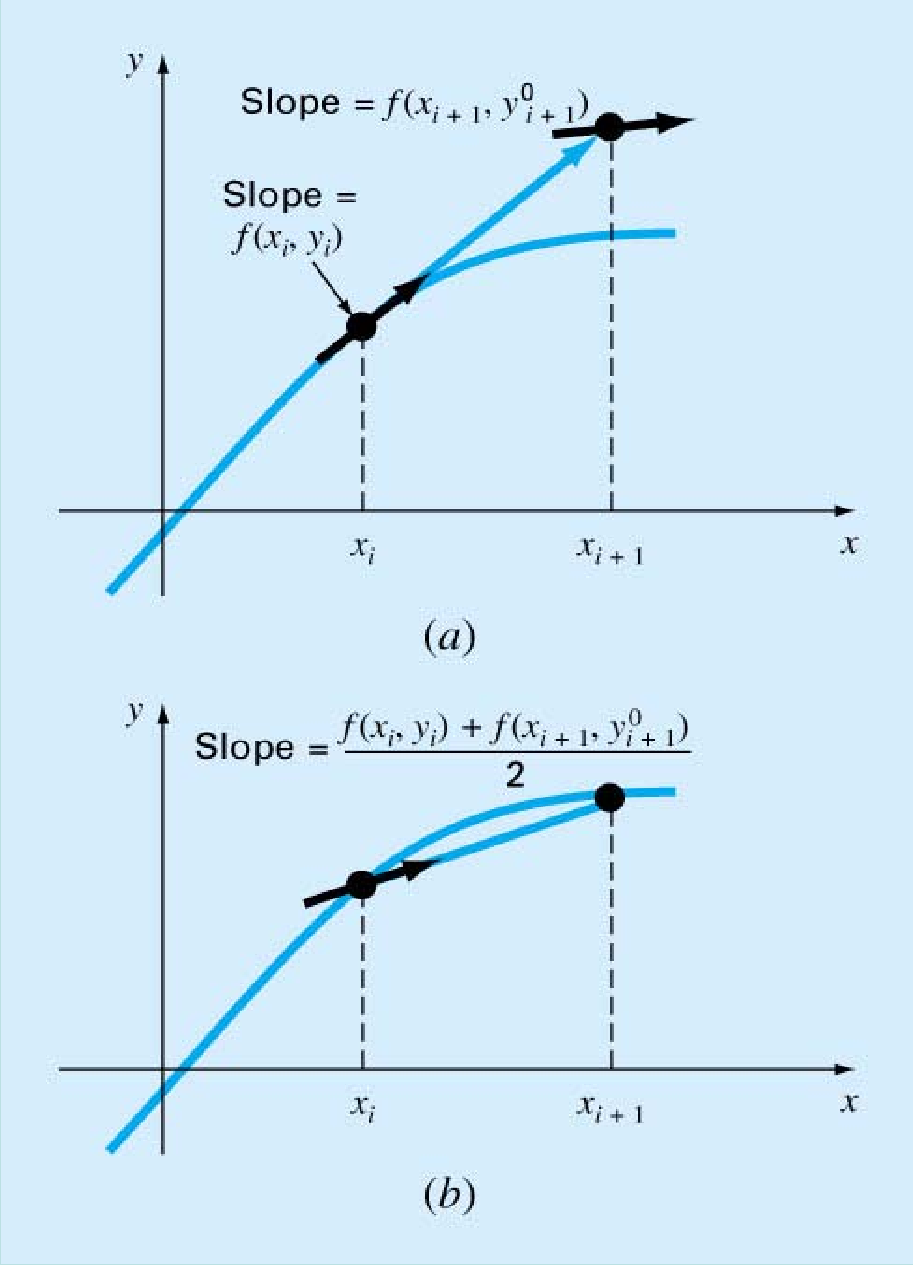
$$\text{Predictor}: \; y_{i+1}^{0} = y_i + h \; f(x_i, y_i)$$

$$\text{Corrector}: \; y_{i+1}^{k+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{k}) \right)$$

Local Truncation Error $O(h^3)$

Global Truncation Error $O(h^2)$

Figure 25.9



Slope = $f(x_{i+1}, y^0_{i+1})$
Slope = $f(x_i, y_i)$
y
x
x_i
x_{i+1}
(a)
Slope = $\frac{f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})}{2}$
y
x
x_i
x_{i+1}
(b)

Midpoint Method

Problem

$$\dot{y}(x) = f(x, y)$$

$$y(x_0) = y_0$$

Midpoint Method

$$y_0 = y(x_0)$$

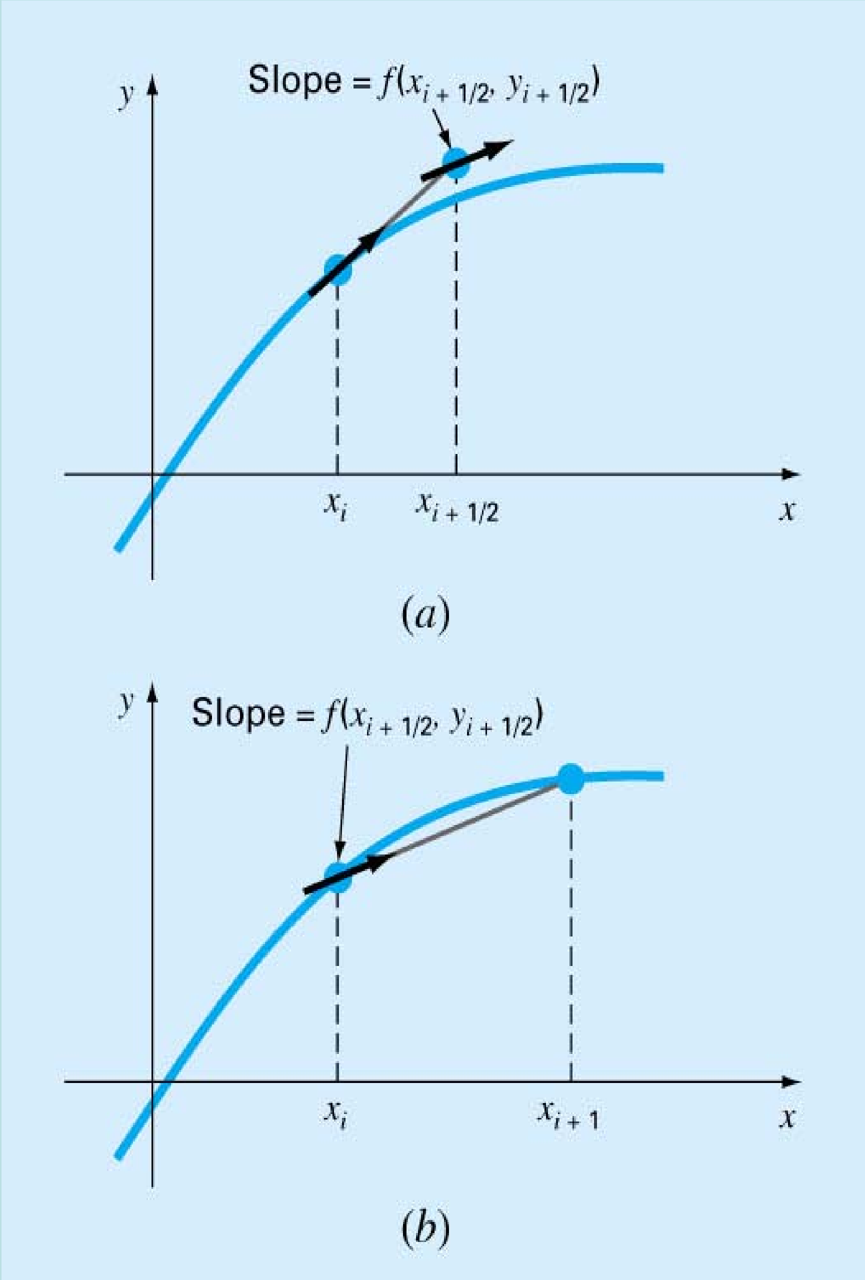
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2}\, f(x_i, y_i)$$

$$y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Local Truncation Error $O(h^3)$

Global Truncation Error $O(h^2)$

Figure 25.12



Lecture 28.
Runge-Kutta Methods

28. Runge-Kutta Methods

Runge-Kutta Methods

- These techniques were developed around 1900 by the German mathematicians C. Runge and M.W. Kutta.

