SE301:Numerical Methods
Topic 6
Numerical Integration

Dr. Samir Al-Amer

Term 053

Lecture 17

Introduction to Numerical Integration

- Definitions
- Upper and Lower Sums
- Trapezoid Method
- Examples

Integration

Indefinite Integrals

$$\int x \, dx = \frac{x^2}{2} + c$$

Indefinite Integrals of a function are functions that differ from each other by a constant.

Definite Integrals

$$\int_0^1 x dx = \left. \frac{x^2}{2} \right|_0^1 = \frac{1}{2}$$

Definite Integrals are numbers.

Fundamental Theorem of Calculus

If f is continuous on an interval $[a,b]$,

F is antiderivative of f (i.e. $F'(x) = f(x)$)

$$\int_a^b f(x)dx = F(b) - F(a)$$

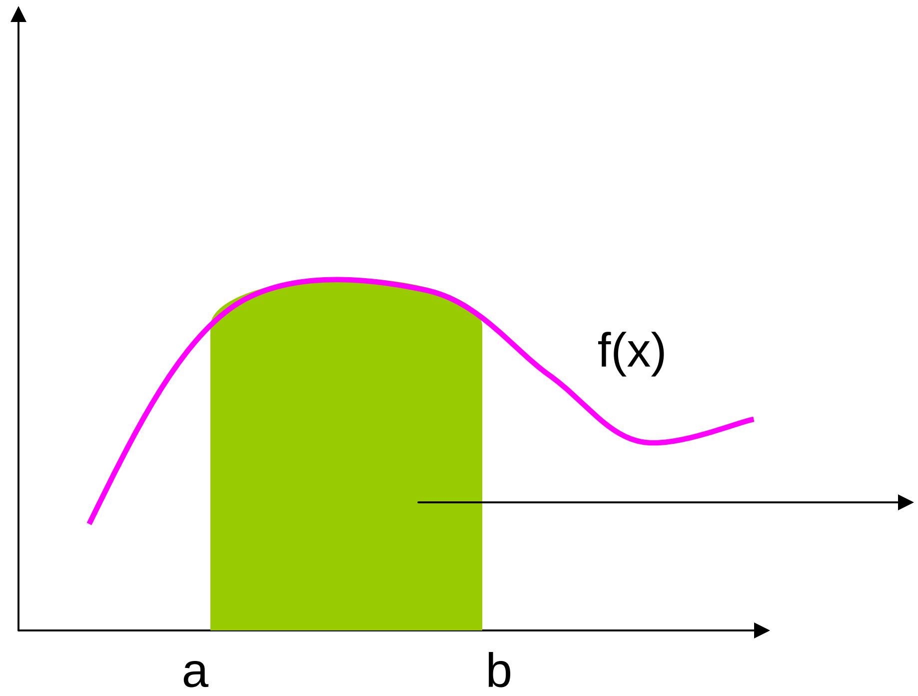
There is no antiderivative for e^{x^2}

No closed form solution for $\int_a^b e^{x^2} dx$

The Area Under the Curve

One interpretation of the definite integral is

Integral = area under the curve



$$Area = \int_a^b f(x)dx$$

Numerical Integration Methods

Numerical integration Methods Covered in this course

- ❑ Upper and Lower Sums
- ❑ Newton-Cotes Methods:
  - ➢ Trapezoid Rule
  - ➢ Simpson Rules
- ❑ Romberg Method
- ❑ Gauss Quadrature

