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\chapter{VECTORS AND FRAMES}

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This chapter introduces the basic concepts of vectors and frames in the three
dimensional space $R^3$.  A frame of reference is defined by using
three orthonormal unit vectors.  This allows deriving the elementary
rotation matrices that define the relation between two frame of
references.  These rotation matrices have interesting properties with
regards to the evaluation of their determinant and their inverse
transformation.  The properties of these rotation
matrices will be outlined.  Further, we obtain the geometric vector
relations between a moving frame $R_1$ and a fixed $R_0$.\\

Next, we introduce the notion of robot arm at the topological
and mathematical levels. Both levels will be dealing with the
geometical description.  A robot arm is intrdouced as
a series of links that are indterconnected by means of joints.  Two
types of joints are used for
interconnecting the robot links.  The topology of robot
arms is decomposed into two subsystems: the
transporter and the effector.  The transporter is the
robot positioning
subsystem,
while the effector is the orienting subsystem.  Several
types of transporter geometry are studied by
the well known coordinate systems.\\

A second look on the orientation matrices is presented with the
objective to investigate the mathematical transformations that
are required for each type of interconnection topology.  Finally, we study
the properties of the global (composite) rotation matrix
of a moving frame.  This presents an extension to the
properties of the set of rotation matrices which has been
studied earlier.  Throughout this book the sentence "frame of reference",
or simply "frame" designate the same
meaning.


\section{Three dimensional space vectors}
A cartesian coordinate system is defined, in the three dimensional
space, by introducing
three orthonormal vectors $X, Y,$ and $Z$.  Figure 2.1 shows a cartesian
coordinate system which is defined using three orthonormal vectors
$X,Y$, and $Z$.  The set of vectors $\{X,Y,Z\}$ is
referred to as orthonormal when the vectors are
orthogonal and each has unit of length.  These
unit vectors can be expressed respectively as:
\begin{eqnarray}
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X & = & 1.X+0.Y+0.Z \nonumber \\
Y & = & 0.X+1.Y+0.Z  \\
Z & = & 0.X+0.Y+1.Z \nonumber
\end{eqnarray}
To simplify this notation, we can write them in the following form:
\begin{eqnarray}
X&=& (1,0,0)^t \nonumber \\
Y&=& (0,1,0)^t  \\
Z&=& (0,0,1)^t \nonumber
\end{eqnarray}
where $(t)$ denotes the transposition.
A cartesian frame of reference $R$ is defined
by using the
three
orthonormal vectors $\{X,Y,Z\}$
and their intersecting origin $O$:
\begin{equation}
R = \{ X,Y,Z  \; \mbox{and} \; 0 \}
\end{equation}
Figure 2.2 shows a cartesian frame of
reference defined by three orthonormal vectors
$\{ X,Y,Z\}$ and $O$ as origin.  Observed in frame
$R$, a point $A$ can be associated a vector $OA_R$,
where the subscript $\theta _R$ denotes
that the vector is observed in
frame $R$.  Let the scalars
$A_x,A_y$, and $A_z$ denotes
the projection of vector $OA_R$ on the
axis $X,Y$, and $Z$ of frame
$R$, respectively.  Therefore we can express
the vector as follows:
\begin{equation}
OA_R = A_x.X+A_y.Y+A_z.Z
\end{equation}
Figure 2.3 shows point $A$ which is observed in frame $R$.
This notation can be shortly reduced to:
\begin{equation}
OA_R = (A_x A_yA_z)^t
\end{equation}
More generally, a vector $U$ whose components are
$U_x, U_y$ and $U_z$ with
respect to axes $X$, $Y$, and $Z$ of
frame $R$ can be expressed as follows:
\[
U = (U_x U_y U_z)^t
\]
{\bf Definition}: The norm or length
of a vector $U$ is denoted
as $|U|$ and is
defined by:
\begin{equation}
|U| = \sqrt{ U_x^2 + U_y^2 + U _z^2 }
\end{equation}
{\bf Definition}: The dot product or scalar product of two
vectors $U$ and $V$ which are
observed in frame $R$, is a scalar defined by:
\[
```

```
U^t. V = U_x . V_x + U_y . V_y + U_z . V_z
\]
The dot product is proportional to the cosine of the
angle between the two vectors and their lengths.  Let $\theta$
be the angle between the vectors $U$ and $V$,
the dot product can then be expressed as follows:
\begin{equation}
U^t . V = |U|.|V|.cos(\theta)
\end{equation}
The dot product
$U^t.V$ is commutative law because:
\begin{equation}
U^t.V = V^t.U = |U|.|V|cos(\theta)=|V||U|.cos(-\theta)
\end{equation}
The dot product $U^t.U$ is nil if the vector $U$ is
nil.  When the vector $U$ is not nil, then the
dot product is always a positive scalar.  The dot product
of two non nil and non-colinear vectors $U$
and $V$ is nil if
and only vectors $U$ and $V$ are
orthogonal.  In this case the angle $\theta$ between
these vectors is equal to $\pm \prod /2$ , i.e. $cos(\theta )=0$.