Runge-Kutta Methods (RK)

- Runge-Kutta methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives.

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$ ***Increment function***

$$a's = \text{constants}$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$ ***p's and q's are constants***

$$k_3 = f(x_i + p_3 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1} h, y_i + q_{n-1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)$$

- *k*'s are recurrence functions. Because each *k* is a functional evaluation, this recurrence makes RK methods efficient for computer calculations.
- Various types of RK methods can be devised by employing different number of terms in the increment function as specified by *n*.
- First order RK method with *n=1* is in fact Euler's method.
- Once *n* is chosen, values of *a*'s, *p*'s, and *q*'s are evaluated by setting general equation equal to terms in a Taylor series expansion.

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

- Values of a_1, a_2, p_1, and q_{11} are evaluated by setting the second order equation to Taylor series expansion to the second order term. Three equations to evaluate four unknowns constants are derived.

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

A value is assumed for one of the unknowns to solve for the other three.

- Because we can choose an infinite number of values for a_2, there are an infinite number of second-order RK methods.
- Every version would yield exactly the same results if the solution to ODE were quadratic, linear, or a constant.
- However, they yield different results if the solution is more complicated (typically the case).
- Three of the most commonly used methods are:
  - Huen Method with a Single Corrector ($a_2=1/2$)
  - The Midpoint Method ($a_2=1$)
  - Raltson's Method ($a_2=2/3$)

Lecture
Taylor Series in Two Variables

The Taylor Series discussed in Chapter 4 is extended to the 2-independent variable case.

This is used to prove RK formula

Taylor Series in One Variable

The Taylor Series expansion of f(x)

$$f(x+h) = \sum_{i=0}^{n-1} \frac{h^i}{i!} f^{(i)}(x) \quad + \quad \frac{h^n}{n!} f^{(n)}(\bar{x})$$

Approximation

Error

$$where\ \bar{x}\ is\ between\ x\ and\ x+h$$

Taylor Series in One Variable another look

Define

$$\left(h\frac{d}{dx}\right)^i f(x) = h^i \frac{d^i f(x)}{dx^i} = f^{(i)}(x)\ \ h^i$$

The Taylor Series expansion of *f(x)*

$$f(x+h) = \sum_{i=0}^{n-1} \frac{1}{i!}\left(h\frac{d}{dx}\right)^i f(x) + \frac{1}{n!}\left(h\frac{d}{dx}\right)^n f(\bar{x})$$

$\bar{x}$ is between x and $x+h$

Definitions

Define

$$\left(h\,\frac{\partial}{\partial x}\right)^{i} f(x,y) = h^{i}\,\frac{\partial^{i} f}{\partial x^{i}}$$

$$\left(h\,\frac{\partial}{\partial x} + k\,\frac{\partial}{\partial y}\right)^{0} f(x,y) = f(x,y)$$

$$\left(h\,\frac{\partial}{\partial x} + k\,\frac{\partial}{\partial y}\right)^{1} f(x,y) = h\,\frac{\partial\ f(x,y)}{\partial x} + k\,\frac{\partial\ f(x,y)}{\partial y}$$

$$\left(h\,\frac{\partial}{\partial x} + k\,\frac{\partial}{\partial y}\right)^{2} f(x,y) = h^{2}\,\frac{\partial^{2}\ f(x,y)}{\partial x^{2}} + 2hk\,\frac{\partial^{2} f(x,y)}{\partial x \partial y} + k^{2}\,\frac{\partial^{2}\ f(x,y)}{\partial y^{2}}$$

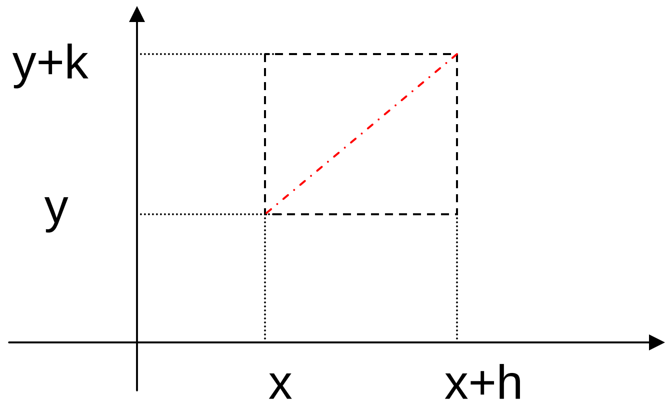
Taylor Series in Two Variables

The Taylor Series expansion of *f(x, y)*

$$f(x+h, y+k) = \sum_{i=0}^{n-1} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x,y) + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(\bar{x},\bar{y})$$

approximation *error*

$(\bar{x},\bar{y})$ is on the line joining between (x, y) and $(x+h, y+k)$



Taylor Series Expansion

$$f(x,y) = (x+1)(x+y+2)^2$$

Taylor Series Expansion   Center of expansion (0,0)

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^0 f(x,y)\bigg|_{(0.0)} = 4$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^1 f(x,y)\bigg|_{(0.0)} = h\ f_x + k\ f_y\big|_{(0,0)}$$

Runge-Kutta Method

Second Order Runge Kutta

$$K_1 = h\ f(t, x)$$

$$K_2 = h\ f(t + \alpha h, x + \beta K_1)$$

$$x(t+h) = x(t) + w_1 K_1 + w_2 K_2$$

Problem:

Find $\alpha, \beta,\ w_1, w_2$

such that $x(t+h)$ is as accurate as possible.

Runge-Kutta Method

Problem : *Find* α, β, w_1, w_2

to match as many terms as possible.

$$x(t+h) = x(t) + hx'(t) + \frac{h^2}{2}x''(t) + \frac{h^3}{6}x'''(t) + ...$$

$$x(t+h) = x(t) + w_1 h \ f(t,x) + w_2 f(t+\alpha h, x+\beta h \ f(t,x))$$

$$f(t+\alpha h, x+\beta h \ f) = f + \alpha h \ f_t + \beta h \ f_x + \frac{1}{2}\left(\alpha h \frac{\partial}{\partial t} + \beta h \frac{\partial}{\partial x}\right)^2 f(\bar{t}, \bar{x})$$

$$x(t+h) = x(t) + (w_1 + w_2)h \ f(t,x) + \alpha w_2 h^2 f_t + \beta w_2 h^2 f \ f_t + O(h^3)$$

Runge-Kutta Method

$$x(t+h) = \boxed{x(t)} + \boxed{hx'(t)} + \boxed{\frac{h^2}{2}x''(t)} + \frac{h^3}{6}x'''(t) + \ldots$$

$$x(t+h) = \boxed{x(t)} + \boxed{(w_1 + w_2)h\ f(t,x)} + \boxed{\alpha\, w_2 h^2 f_t + \beta\, w_2 h^2 f\ f_t} + O(h^3)$$

$$\Rightarrow w_1 + w_2 = 1, \quad \alpha\, w_2 = 0.5, \quad \beta\, w_2 = 0.5$$

One possible solution

$$w_1 = 0.5, \quad w_2 = 0.5, \quad \alpha = 1, \quad \beta = 1$$

Runge-Kutta Method

Second Order Runge Kutta

$$K_1 = f(t, x)$$

$$K_2 = f(t + h, x + K_1 h)$$

$$x(t + h) = x(t) + \frac{1}{2}\left(K_1 + K_2\right)h$$

Runge-Kutta Method

Alternative Formula

Second Order Runge Kutta

$$F_1 = f(t, x)$$

$$F_2 = f(t + h, x + hF_1)$$

$$x(t+h) = x(t) + \frac{h}{2}\left(F_1 + F_2\right)$$

Runge-Kutta Method
Alternative Formula

$$\Rightarrow w_1 + w_2 = 1, \quad \alpha\, w_2 = 0.5, \quad \beta\, w_2 = 0.5$$

another solution

Pick α any non - zero number

$$\beta = \alpha, \quad w_1 = 1 - \frac{1}{2\alpha}, \quad w_2 = \frac{1}{2\alpha}$$