Upper and Lower Sums

The interval is divided into subintervals

$$Partition\ P = \{a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b\}$$

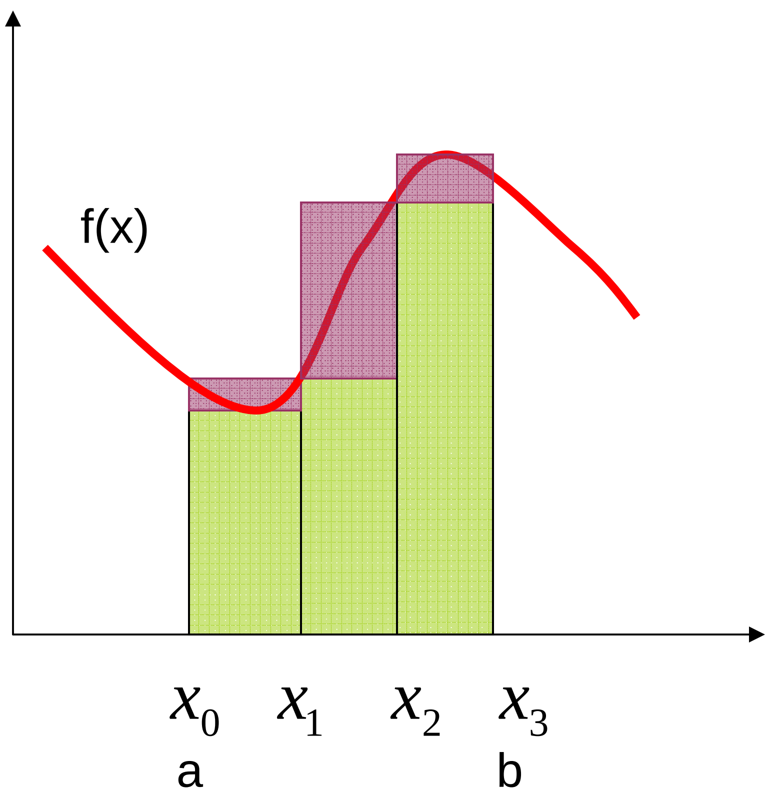
Define

$$m_i = \min\{f(x) : x_i \le x \le x_{i+1}\}$$

$$M_i = \max\{f(x) : x_i \le x \le x_{i+1}\}$$

Lower sum $$L(f,P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$$

Upper sum $$U(f,P) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$$



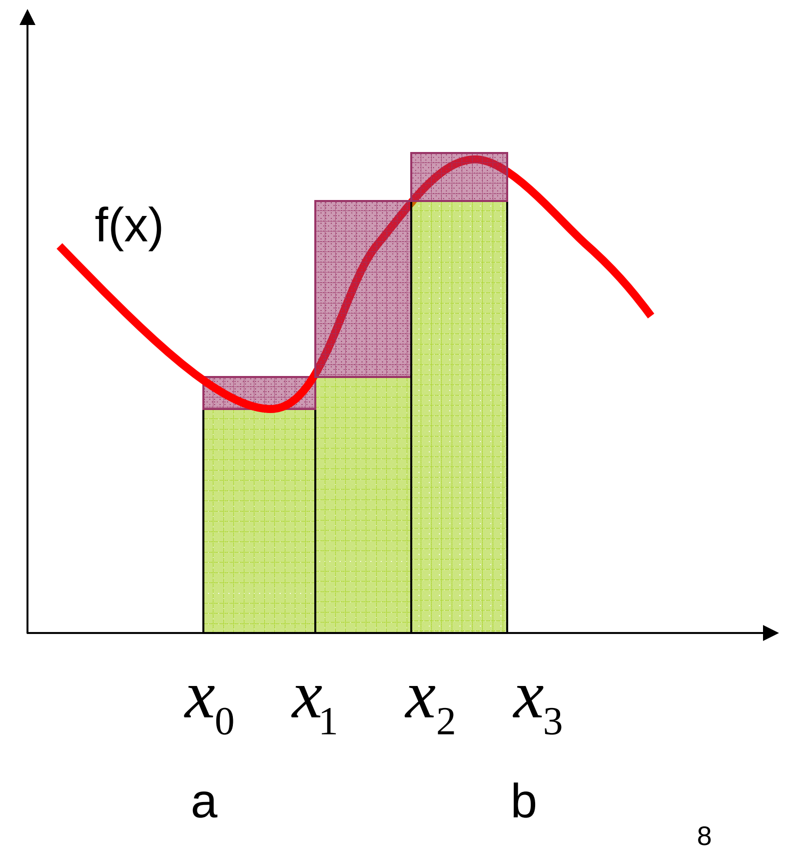
Upper and Lower Sums

Lower sum $$L(f,P)=\sum_{i=0}^{n-1} m_i\left(x_{i+1}-x_i\right)$$

Upper sum $$U(f,P)=\sum_{i=0}^{n-1} M_i\left(x_{i+1}-x_i\right)$$

$$\text{Estimate of the integral}=\frac{L+U}{2}$$

$$Error \le \frac{U-L}{2}$$



Example

$$\int_0^1 x^2 \, dx$$

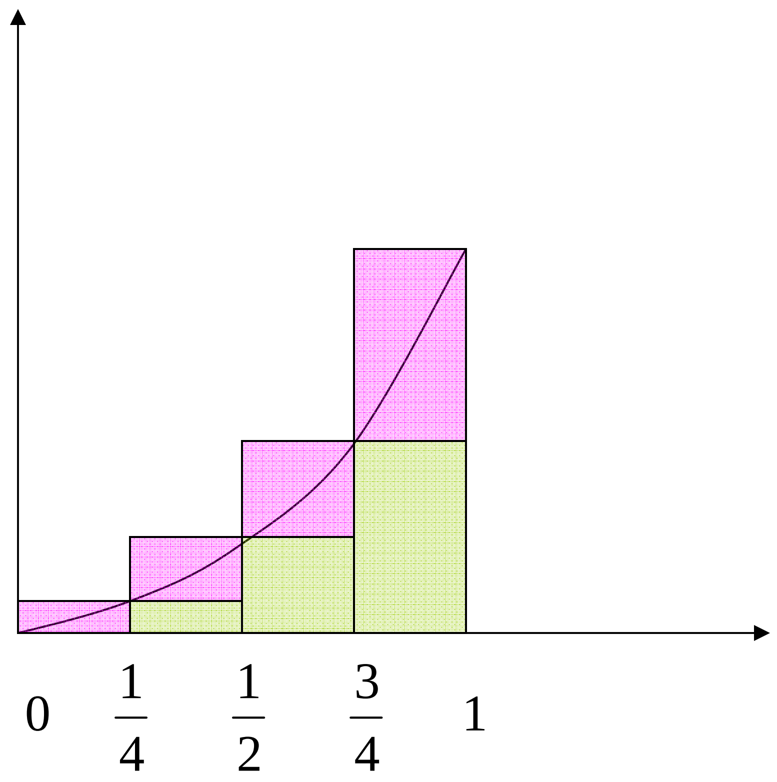
$$Partition\ P = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$$

$n = 4$ (four equal intervals)

$$m_0 = 0, \quad m_1 = \frac{1}{16}, \quad m_2 = \frac{1}{4}, \quad m_3 = \frac{9}{16}$$

$$M_0 = \frac{1}{16}, \quad M_1 = \frac{1}{4}, \quad M_2 = \frac{9}{16}, \quad M_3 = 1$$

$$x_{i+1} - x_i = \frac{1}{4} \qquad for \quad i = 0,1,2,3$$



Example

Lower sum $L(f,P)=\sum_{i=0}^{n-1} m_i\left(x_{i+1}-x_i\right)$

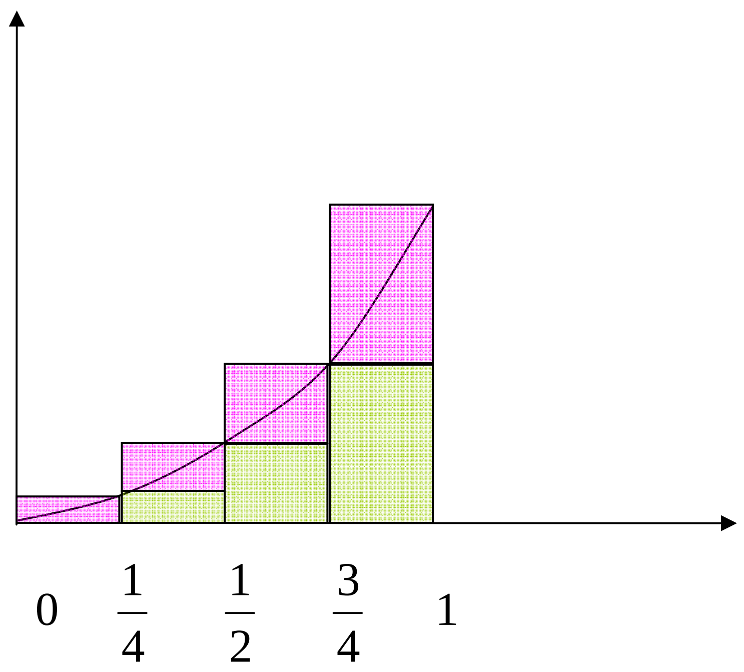
$$L(f,P)=\frac{1}{4}\left[0+\frac{1}{16}+\frac{1}{4}+\frac{9}{16}\right]=\frac{14}{64}$$