The dot product is linear function
because we have:
\[
(\alpha.U+\beta.V).W=\alpha (U.W)+\beta.(V.W)
\]
where $\alpha$ and $\beta$ are two arbitrary scalars. \\
\ \\
{\bf Definition}: The cross product of two vectors $U$ and $V$ , that
are observed in framed $R$, is
a vector $U\times V$ which is orthogonal to $U$
and $V$ in the right-handed sense.  The
cross product vector is defined as follows:
\begin{equation}
U\times V =
\left[
\begin{array}{c}
U_x \\
U_y \\
U_z
\end{array}
\right]
\times
\left[
\begin{array}{c}
V_x \\
V_y \\
V_z
\end{array}
\right]
=
\left[
\begin{array}{c}
U_y.V_z-U_z.V_y \\
U_z.V_x-U_x.V_z \\
```

```
U_x.V_y-U_y.V_x
\end{array}
\right]
\end{equation}
The cross product $U \times V$ can also be
expressed as function off the product of
the vector lengths and the sine of the
angle from $U$ to $V$:
\begin{equation}
|U \times V| = |U|.|V|.sin(\theta)
\end{equation}
Figure 2.4 shows vectors $U$ and $V$ and the
vector that defines their cross product.  The
cross product $U \times V$ defines a vector
$W=U \times V$ which is
orthogonal to both vectors $U$ and $V$,
we can easily verify that:
\[
W^t.U = W^t.V=0
\]

\section{Frame rotation}
The prime interest of finding the
rotation matrix between two frame of references
$R_1$ and $R_0$ is to determine a
transfer
matrix which
allows converting the coordinate of a
vector, that is observed in a frame,
into the coordinate of the
same vector with respect to the
other frame.  We assume the
frame
$R_1 = \{ X_1.Y_1.Z_1 \; \mbox{and} \; 0_1\}$
is subject to some motion with
respect to a fixed reference coordinate
system
$R_0 = \{ X_0.Y_0.Z_0 \; \mbox{and} \; \theta_0\}$

Initially we assume the origin
$\theta_1$ coincides with $\theta_0$ and
vectors $X_1,Y_1$ and $Z_1$ coincide
with vectors $X_0,Y_0$ and $Z_0$, respectively.
Figure 2.5 shows two initially coinciding frame
of references. \\

Assume the frame $R_1$ is rotated by an
angle $\theta$ about the $Z_0$ axis.  As frame
$R_1$ is only
rotated, both origin still coincide but the orientation
of $R_1$ is now changing as a result of
the rotation.  Figure 2.6 shows the
frame $R_1$ after rotating this frame
by an angle $\theta$ about axis $Z$
can be expressed with respect to
frame $R_0$ as follows:
\begin{eqnarray}
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X_{1,0} & = & (cos(\theta)\,\, sin(\theta ) \,\, 0)^t \nonumber \\
Y_{1,0} & = & (-sin(\theta) \, cos(\theta) \,\, 0)^t \nonumber \\
X_{1,0} & = & (0 \,\, 0 \,\, 1)^t \nonumber
\end{eqnarray}
where $X_{1,0}$ , $Y_{1,0}$, and $Z_{1,0}$ denote
the orthonormal vectors of frame $R_1$.
Note that the first index is used as the
vector identifier and the second index is used
to state that these vectors are
observed with respect to frame $R_0$.

Considering now a vector $O_1A$ whose components are
$A_{x1}$, $A_{y1}$, and $A_{z1}$ with respect to
frame $R_1$.  By definition we have:
\begin{equation}
O_1A_{R1}=A_{x1}.X_{1,0} + A_{y1}.Y_{1,0} + A_{z1}.Z_{1,0}
\end{equation}
To express this vector in $R_0$ we can substitute
the vectors $X_{1,0}$, $Y_{1,0}$, and
$Z_{1,0}$ with their
expressions that give the coordinate of
that vector with respect to
$R_0$:
\begin{eqnarray}
O_0A_{R0} & = & A_{x1}.(cos(\theta ). X_{0,0} + sin (\theta ).
Y_{0,0})+\nonumber \\
        &   & A_{y1}.(-sin(\theta ).X_{0,0}+cos(\theta ).Y_{0,0})+A_{z1}.Z_{0,0}
\end{eqnarray}
where vectors $X_{0,0}$, $Y_{0,0}$, and $Z_{0,0}$ are
the orthonormal vectors of frame $R_0$ which are
observed with respect to the same frame.  By grouping the
terms with respect to vectors
$Z_{0,0}$, $Y_{0,0}$ and $Z_{0,0}$ ob obtain:
\begin{eqnarray}
O_0 A_{R0} & = & ( A_{x1}. cos (\theta )-A_{y1}.sin(\theta)).X_{0,0} + \nonumber
\\
         &   & (A_{x1}.sin(\theta)+A_{y1}.cos (\theta )).Y_{0,0}+ A_{z1}.Z_{0,0}
\end{eqnarray}
Vector $O_oA$ is represented in $R_0$ by the general rleation:
\[
O_0A_{R0} = A_{x0}.X_0 + A_{y0}.Y_0+ A_{z0}.Z_0
\]
This vector can be expressed in matrix form:
\[
O_0A_{R0}=
\left[
\begin{array}{ccc}
cos(\theta ) & - sin(\theta) & 0 \\
sin(\theta ) & cos(\theta) & 0 \\
0 & 0 & 1
\end{array}
\right]
.
\left[
\begin{array}{c}
A_{x1} \\
A_{y1} \\
```