Second Order Runge Kutta Formulas (select $\alpha \neq 0$)

$$K_1 = h\, f(t, x)$$

$$K_2 = h\, f(t + \alpha\, h, x + \alpha\, K_1)$$

$$x(t+h) = x(t) + \left(1 - \frac{1}{2\alpha}\right) F_1 + \frac{1}{2\alpha} F_2$$

Runge-Kutta Method

Fourth Order Runge Kutta

$$K_1 = f(t, x)$$

$$K_2 = f(t + \frac{1}{2}h, x + \frac{1}{2}K_1 h)$$

$$K_3 = f(t + \frac{1}{2}h, x + \frac{1}{2}K_2 h)$$

$$K_4 = f(t + h, x + K_3 h)$$

$$x(t+h) = x(t) + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)h$$

Second order Runge-Kutta Method

Example

Solve the following system to find x(1.02) using RK2

$$\dot{x}(t) = 1 + x^2(t) + t^3, \qquad x(1) = -4, h = 0.01$$

STEP1:

$$K_1 = h\ f(t,x) = 0.01(1 + x^2 + t^3) = 0.18$$

$$K_2 = h\ f(t+h, x+K_1) = 0.01(1 + (x+0.18)^2 + (t+.01)^3) = 0.1692$$

$$x(1+0.01) = x(1) + \frac{1}{2}\left(K_1 + K_2\right) = -4 + \frac{1}{2}(0.18 + 0.1692) = -3.8254$$