Upper sum $U(f,P)=\sum_{i=0}^{n-1} M_i\left(x_{i+1}-x_i\right)$

$$U(f,P)=\frac{1}{4}\left[\frac{1}{16}+\frac{1}{4}+\frac{9}{16}+1\right]=\frac{30}{64}$$

$$\text{Estimate of the integral}=\frac{1}{2}\left(\frac{30}{64}+\frac{14}{64}\right)=\frac{11}{32}$$

$$Error<\frac{1}{2}\left(\frac{30}{64}-\frac{14}{64}\right)=\frac{1}{8}$$



Upper and Lower Sums

- Estimates based on Upper and Lower Sums are easy to obtain for **<u>monotonic</u>** functions (always increasing or always decreasing).
- For non-monotonic functions, finding maximum and minimum of the function can be difficult and other methods can be more attractive.

Newton-Cotes Methods

- In Newton-Cote Methods, the function is approximated by a polynomial of order *n*
- Computing the integral of a polynomial is easy.

$$\int_a^b f(x)dx \approx \int_a^b \left(a_0 + a_1 x + \ldots + a_n x^n\right)dx$$

$$\int_a^b f(x)dx \approx a_0(b-a) + a_1 \frac{(b^2-a^2)}{2} + \ldots + a_n \frac{(b^{n+1}-a^{n+1})}{n+1}$$

Newton-Cotes Methods

- Trapezoid Method (First Order Polynomial are used)

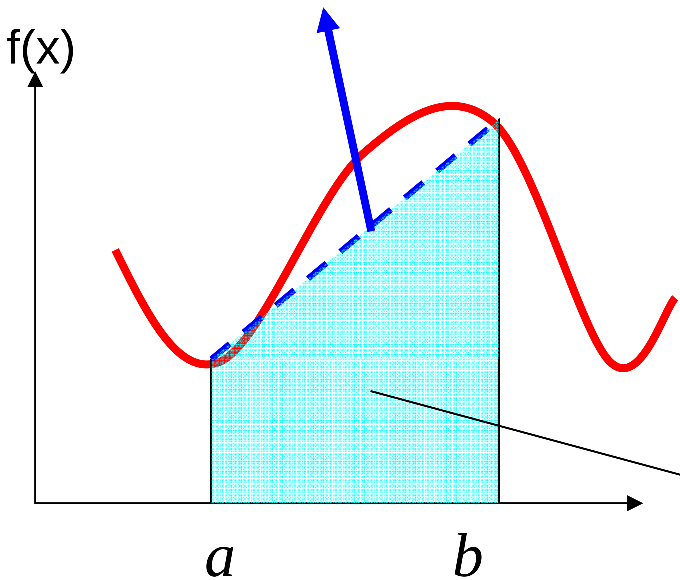
$$\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1 x)dx$$

- Simpson 1/3 Rule (Second Order Polynomial are used),

$$\int_a^b f(x)dx \approx \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

Trapezoid Method

$$f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$$



$$I=\int_a^b f(x)dx$$

$$I\approx\int_a^b\left(f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\right)dx$$

$$=\left(f(a)-a\frac{f(b)-f(a)}{b-a}\right)x\Big|_a^b$$

$$+\frac{f(b)-f(a)}{b-a}\frac{x^2}{2}\Big|_a^b$$

$$=(b-a)\frac{f(b)+f(a)}{2}$$

Trapezoid Method

Derivation-One interval

$$I = \int_a^b f(x)dx \approx \int_a^b \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right) dx$$

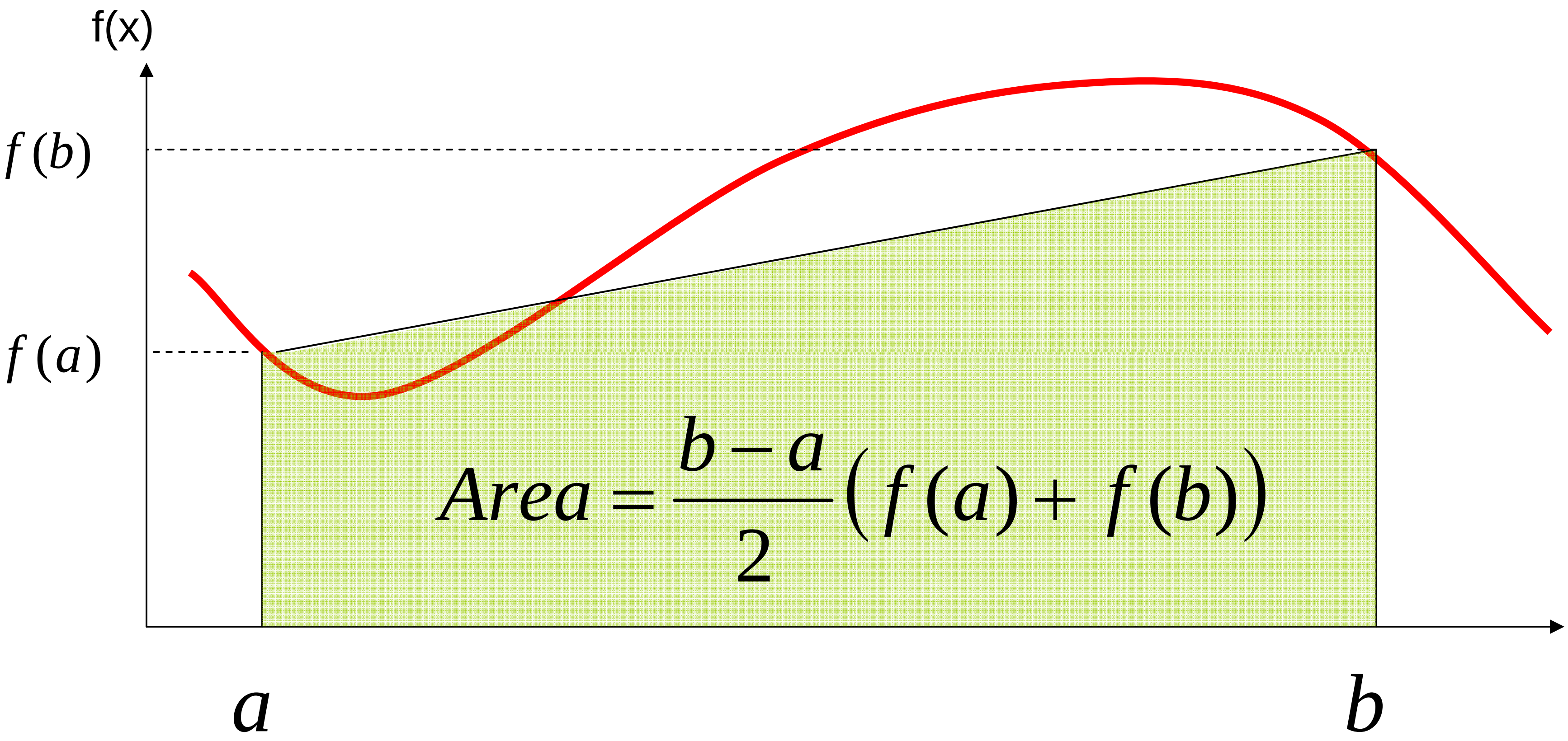
$$I \approx \int_a^b \left(f(a) - a\frac{f(b) - f(a)}{b - a} + \frac{f(b) - f(a)}{b - a} x \right) dx$$