Second order Runge-Kutta Method

Example

STEP 2

$$K_1 = h\ f(t,x) = 0.01(1+x^2+t^3) = 0.1666$$

$$K_2 = h\ f(t+h, x+K_1) = 0.01(1+(x+0.1666)^2+(t+.01)^3) = 0.1545$$

$$x(1.01+0.01) = x(1.01) + \frac{1}{2}\left(K_1+K_2\right)$$

$$= -3.8254 + \frac{1}{2}(0.1666+0.1545) = -3.6648$$

Runge-Kutta Method

Fourth Order Runge Kutta

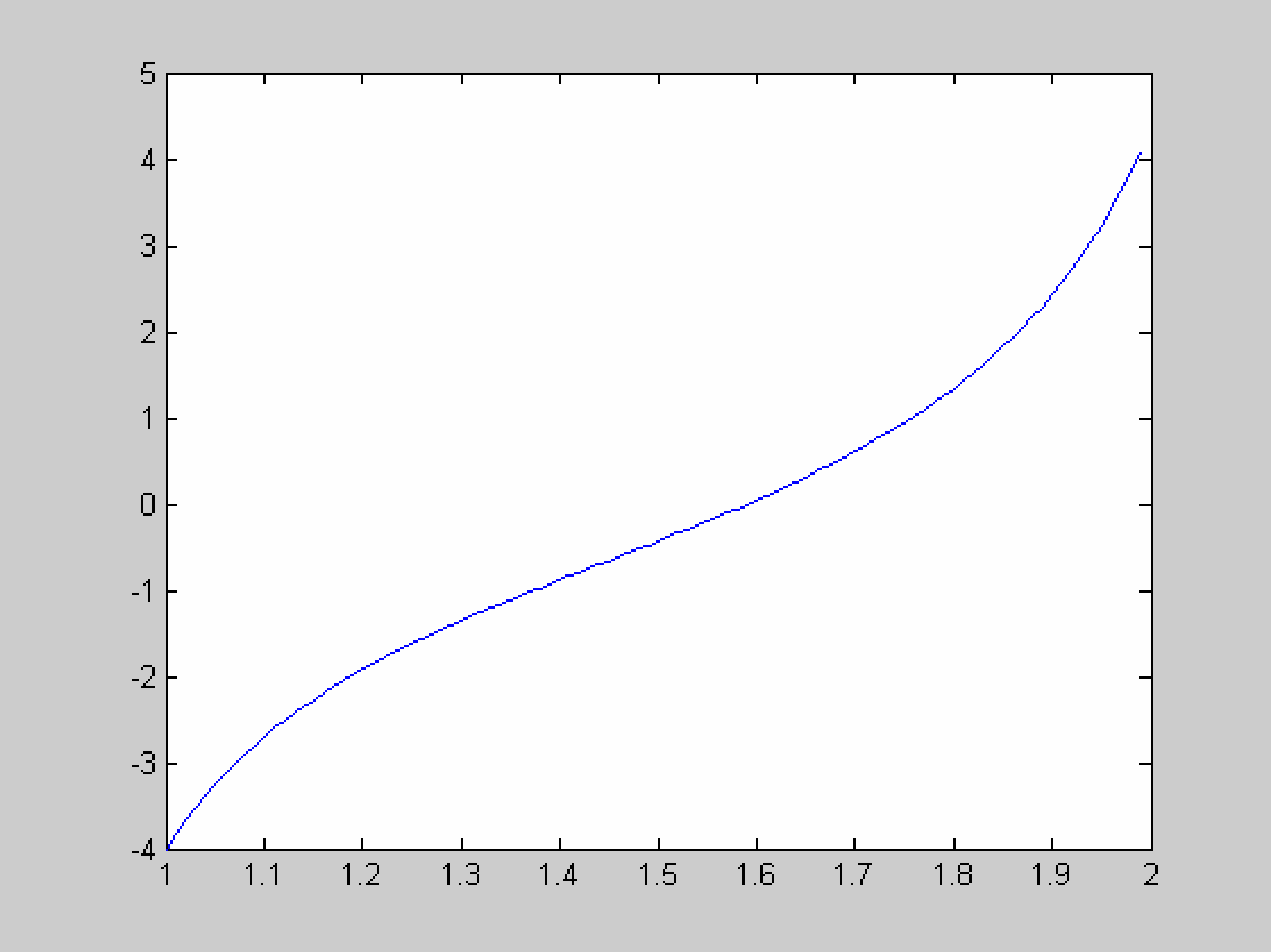
$$K_1 = f(t, x)$$

$$K_2 = f(t + \frac{1}{2}h, x + \frac{1}{2}K_1 h)$$

$$K_3 = f(t + \frac{1}{2}h, x + \frac{1}{2}K_2 h)$$

$$K_4 = f(t + h, x + K_3 h)$$

$$x(t+h) = x(t) + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)h$$



5
4
3
2
1
0
-1
-2
-3
-4
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2

Example

4-order Runge-Kutta Method

$$\frac{dy}{dx} = 1 + y - x^2$$

$$y(0) = 0.5$$

$$h = 0.2$$

$$Compute \quad y(0.2) \ and \ y(0.4)$$

Example

4-order Runge-Kutta Method

$$\text{Iteration 1 of the RK4} \quad (\mathrm{x} = 0, \mathrm{y} = 0.5)$$

$$K_1 = h \ f(x, y) = 0.3000$$

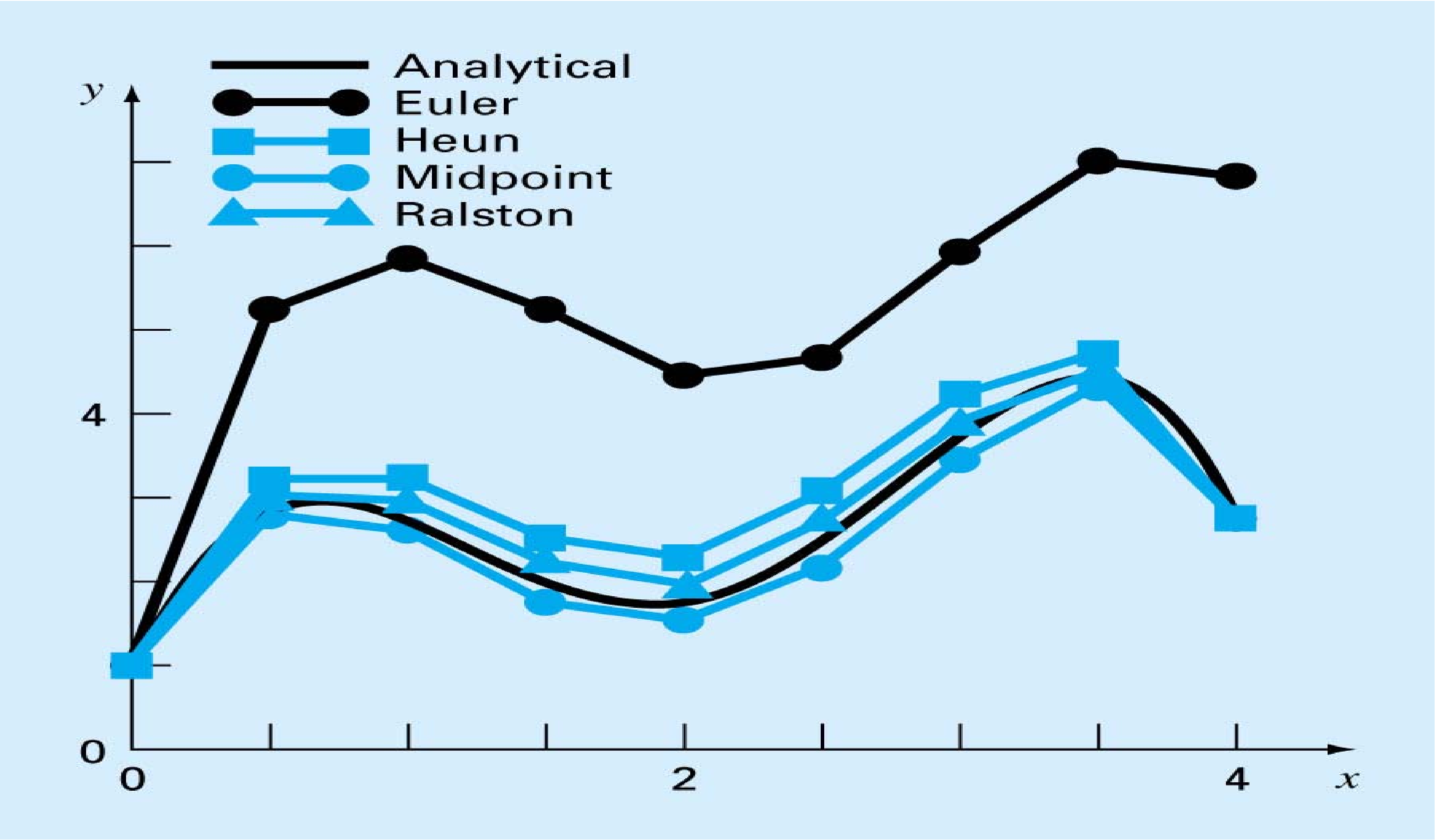
$$K_2 = h \ f(x + \frac{1}{2}h, y + \frac{1}{2}K_1) = 0.3280$$

$$K_3 = h \ f(x + \frac{1}{2}h, y + \frac{1}{2}K_2) = 0.3308$$

$$K_4 = h \ f(x + h, y + K_3) = 0.3482$$

$$y(x+h) = y(x) + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right) = 0.8276$$

Figure 25.14



High Order ODE

- High order ODE
- Systems of High order ODE
- Procedure
- Examples

High Order ODE

- Methods discussed earlier such as Euler ,Runge-Kutta,...are used for first order ordinary differential equations
- How do solve second order, or higher ODE?

- The approach: Convert the high order differential equation into a system of first order Differential equation

$$\dot{X}(t) = F(t, X)$$

Example of converting High order ODE to first order ODEs

Convert

$$\ddot{x} + 3\dot{x} + 6x = 1$$

$$\dot{x}(0) = 1; \quad x(0) = 4$$

to a system of first order ODE

1. Select a new set of variable

$$z_1 = x$$

$$z_2 = \dot{x}$$

One degree less than the highest order derivative

Example of converting High order ODE to first order ODEs

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	1	$\dot{z}_2 = 1 - 3z_2 - 6z_1$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 1 - 3z_2 - 6z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Example of converting High order ODE to first order ODEs

Convert

$$\dddot{x} + 2\ddot{x} + 7\dot{x} + 8x = 0$$

$$\ddot{x}(0) = 9,\ \dot{x}(0) = 1;\quad x(0) = 4$$

1. Select a new set of variable

$$z_1 = x$$

$$z_2 = \dot{x}$$

$$z_3 = \ddot{x}$$

One degree less than the highest order derivative

Example of converting High order ODE to first order ODEs

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	1	$\dot{z}_2 = z_3$
$\ddot{x}$	z_3	9	$\dot{z}_3 = -2z_3 - 7z_2 - 8z_1$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ -2z_3 - 7z_2 - 8z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$$

Example of converting High order ODE to first order ODEs

Convert

$$\dddot{x} + 5\ddot{x} + 2\dot{x} + 8y = 0$$

$$\ddot{y} + 2xy + \dot{x} = 2$$

$$x(0) = 4;\ \dot{x}(0) = 2;\ \ddot{x}(0) = 9;\ y(0) = 1;\ \dot{y}(0) = -3$$

1. Select a new set of variable

$$z_1 = x$$

$$z_2 = \dot{x}$$

$$z_3 = \ddot{x}$$

$$z_4 = y$$

$$z_5 = \dot{y}$$

One degree less than the highest order derivative

Example of converting High order ODE to first order ODEs

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	2	$\dot{z}_2 = z_3$
$\ddot{x}$	z_3	9	$\dot{z}_3 = -5z_3 - 2z_2 - 8z_4$
y	z_4	1	$\dot{z}_4 = z_5$
$\dot{y}$	z_5	-3	$\dot{z}_5 = 2 - z_2 - 2z_1 z_4$

Solution of a second order ODE

- Solve the equation using Euler method. Use h=0.1

$$\ddot{x} + 2\dot{x} + 8x = 2$$

$$x(0) = 1; \dot{x}(0) = -2$$

Select a new set of variable $z_1 = x, z_2 = \dot{x}$

The second order equation is expressed as

$$\dot{Z} = F(Z) = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 2 - 2z_2 - 8z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Solution of a second order ODE

$$F(Z)=\begin{bmatrix} z_2 \\ 2-2z_2-8z_1 \end{bmatrix}, Z(0)=\begin{bmatrix} 1 \\ -2 \end{bmatrix}, h=0.1$$

$$Z(0+0.1)=Z(0)+hF(Z(0))$$

$$=\begin{bmatrix} 1 \\ -2 \end{bmatrix}+0.1\begin{bmatrix} -2 \\ 2-2(-2)-8(1) \end{bmatrix}=\begin{bmatrix} 0.8 \\ -2.2 \end{bmatrix}$$

$$Z(0.2)=Z(0.1)+hF(Z(0.1))$$

$$=\begin{bmatrix} 0.8 \\ -2.2 \end{bmatrix}+0.1\begin{bmatrix} -2.2 \\ 2-2(-2.2)-8(0.8) \end{bmatrix}=\begin{bmatrix} 0.58 \\ -2.2 \end{bmatrix}$$

Systems of Equations

- Many practical problems in engineering and science require the solution of a system of simultaneous ordinary differential equations rather than a single equation:

$$
\begin{aligned}
\frac{dy_1}{dx} &= f_1(x, y_1, y_2, \ldots, y_n) \\
\frac{dy_2}{dx} &= f_2(x, y_1, y_2, \ldots, y_n) \\
&\vdots \\
\frac{dy_n}{dx} &= f_n(x, y_1, y_2, \ldots, y_n)
\end{aligned}
$$

- Solution requires that n initial conditions be known at the starting value of x.

SE301:Numerical Methods

29. Adam-Moulton Multi-step Predictor-Corrector Methods

Multi-step Method

- Single Step Methods
  - Euler, Runge-Kutta are single step methods
  - Information about x(t) is used to estimate x(t+h)
- Multistep Methods
  - Adam-Moulton method is a multi-step method
  - To estimate x(t+h) information about x(t),x(t-h), x(t-2h)... are used

Heun's Predictor Corrector method

Original Heun's predictor corrector method is not a multi-step method, but the non-self starting method is a multi-step method

Multistep Methods

The Non-Self-Starting Heun Method/

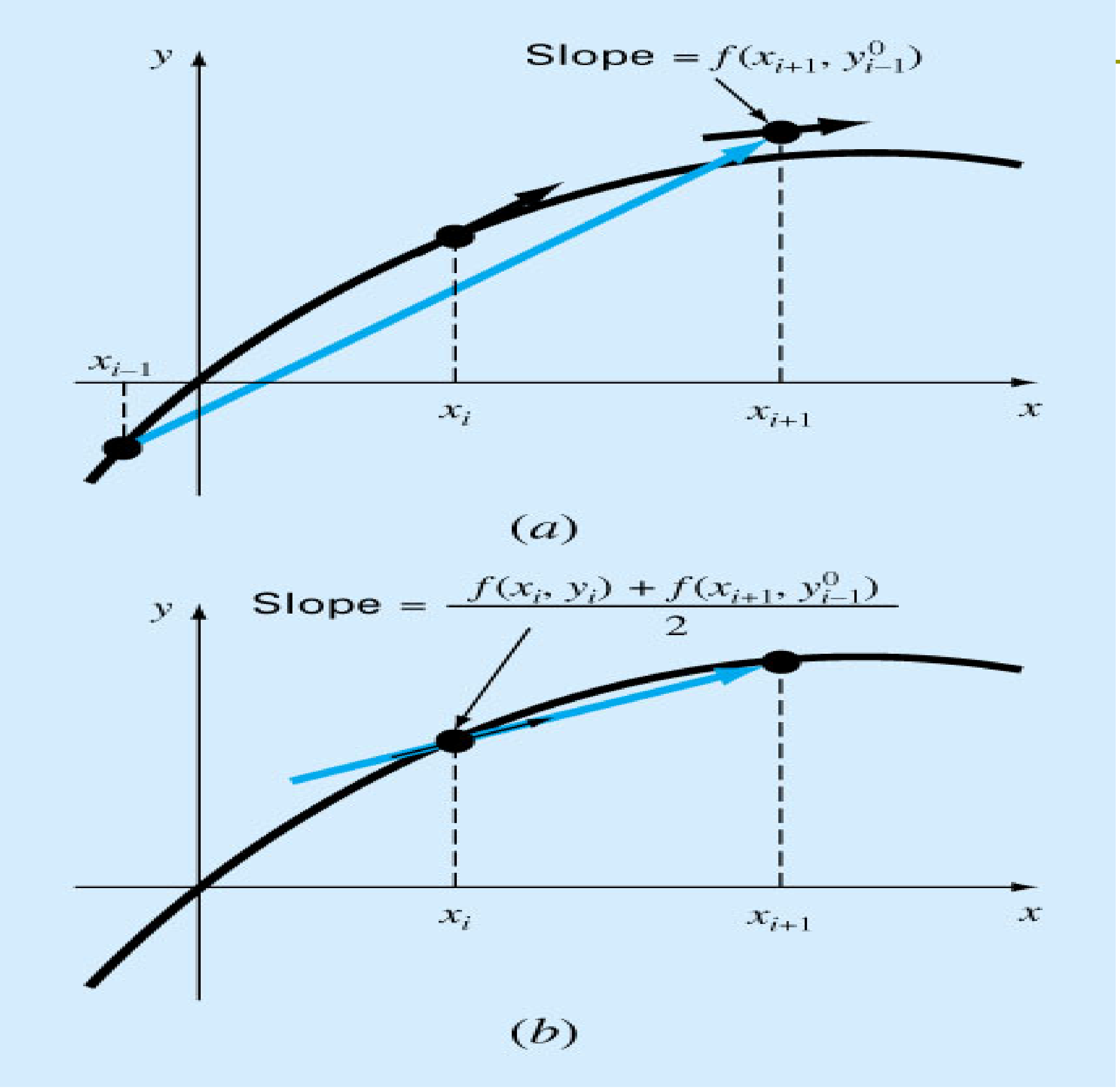
- Huen method uses *Euler's method* as a *predictor* and *trapezoidal rule* as a *corrector*.
- Predictor is the weak link in the method because it has the greatest error, $O(h^2)$.
- One way to improve Heun's method is to develop a predictor that has a local error of $O(h^3)$.

$$y_{i+1}^{0} = y_{i-1}^{m} + f(x_i, y_i^m)2h$$

$$y_{i+1}^{j} = y_i^m + \frac{f(x_i, y_i^m) + f(x_{i+1}, y_{i+1}^{j+1})}{2}h$$

$$for \quad j = 1,2,\ldots, m$$

Figure 26.4



Slope = $f(x_{i+1}, y^0_{i-1})$
y
x
x_{i-1}
x_i
x_{i+1}
(a)
Slope = $\frac{f(x_i, y_i) + f(x_{i+1}, y^0_{i-1})}{2}$
y
x
x_i
x_{i+1}
(b)

Error Analysis

- Both the predictor and corrector local error are of order $O(h^3)$
- Error estimate for the corrector:

$$E_c = -\frac{y_{i+1}^0 - y_{i+1}^m}{5}$$

- Modifiers:

$$y_{i+1}^m \leftarrow y_{i+1}^m - \frac{y_{i+1}^0 - y_{i+1}^m}{5}$$

$$y_{i+1}^0 \leftarrow y_{i+1}^0 + \frac{4}{5}\left(y_i^m - y_i^0\right)$$

Step-Size Control/

- *Constant Step Size*.
  - A value for *h* must be chosen prior to computation.
  - It must be small enough to yield a sufficiently small truncation error.
  - It should also be as large as possible to minimize run time cost and round-off error.

- *Variable Step Size*.
  - If the corrector error is greater than some specified error, the step size is decreased.
  - A step size is chosen so that the convergence criterion of the corrector is satisfied in two iterations.
  - A more efficient strategy is to increase and decrease by doubling and halving the step size.

Integration Formulas/ *Newton-Cotes Formulas*.

Open Formulas.

$$y_{i+1} = y_{i-n} + \int_{x_{i-n}}^{x_{i+1}} f_n(x)dx$$

$f_n(x)$ is an n^{th} order interpolating polynomial.

Closed Formulas.

$$y_{i+1} = y_{i-n+1} + \int_{x_{i-n+1}}^{x_{i+1}} f_n(x)dx$$

Adams Formulas (Adams-Bashforth). Open Formulas.

- The Adams formulas can be derived in a variety of ways. One way is to write a forward Taylor series expansion around x_i. A second order open Adams formula:

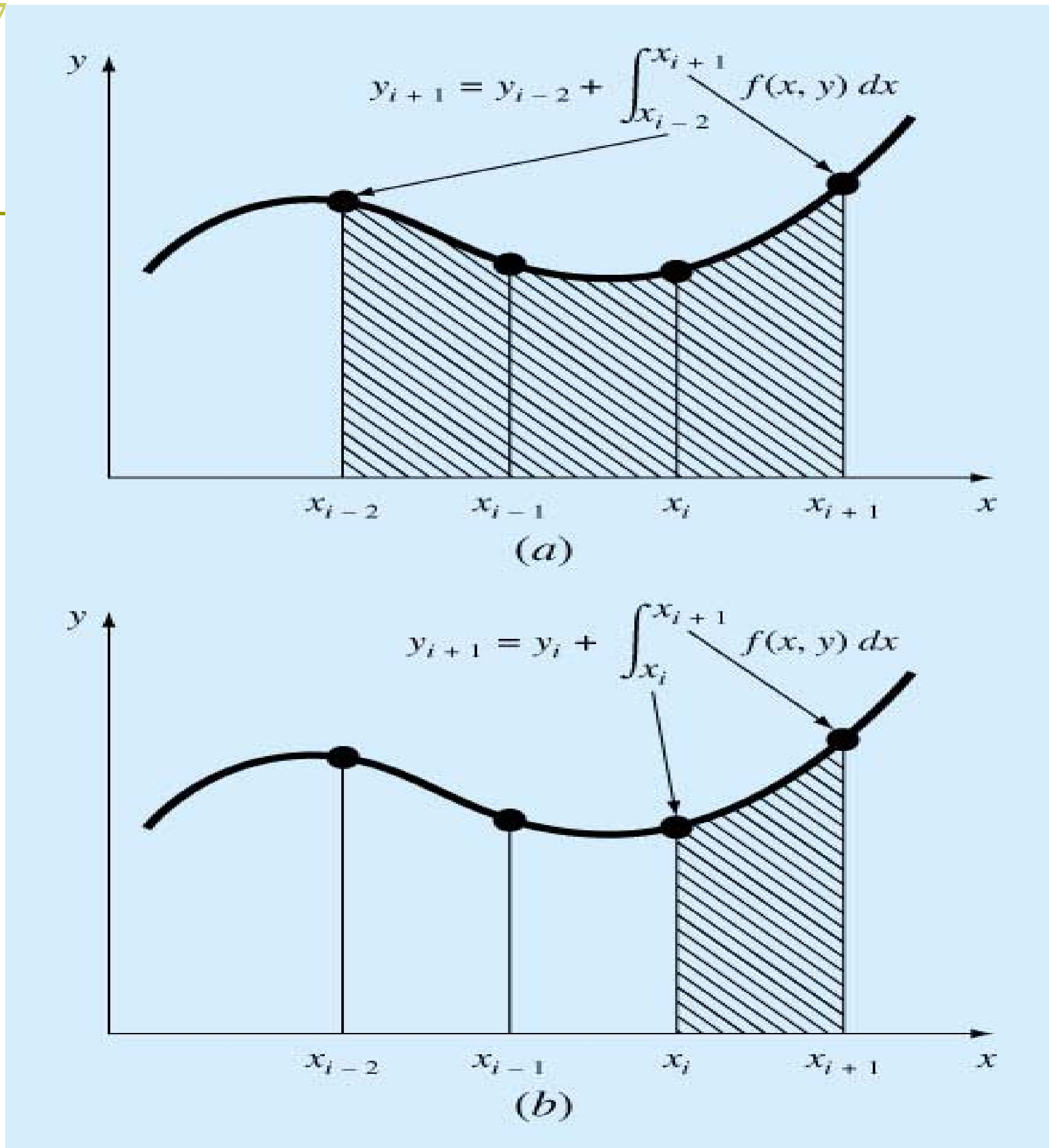
$$y_{i+1} = y_i + h\left(\frac{3}{2} f_i - \frac{1}{2} f_{i-1}\right) + \frac{5}{12} h^3 f_i'' + O(h^4)$$