$$= \left(f(a) - a\frac{f(b) - f(a)}{b - a} \right) x \Big|_a^b + \frac{f(b) - f(a)}{b - a} \frac{x^2}{2} \Big|_a^b$$

$$= \left(f(a) - a\frac{f(b) - f(a)}{b - a} \right) (b - a) + \frac{f(b) - f(a)}{2(b - a)} (b^2 - a^2)$$

$$= (b - a)\frac{f(b) + f(a)}{2}$$

Trapezoid Method



$$Area = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

Trapezoid Method

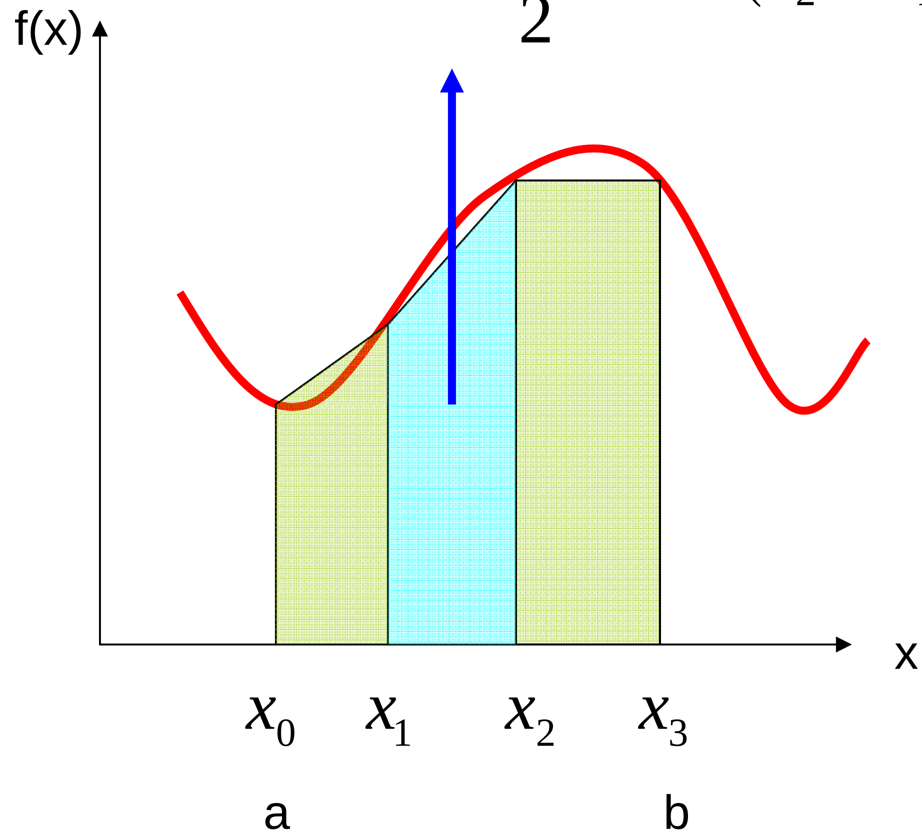
Multiple Application Rule

The interval $[a, b]$ is partitioned into n segments

$$a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b$$

$$\int_a^b f(x)dx = \text{sum of the areas of the trapezoids}$$

$$Area = \frac{f(x_2) + f(x_1)}{2}(x_2 - x_1)$$



Trapezoid Method

General Formula and special case

If the interval is divided into n segments(not necessarily equal)

$$a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b$$

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \frac{1}{2}\left(x_{i+1} - x_i\right)\left(f(x_{i+1}) + f(x_i)\right)$$

Special Case (Equaliy spaced base points)

$$x_{i+1} - x_i = h \quad \text{for all} \quad i$$

$$\int_a^b f(x)dx \approx h\left[\frac{1}{2}\left[f(x_0) + f(x_n)\right] + \sum_{i=1}^{n-1} f(x_i)\right]$$

Example

Given a tabulated values of the velocity of an object.

Obtain an estimate of the distance traveled in the interval [0,3].

Time (s)	0.0	1.0	2.0	3.0
Velocity (m/s)	0.0	10	12	14

Distance = integral of the velocity

$$\text{Distance} = \int_0^3 V(t)\, dt$$

Example 1

The interval is divided into 3 subintervals

Base points are $\{0,1,2,3\}$

Time (s)	0.0	1.0	2.0	3.0
Velocity (m/s)	0.0	10	12	14

Trapezoid Method

$$h = x_{i+1} - x_i = 1$$

$$T = h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\left(f(x_0) + f(x_n)\right)\right]$$

$$\text{Distance} = 1\left[(10+12) + \frac{1}{2}(0+14)\right] = 29$$

Estimating the Error

For Trapezoid method

How many equally spaced intervals are needed to compute $\int_0^{\pi} \sin(x)dx$ to 5 decimal digit accurcy?

Error in estimating the integral
Theorem

Assumption: $f''(x)$ is continuous on $[a,b]$

Equal intervals (width $= h$)

Theorem: If Trapezoid Method is used to approximate $\int_a^b f(x)dx$ then

$$Error = -\frac{b-a}{12} h^2 f''(\xi) \quad \text{where } \xi \in [a,b]$$

$$|Error| \le \frac{b-a}{12} h^2 \max_{x\in[a,b]} |f''(x)|$$

Example

$$\int_0^{\pi} \sin(x)dx, \quad find\ h\ so\ that\ |\text{error}| \le \frac{1}{2}\times 10^{-5}$$

$$|Error| \le \frac{b-a}{12}\, h^2 \max_{x\in[a,b]} |f''(x)|$$

$$b=\pi;\ a=0; \quad f'(x)=\cos(x); \quad f''(x)=-\sin(x)$$

$$|f''(x)| \le 1 \quad \Rightarrow |Error| \le \frac{\pi}{12} h^2 \le \frac{1}{2}\times 10^{-5}$$

$$\Rightarrow \qquad h^2 \le \frac{6}{\pi}\times 10^{-5}$$

Example

x	1.0	1.5	2.0	2.5	3.0
f(x)	2.1	3.2	3.4	2.8	2.7

Use Trapezoid method to Compute $\int_1^3 f(x)dx$

$$Trapezoid \quad T(f,P)=\sum_{i=0}^{n-1}\frac{1}{2}\left(x_{i+1}-x_i\right)\left(f(x_{i+1})+f(x_i)\right)$$

$$Special\ Case: \quad h=x_{i+1}-x_i \quad \text{for all} \quad i,$$

$$T(f,P)=h\left[\sum_{i=1}^{n-1}f(x_i)\ +\frac{1}{2}\left(f(x_0)+f(x_n)\right)\right]$$

Example

x	1.0	1.5	2.0	2.5	3.0
f(x)	2.1	3.2	3.4	2.8	2.7

$$\int_1^3 f(x)dx \approx h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\left(f(x_0) + f(x_n)\right)\right]$$