- A backward Taylor series around x_{i+1} can be written:

$$y_{i+1} = y_i + h \sum_{k=0}^{n-1} \beta_k f_{i+1-k} + O(h^{n+1})$$

Listed in Table 26.2

Figures 26.7



$y_{i+1} = y_{i-2} + \int_{x_{i-2}}^{x_{i+1}} f(x, y)\,dx$
y
x
x_{i-2}
x_{i-1}
x_i
x_{i+1}
(a)
$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x, y)\,dx$
y
x
x_{i-2}
x_{i-1}
x_i
x_{i+1}
(b)

Higher-Order multistep Methods/ *Milne's Method.*

- Uses the three point Newton-Cotes open formula as a predictor and three point Newton-Cotes closed formula as a corrector.

Fourth-Order Adams Method.

- Based on the Adams integration formulas. Uses the fourth-order Adams-Bashforth formula as the predictor and fourth-order Adams-Moulton formula as the corrector.

Predictor-Corrector

Two - step

$$\text{Predictor} : y_{i+1} = y_i + h\left(\frac{3}{2} f(y_i) - \frac{1}{2} f(y_{i-1})\right)$$

$$\text{Corrector} : y_{i+1} = y_i + h\left(\frac{1}{2} f(y_{i+1}) + \frac{1}{2} f(y_i)\right)$$

Three - Step

$$\text{Predictor} : y_{i+1} = y_i + h\left(\frac{23}{12} f(y_i) + \frac{-16}{12} f(y_{i-1}) + \frac{5}{12} f(y_{i-2})\right)$$

$$\text{Corrector} : y_{i+1} = y_i + h\left(\frac{5}{12} f(y_{i+1}) + \frac{8}{12} f(y_i) + \frac{-1}{12} f(y_{i-1})\right)$$

See pages 744 (*predictor*), 746 (*corrector*) *formulas*

4-Step Adams-Moulton Predictor-Corrector

Predictor: (Adams-Bashforth Predictor)

$$y_{i+1} = y_i + \frac{h}{24}\left(55 f(y_i) - 59 f(y_{i-1}) + 37 f(y_{i-2}) - 9 f(y_{i-3})\right)$$

Corrector: (Adams-Moulton Corrector)

$$y_{i+1} = y_i + \frac{h}{24}\left(9 f(y_{i+1}) + 19 f(y_i) + -5 f(y_{i-1}) + f(y_{i-2})\right)$$

See pages 744(*predictor*), 746(*corrector*) *formulas*

4-Step Adams-Moulton Predictor-Corrector

Predictor : (Adams - Bashforth Predictor)

$$y_{i+1} = y_i + \frac{h}{24}\left(55 f(y_i) - 59 f(y_{i-1}) + 37 f(y_{i-2}) - 9 f(y_{i-3})\right)$$

Corrector : (Adams - Moulton Corrector)

$$y_{i+1} = y_i + \frac{h}{24}\left(9 f(y_{i+1}) + 19 f(y_i) + -5 f(y_{i-1}) + f(y_{i-2})\right)$$

See pages 744(*predictor*), 746(*corrector*) *formulas*

Example

Solve

$$\frac{dy}{dx} = 2x + y^2 x \qquad y(0) = 2$$

$h = 0.1,\ Use\ \ 2-step$ Predictor corrector Method

compute y(0.4)

We need two initial conditions to use the

$2-step$ Predictor corrector Method

We will first use use RK2 to estimate y(0.1)

Example

We need two initial conditions

Use $RK2$ *to compute* $y(0.1)$ *then we can use*

the Predictor corrector Method

$$\frac{dy}{dx} = 2x + y^2 x \qquad y(0) = 2, \quad h = 0.1,$$

$$K1 = 0.1(0) = 0$$

$$K2 = 0.1(0.2 + 0.4) = 0.06$$

$$y(0.1) = 2 + 0.5(0.06) = 2.03$$

Example

$$\frac{dy}{dx} = 2x + y^2 x \qquad y_{i-1} = y(0) = 2, y_i = y(0.1) = 2.03, \qquad h = 0.1$$

$$\text{Predictor}: y_{i+1} = y_i + h\left(\frac{3}{2} f(y_i) - \frac{1}{2} f(y_{i-1})\right)$$

$$= 2.03 + 0.1\left(\frac{3}{2}\left(2(0.1) + 2.03^2(0.1)\right) - \frac{1}{2}(0+0)\right) =$$

$$\text{Corrector}: y_{i+1} = y_i + h\left(\frac{1}{2} f(y_{i+1}) + \frac{1}{2} f(y_i)\right)$$

$$= 2.03 + 0.1\left(\frac{1}{2} f(y_{i+1}) + \frac{1}{2}\left(2(0.1) + 2.03^2(0.1)\right)\right)$$

of steps

- at each iteration one prediction step is done and as many correction steps as needed.
- Usually few corrections steps are done (1 to 3)
- It is usually better (in terms of accuracy) to use smaller steps size than corrections beyond few steps.

SE301:Numerical Methods

31. Finite Difference methods for solving Boundary Value problems

Outlines

- Boundary Value Problem
- Shooting Method
- Finite Difference Method

Boundary-Value and Initial value Problems

Boundary-Value Problems

- The auxiliary conditions are not at one point of time
- More difficult to solve than initial value problem

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \;\; x(2) = 1.5$$

Initial-Value Problems

- The auxiliary conditions are at one point of time

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \;\; \dot{x}(0) = 2.5$$

Solution of Boundary-Value Problems

Shooting method

Methods for Boundary-Value Problems

1. Shooting method:
   - Guess a values for The auxiliary conditions at one point of time
   - Solve the initial value problem using Euler,Runge-Kutta, …
   - Check if the boundary conditions is satisfied otherwise modify the guess and resolve the problem.