$$= 0.5\left[3.2 + 3.4 + 2.8 + \frac{1}{2}(2.1 + 2.7)\right]$$

$$= 5.9$$

SE301: Numerical Method

Lecture 18

Recursive Trapezoid Method

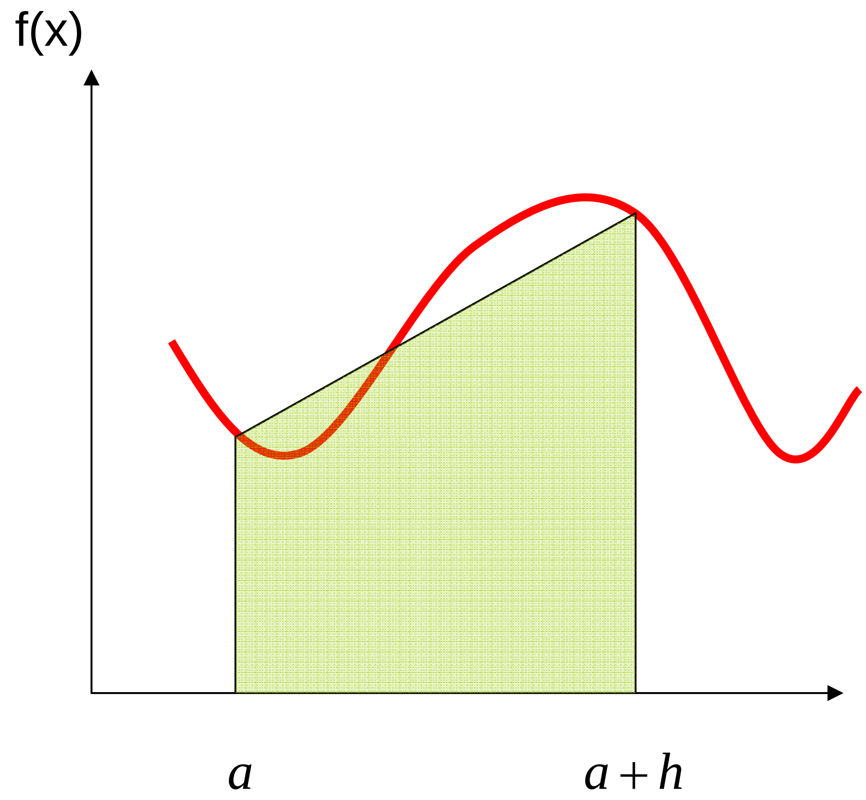
Recursive formula is used

Recursive Trapezoid Method

Estimate based on one interval

$$h = b - a$$

$$R(0,0) = \frac{b-a}{2}\left(f(a) + f(b)\right)$$



Recursive Trapezoid Method

Estimate based on 2 intervals

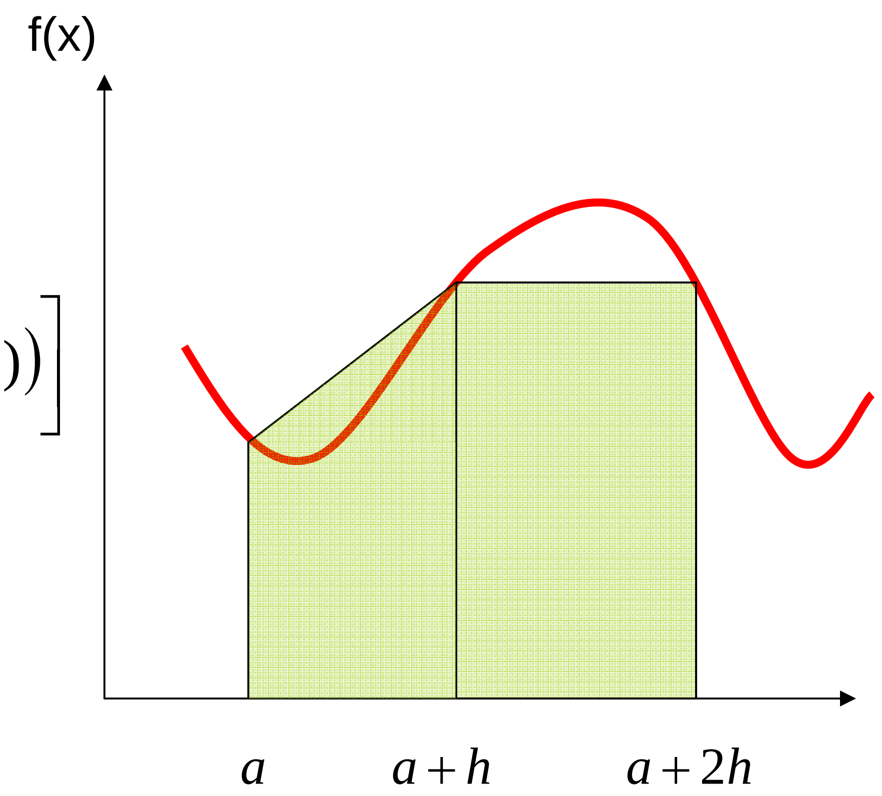
$$h = \frac{b-a}{2}$$

$$\mathrm{R}(1,0) = \frac{b-a}{2}\left[f(a+h) + \frac{1}{2}\left(f(a) + f(b)\right)\right]$$

$$R(1,0) = \frac{1}{2}R(0,0) + h\left[f(a+h)\right]$$

Based on previous estimate

Based on new point



Recursive Trapezoid Method

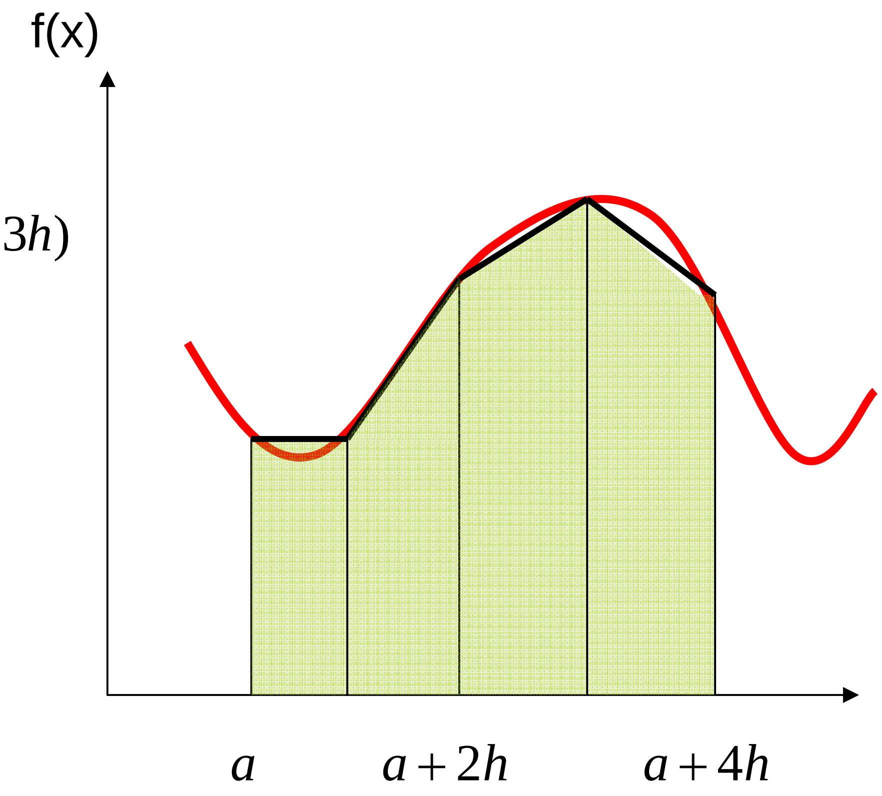
$$h = \frac{b-a}{4}$$

$$R(2,0) = \frac{b-a}{4}\Big[f(a+h) + f(a+2h) + f(a+3h) + \frac{1}{2}\big(f(a) + f(b)\big)\Big]$$

$$R(2,0) = \frac{1}{2}R(1,0) + h\big[f(a+h) + f(a+3h)\big]$$

Based on previous estimate

Based on new points



Recursive Trapezoid Method
Formulas

$$\mathrm{R}(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\big(a + (2k-1)h\big)\right]$$

$$h = \frac{b-a}{2^n}$$

Recursive Trapezoid Method

$$h = b - a, \qquad \mathrm{R}(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$h = \frac{b-a}{2}, \qquad R(1,0) = \frac{1}{2}R(0,0) + h\left[\sum_{k=1}^{1} f\big(a + (2k-1)h\big)\right]$$

$$h = \frac{b-a}{2^2}, \qquad R(2,0) = \frac{1}{2}R(1,0) + h\left[\sum_{k=1}^{2} f\big(a + (2k-1)h\big)\right]$$

$$h = \frac{b-a}{2^3}, \qquad R(3,0) = \frac{1}{2}R(2,0) + h\left[\sum_{k=1}^{2^2} f\big(a + (2k-1)h\big)\right]$$

....

$$h = \frac{b-a}{2^n}, \qquad R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\big(a + (2k-1)h\big)\right]$$

Advantages of Recursive Trapezoid

Recursive Trapezoid:

- Gives the same answer as the standard Trapezoid method.
- Make use of the available information to reduce computation time.
- Useful if the number of iterations is not known in advance.

SE301:Numerical Methods
19. Romberg Method

Motivation

Derivation of Romberg Method

Romberg Method

Example

When to stop?

Motivation for Romberg Method

- Trapezoid formula with an interval h gives error of the order $O(h^2)$
- *We can combine two Trapezoid estimates with intervals 2h and h to get a better estimate.*

Romberg Method

Estimates using Trapezoid method with intervals of size h, 2h, 4h,8h,...

are combined to improve the approximation of $\int_a^b f(x)\,dx$

First column is obtained using Trapezoid Method

R(0,0)			
R(1,0)	R(1,1)		
R(2,0)	R(2,1)	R(2,2)	
R(3,0)	R(3,1)	R(3,2)	R(3,3)

The other elements are obtained using the Romberg Method

First Column
Recursive Trapezoid Method

$$\mathrm{R}(0,0) = \frac{b-a}{2}\left[f(a) + f(b) \right]$$

$$R(n,0) = \frac{1}{2} R(n-1,0) + h \left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right) \right]$$

$$h = \frac{b-a}{2^n}$$

Derivation of Romberg Method

$$\int_a^b f(x)dx = R(n-1,0) + O(h^2) \qquad \text{Trapezoid method} \quad with\ h = \frac{b-a}{2^{n-1}}$$

$$\int_a^b f(x)dx = R(n-1,0) + a_2h^2 + a_4h^4 + a_6h^6 + \ldots \qquad (eq1)$$

More accurate estimate is obtained by R(n,0)

$$\int_a^b f(x)dx = R(n,0) + \frac{1}{4}a_2h^2 + \frac{1}{16}a_4h^4 + \frac{1}{64}a_6h^6 + \ldots \qquad (eq2)$$

$$eq1 - 4 * eq2 \qquad gives$$

$$\int_a^b f(x)dx = R(n,0) + \frac{1}{3}\left[R(n,0) - R(n-1,0)\right] + b_4h^4 + b_6h^6 + \ldots$$

Romberg Method

$$R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$h = \frac{b-a}{2^n},$$

R(0,0)			
R(1,0)	R(1,1)		
R(2,0)	R(2,1)	R(2,2)	
R(3,0)	R(3,1)	R(3,2)	R(3,3)

$$R(n,0) = \frac{1}{2}R(n-1,0) + h\left[\sum_{k=1}^{2^{(n-1)}} f\left(a + (2k-1)h\right)\right]$$

$$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}\left[R(n,m-1) - R(n-1,m-1)\right]$$

$$for \quad n \geq 1, \quad m \geq 1$$

Property of Romberg Method

Theorem

$$\int_a^b f(x)dx = R(n,m) + O(h^{2m+2})$$

R(0,0)			
R(1,0)	R(1,1)		
R(2,0)	R(2,1)	R(2,2)	
R(3,0)	R(3,1)	R(3,2)	R(3,3)

Error Level $O(h^2)$ $O(h^4)$ $O(h^6)$ $O(h^8)$

Example 1

0.5	
3/8	1/3

Compute $\int_0^1 x^2 dx$

$$h = 1,\ \ R(0,0) = \frac{b-a}{2}\left[f(a) + f(b)\right] = \frac{1}{2}\left[0+1\right] = 0.5$$

$$h = \frac{1}{2},\ R(1,0) = \frac{1}{2}R(0,0) + h(f(a+h)) = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{3}{8}$$

$$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}\left[R(n,m-1) - R(n-1,m-1)\right]$$

$$for \quad n \geq 1, \quad m \geq 1$$

$$R(1,1) = R(1,0) + \frac{1}{4^1 - 1}\left[R(1,0) - R(0,0)\right] = \frac{3}{8} + \frac{1}{3}\left[\frac{3}{8} - \frac{1}{2}\right] = \frac{1}{3}$$

Example 1 cont.

0.5		
3/8	1/3	
11/32	1/3	1/3

$$h = \frac{1}{4},\ R(2,0) = \frac{1}{2}R(1,0) + h(f(a+h) + f(a+3h)) = \frac{1}{2}\left(\frac{3}{8}\right) + \frac{1}{4}\left(\frac{1}{16} + \frac{9}{16}\right) = \frac{11}{32}$$

$$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}\left[R(n,m-1) - R(n-1,m-1)\right]$$

$$R(2,1) = R(1,0) + \frac{1}{4^1 - 1}\left[R(2,0) - R(1,0)\right] = \frac{11}{32} + \frac{1}{3}\left[\frac{11}{32} - \frac{3}{8}\right] = \frac{1}{3}$$

$$R(2,2) = R(2,1) + \frac{1}{4^2 - 1}\left[R(2,1) - R(1,1)\right] = \frac{1}{3} + \frac{1}{15}\left[\frac{1}{3} - \frac{1}{3}\right] = \frac{1}{3}$$

When do we stop?

$$STOP \quad if$$

$$\left| R(n, m-1) - R(n-1, m-1) \right| \leq \varepsilon$$

or

after a given number of steps

for example STOP at R(4,4)

SE301:Numerical Methods

20. Gauss Quadrature

Motivation

General integration formula

Read 22.3-22.3

Motivation

Trapezoid Method

$$\int_a^b f(x)dx = h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\big(f(x_0) + f(x_n)\big)\right]$$

It can be expressed as

$$\int_a^b f(x)dx = \sum_{i=0}^{n} c_i \;\; f(x_i)$$

$$where \quad c_i = \begin{cases} h & i = 1,2,\ldots,n-1 \\ 0.5\,h & i = 0 \;\; and \;\; n \end{cases}$$

General Integration Formula

$$\int_a^b f(x)dx = \sum_{i=0}^{n} c_i \; f(x_i)$$

c_i : *Weights* x_i : *Nodes*

Problem :

How do we select c_i *and* x_i so that the formula gives good approximation of the integral

Lagrange Interpolation

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx$$

where $P_n(x)$ is a polynomial that interpolate f(x)

at the nodes $x_0, x_1, ..., x_n$

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx = \int_a^b \left(\sum_{i=0}^{n} \ell_i(x) f(x_i) \right) dx$$

$$\Rightarrow \int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i \; f(x_i) \quad \textit{where} \;\; c_i = \int_a^b \ell_i(x)\, dx$$

Question

What is the best way to choose the nodes and the weights?