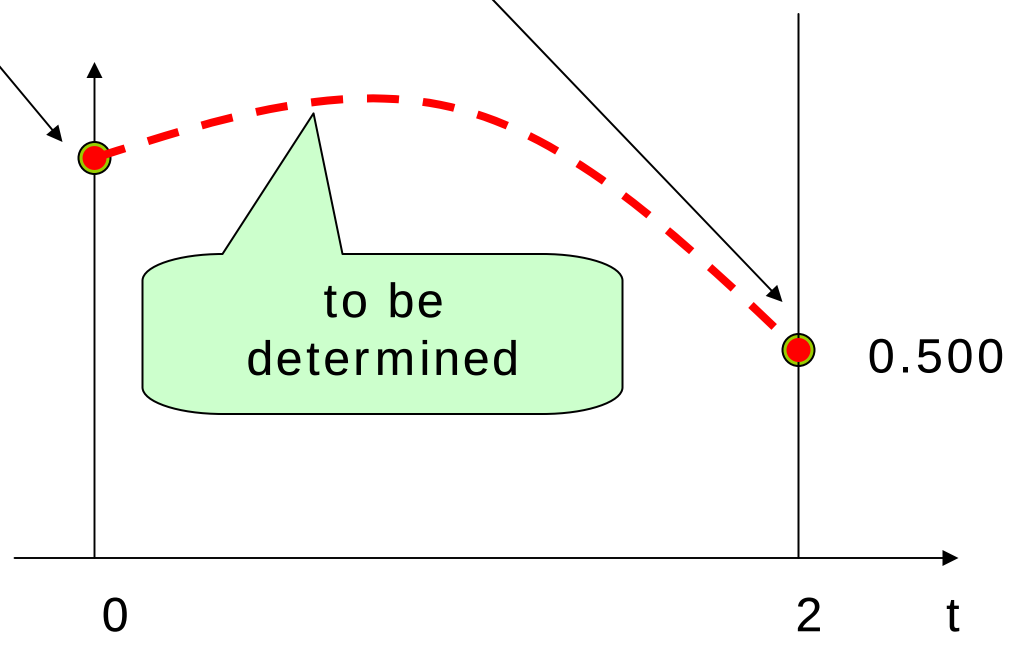
- Use interpolation in updating the guess
- It is an iterative procedure and can be efficient in solving the BVP

Shooting method

Example

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 0.5$$



Shooting method

Example

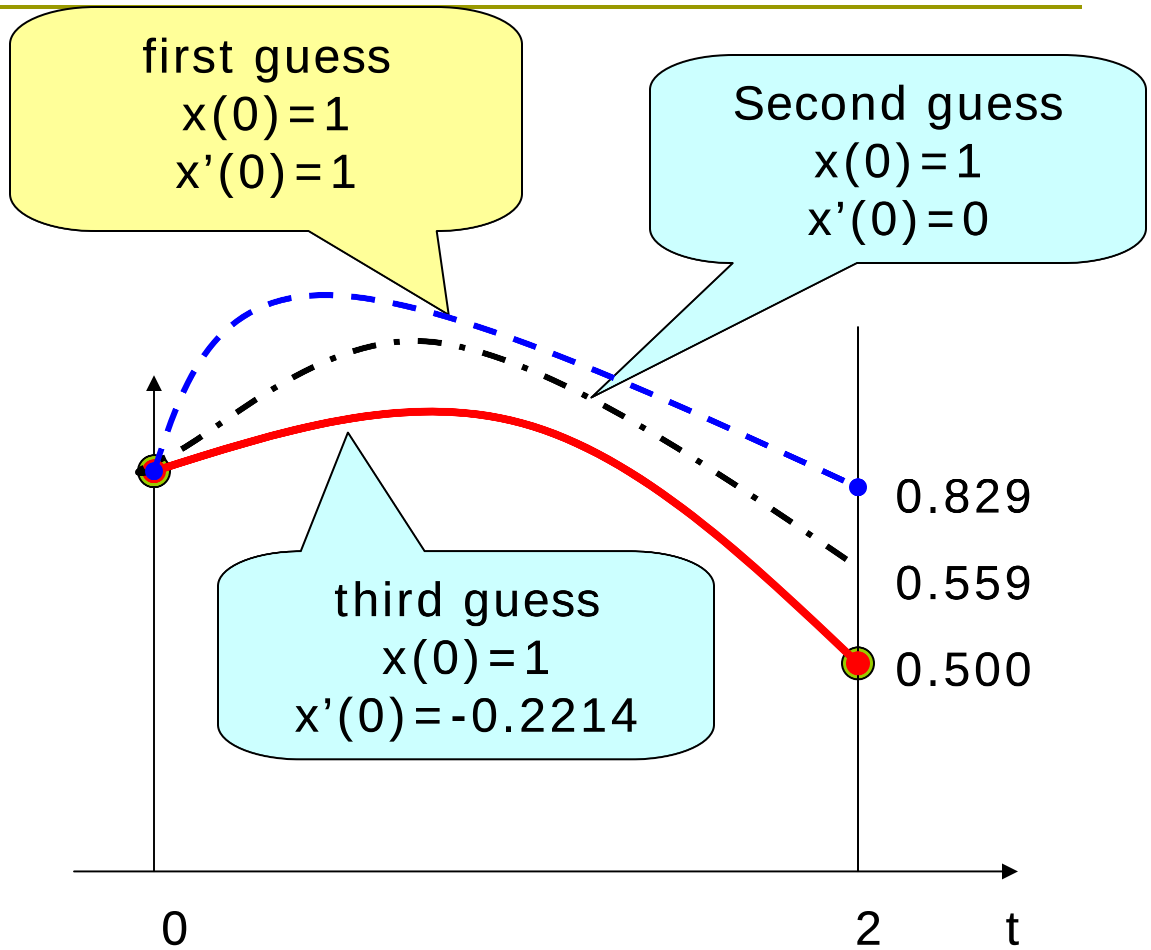
- Example

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \ \ x(2) = 0.5$$

Using linear interpolation of first and second guesses

x'(0)=-0.2214



Solution of Boundary-Value Problems

Finite Difference Method

Methods for Boundary-Value Problems

2. Finite Difference Method :

- Divide the interval into n intervals
- The solution of the BVP is converted to the problem of determining the value of function at the base points.
- Use finite approximations to replace the derivatives
- This approximation results in a set of algebraic equations.
- Solve the equations to obtain the solution of the BVP