Theorem

Let q be a nontrivial polynomial of degree $\mathrm{n}+1$ such that

$$\int_a^b x^k q(x)dx = 0 \qquad 0 \le k \le n$$

Let $x_0, x_1, x_2, ..., x_n$ are the zeros of $q(x)$ then

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i) \quad where \quad c_i = \int_a^b \ell_i(x)dx$$

The formula will be exact for all polynomials of order $\le 2\mathrm{n}+1$

Weighted Gaussian Quadrature

Theorem

Let q be a nontrivial polynomial of degree $n+1$ such that

$$\int_a^b x^k \; q(x) \; w(x)\, dx = 0 \qquad\qquad 0 \le k \le n$$

Let $x_0, x_1, x_2, ..., x_n$ are the zeros of $q(x)$ then

$$\int_a^b f(x)\, w(x)\, dx \approx \sum_{i=0}^{n} c_i\, f(x_i) \quad where$$

$$c_i = \int_a^b \ell_i(x) w(x)\, dx, \qquad\qquad \ell_i(x) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{x - x_j}{x_i - x_j}$$

The formula will be exact whenever *f(x)* is a polynomial of order $\le 2n+1$

Determining The Weights and Nodes

$$\int_{-1}^{1} f(x)\, dx = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$$

How do we select the nodes and the weights so that the formula is exact for all polynomials of order ≤ 5?

Determining The Weights and Nodes

Solution

$Let\ \ q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$q(x)$ must satisfy

$$\int_{-1}^{1} q(x)dx = 0$$

$$\int_{-1}^{1} q(x)xdx = 0$$

$$\int_{-1}^{1} q(x)x^2 dx = 0$$

one possible solution $a_0 = a_2 = 0, a_1 = -3,\ a_3 = 5$

Theorem

Let q be a nontrivial polynomial of degree $\mathrm{n}+1$ such that

$$\int_a^b x^k q(x)dx = 0 \qquad 0 \le k \le n$$

Let $x_0, x_1, x_2, \ldots, x_n$ are the zeros of $q(x)$ then

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i) \quad where \quad c_i = \int_a^b \ell_i(x)dx$$

The formula will be exact for all polynomials of order $\le 2\mathrm{n}+1$

Determining The Weights and Nodes Solution

$$Let\ \ q(x) = -3x + 5x^3$$

$$roots\ \ \ of\ \ q(x)\ \text{are}\ \ \ 0, \pm\sqrt{\frac{3}{5}}$$

$$\text{The nodes are } x_0 = -\sqrt{\frac{3}{5}}, \quad x_1 = 0, \, x_2 = \sqrt{\frac{3}{5}}$$

To obtain the weights we use

$$\int_{-1}^{-1} f(x)dx = A_0 f\left(-\sqrt{\frac{3}{5}}\right) + A_1 f\left(0\right) + A_2 f\left(\sqrt{\frac{3}{5}}\right)$$

Determining The Weights and Nodes
Solution

The nodes are $x_0 = -\sqrt{\frac{3}{5}}, \quad x_1 = 0, x_2 = \sqrt{\frac{3}{5}}$

The weights are $A_0 = \frac{5}{9}, \quad A_1 = \frac{8}{9}, A_2 = \frac{5}{9}$

The Gauss Quadrature (n = 2)

$$\int_{-1}^{1} f(x)dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

Gaussian Quadrature

See more in Table 22.1 (page 626)

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{n} c_i \; f(x_i)$$

$n=1$ $x_0 = -0.57735,\; x_1 = 0.57735$

$c_0 = 1,\; c_1 = 1$

$n=2$ $x_0 = -0.774596,\; x_1 = 0.000000,\; x_2 = 0.774596$

$c_0 = 0.555556,\; c_1 = 0.888889,\; c_2 = 0.555556$

$n=3$ $x_0 = -0.86113,\; x_1 = -0.33998,\; x_2 = 0.33998,\; x_3 = 0.86113$

$c_0 = 0.34785,\; c_1 = 0.65214,\; c_2 = 0.65214,\; c_3 = 0.34785$

$n=4$ $x_0 = -0.906179,\; x_1 = -0.538469,\; x_2 = 0.00000,\; x_3 = 0.538469, x_4 = 0.906179$

$c_0 = 0.236926,\; c_1 = 0.478628,\; c_2 = 0.568889,\; c_3 = 0.478628, c_4 = 0.236926$

Error Analysis for Gauss Quadrature

Let the integral $\int_a^b f(x)dx$ be approximated by

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i)$$

where c_i *and* x_i are selected according to

Gauss Quadrature formula then the true error satisfies

$$\text{Error} = \frac{2^{2n+3}\left[(n+1)!\right]^4}{(2n+3)\left[(2n+2)!\right]^3} \quad f^{(2n+2)}(\xi), \qquad \xi \in [-1,1]$$

The formula will be exact for all polynomials of order $\leq 2n+1$

Example

Evaluate $\int_0^1 e^{-x^2} dx$ using Gaussian Quadrature with $\text{n} = 1$

$$\int_{-1}^{1} f(x)dx = f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right)$$

How do we use the formula to compute $\int_a^b f(x)dx$

for arbitrary a and b.

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)dt$$

Example

$$\int_0^1 e^{-x^2} dx = \frac{1}{2} \int_{-1}^{1} e^{-(.5t+.5)^2} dt$$

$$= \frac{1}{2}\left[e^{-\left(-0.5\sqrt{\frac{1}{3}}+.5\right)^2} + e^{-\left(0.5\sqrt{\frac{1}{3}}+.5\right)^2} \right]$$

Improper Integrals

Methods discussed earlier can not be used directly to approximate improper integrals (one of the limits is ∞ or $-\infty$)

$\Rightarrow$ Use a transformation like the following

$$\int_{a}^{b} f(x) = \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt, \quad (\text{assuming } ab > 0)$$

and apply the method on the new function

Example:

$$\int_{1}^{\infty} \frac{1}{x^2}\, dx = \int_{0}^{1} \frac{1}{t^2} \left(\frac{1}{\left(\frac{1}{t}\right)^2} \right) dx$$

Quiz

x	1.0	1.5	2.0	2.5	3.0
f(x)	2.1	3.2	3.4	2.8	2.7

Use Trapezoid method to Compute $\int_1^3 f(x)dx$

$$Trapezoid \quad T(f,P) = \sum_{i=0}^{n-1} \frac{1}{2}(x_{i+1} - x_i)(f(x_{i+1}) + f(x_i))$$

$$Special\ Case: \quad h = x_{i+1} - x_i \quad \text{for all} \quad i,$$

$$T(f,P) = h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}(f(x_0) + f(x_n))\right]$$