Finite Difference Method
Example

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

Divide the interval [0,1] into n = 4 intervals

Base points are

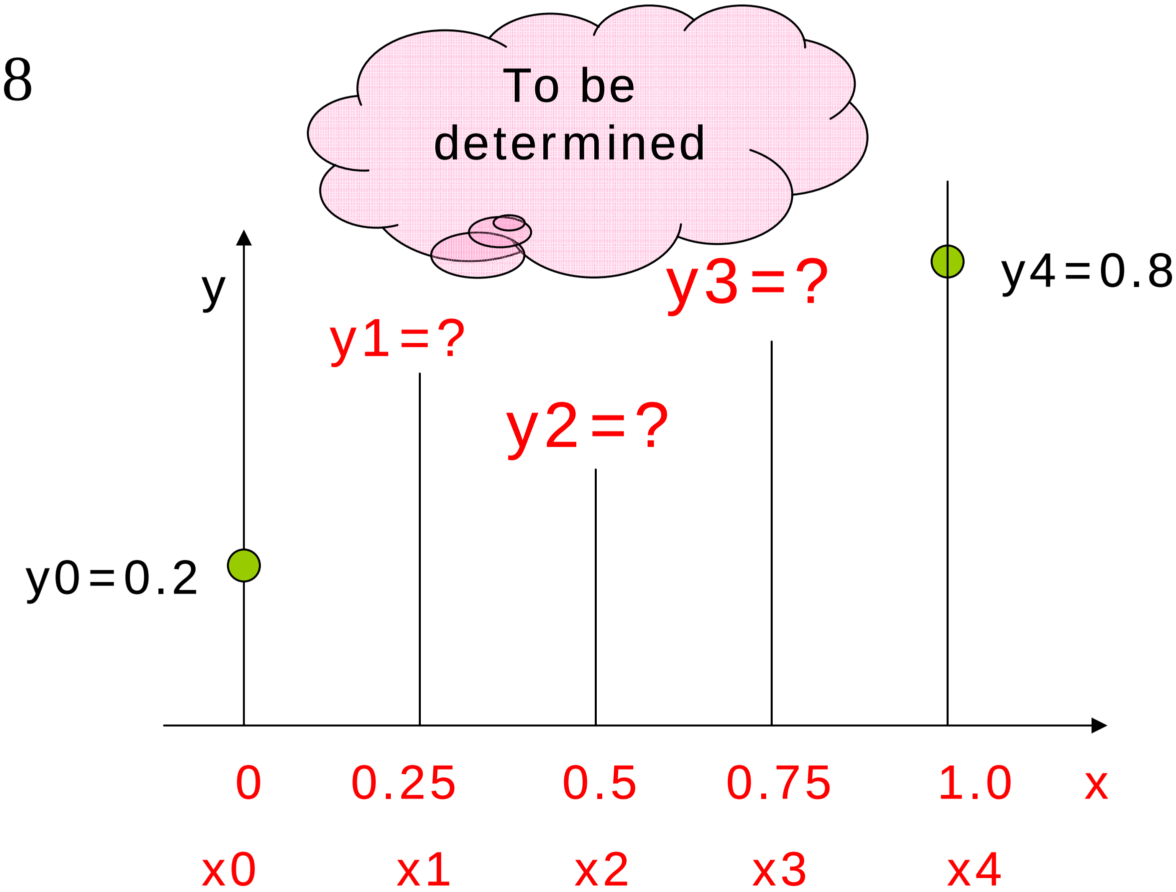
x0=0

x1=0.25

x2=.5

x3=0.75

x4=1.0



Finite Difference Method
Example

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

Divide the interval [0,1] into n = 4 intervals

Base points are

x0=0

x1=0.25

x2=.5

x3=0.75

x4=1.0

Replace

$$\ddot{y} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \qquad \textit{central difference formula}$$

$$\dot{y} = \frac{y_{i+1} - y_{i-1}}{2h} \qquad \textit{central difference formula}$$

$$\ddot{y} + 2\dot{y} + y = x^2$$

Becomes

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_{i-1}}{2h} + y_i = x_i^{\,2}$$

Second Order BVP

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 \quad with \;\; y(0) = 0.2, \qquad y(1) = 0.8$$

$$Let \;\; h = 0.25$$

Base Points

$$x_0 = 0, \; x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$$

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h} = \frac{y_{i+1} - y_i}{h}$$

$$\frac{d^2y}{dx^2} \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Second Order BVP

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_i}{h} + y_i = x_i^{\ 2} \quad i = 1,2,3$$

$$x_0 = 0,\ x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$$

$$y_0 = 0.2,\ y_1 = ?, y_2 = ?, y_3 = ?, \quad y_4 = 0.8$$

$$16(y_{i+1} - 2y_i + y_{i-1}) + 8(y_{i+1} - y_i) + y_i = x_i^{\ 2}$$

$$24y_{i+1} - 39y_i + 16y_{i-1} = x_i^{\ 2}$$

Second Order BVP

$$24y_{i+1} - 39y_i + 16y_{i-1} = x_i^{\ 2}$$

$$i = 1 \quad 24y_2 - 39y_1 + 16y_0 = x_1^{\ 2}$$

$$i = 2 \quad 24y_3 - 39y_2 + 16y_1 = x_2^{\ 2}$$

$$i = 3 \quad 24y_4 - 39y_3 + 16y_2 = x_3^{\ 2}$$

$$\begin{bmatrix} -39 & 24 & \\ 16 & -39 & 24 \\ & 16 & -39 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.25^2 - 16(0.2) \\ 0.5^2 \\ 0.75^2 - 24(0.8) \end{bmatrix}$$

$$\textit{Solution} \quad y_1 = 0.4791, y_2 = 0.6477, y_3 = 0.7436$$

Second Order BVP

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$$

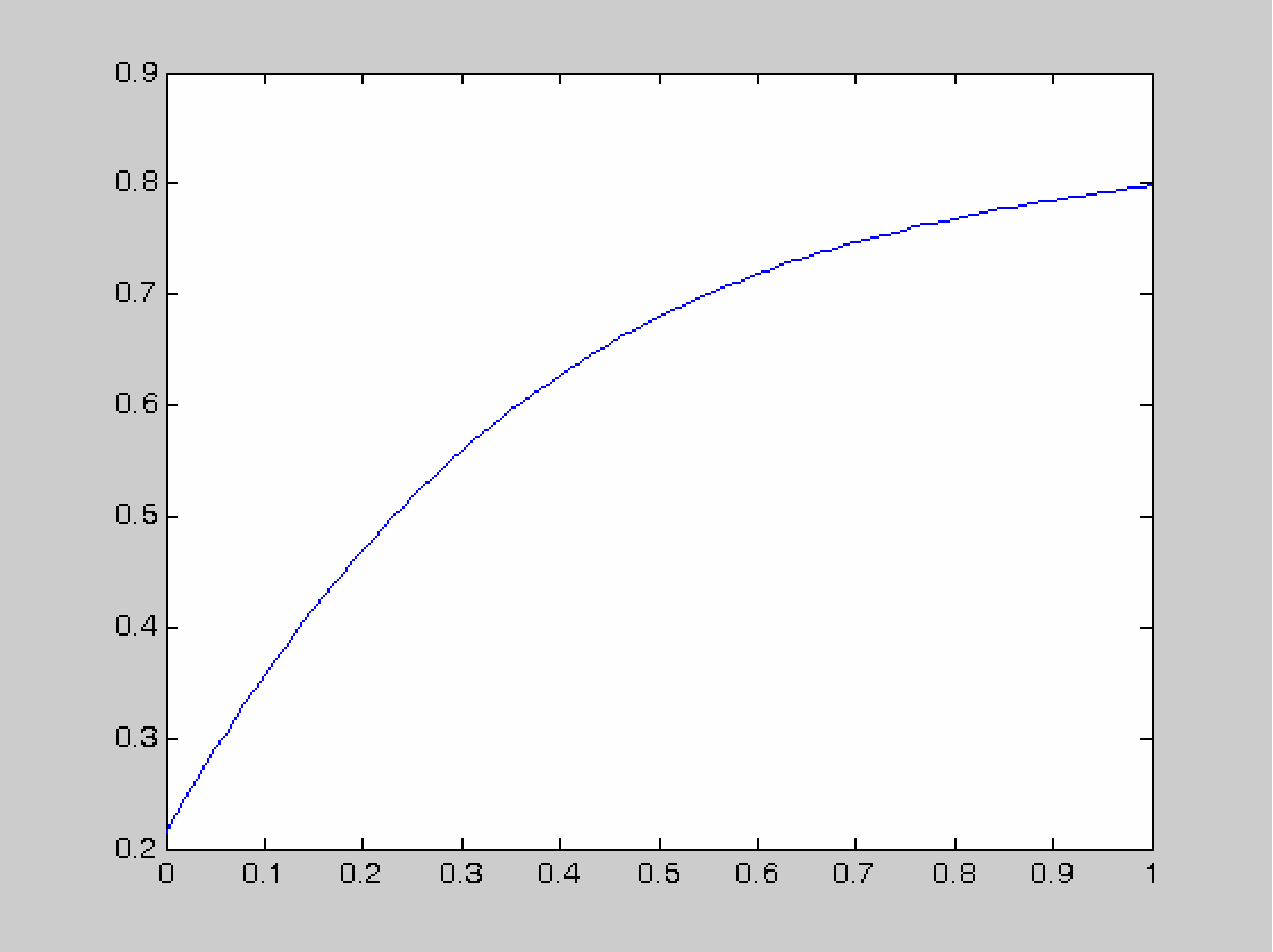
$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_i}{h} + y_i = x_i^{\ 2} \quad i = 1,2,\ldots,100$$

$$x_0 = 0,\ x_1 = 0.01, x_2 = 0.02, x_{99} = 0.99, x_{100} = 1$$

$$y_0 = 0.2,\ y_1 = ?, y_2 = ?, y_3 = ?, \quad y_{100} = 0.8$$

$$10000\left(y_{i+1} - 2y_i + y_{i-1}\right) + 200\left(y_{i+1} - y_i\right) + y_i = x_i^{\ 2}$$

$$10200y_{i+1} - 20199y_i + 10000y_{i-1} = x_i^{\ 2}$$


0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Solution of Boundary-Value Problems

Finite Difference Method

<u>Finite Difference Method :</u>

- Other formulas can be used for approximating the derivatives
- For some linear cases this reduces to tri-diagonal system.