```
A_{z1}
\end{array}
\right]
=
\left[
\begin{array}{c}
A_{x0} \\
A_{y0} \\
A_{z0}
\end{array}
\right]
\]
The transfer matrix from $R_1$ into frame $R_0$
represents the rotation matrix about the
$Z_0$ axis by an angle $\theta$.  The
rotation matrix is denoted by $ROTZ(\theta)$,
we have:
\[
O_0A_{R0} = ROTZ(\theta).O_1A_{R1}
\]
This matrix can be derived from the expression
of vectors  $X_{1,0}, Y_{1,0}$, and $Z_{1,0}$.  It is
important to note that the columns of $ROTZ(\theta)$ are
the vectors $X_{1,0},Y_{1,0}$ and
$Z_{1,0}$, respectively.
\[
ROTZ(\theta) =
\left[
\begin{array}{ccc}
cos(\theta) & - sin(\theta) & 0 \\
sin(\theta) &  cos(\theta) & 0 \\
0 & 0 & 1
\end{array}
\right]
=
\left[
\begin{array}{ccc}
C(\theta) & -S(\theta) & 0 \\
S(\theta) & C(\theta) & 0 \\
0 & 0 & 1
\end{array}
\right]
\]
where $C(\theta)$ denotes $cos(\theta)$ and $S(\theta)$
is $sin(\theta)$, respectively.  Similarly, we denote
by $ROTX(\alpha)$ and $ROTY(\beta)$ the
rotation matrices about the $X_0$ by an angle
$\alpha$ and $Y_0$ by an
angle $\beta$, respectively.  Figures 2.7 and 2.8
show the effect on frame $R_1$ of these two
rotations.  These figures assume that in both cases
the frame $R_1$ was
initially coinciding with
the fixed frame of reference, we have therefore:
\begin{equation}
ROTX(\alpha) = \left[
\begin{array}{ccc}
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1&0&0  \\
0&C(\alpha)&-S(\alpha) \\
0&S(\alpha) & C(\alpha) \\
\end{array}
\right]
\end{equation}
The rotation matrix $ROTY(\beta)$ is defined by:
\begin{equation}
ROTY(\beta) = \left[
\begin{array}{ccc}
C(\beta) & 0 & S(\beta) \\
0&1&0 \\
-S(0)&0&C(\beta)
\end{array}
\right]
\end{equation}
The three rotation matrices
$ROTX(\alpha)$, $ROTY(\beta)$, and $ROTZ(\theta)$ define
the set of fundamental rotations in the
cartesian coordinate system.  Note that
every rotation matrix becomes the identity matrix
whenever the rotation angle becomes nil.

\section{Rotation matrices}
For ease of discussion we denote by $ROTW$ an
arbitrary rotation matrix, where $W$ may
be $X$, $Y$ or $Z$.  The rotation matrices
define the set $S=\{ROTW:W=X,Y,Z\}$ of
fundamental rotations in the three dimensional space.  There
are three fundamental properties for the set $S$ of
rotation matrices:
\begin{itemize}
\item
The determinant
\item
The inverse
\item
The product
\end{itemize}
The {\bf determinant} of any rotation matrix
$ROTW(\alpha)$, of the set $S$, is
equal to the unit:
\[
det(ROTW(\alpha))=1 \; \; \; \; \mbox{for} \; \; W:X,Y \; \mbox{or} \; Z
\]
We can easily see that for an arbitrary angle $\alpha$ and; for $W=X,Y$, or $Z$,
we
have:
\begin{equation}
det (ROTW(\alpha)) = S(\alpha).S(\alpha) + C(\alpha).C(\alpha)=1
\end{equation}
The {\em inverse matrix} of every rotation
matrix $ROTW(\alpha)$, of
the set $S$, is a matrix
whose transpose is identical to $ROTW(\alpha)$,
\begin{equation}
[ROTW(\alpha)]^{-1}=[ROTW(\alpha)]^t
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\end{equation}
where $\theta^t$ denotes the transpose of a matrix.
To prove this relation, consider a
matrix $M$ and let $M^{-1}$ denotes its
inverse matrix, we have:
\begin{equation}
M^{-1} = \frac{1} {det(M)}[Co(M)]^t
\end{equation}
where $Co(M)$ denotes the co-factor matrix associated to $M$.  Using
this formula for a rotation matrix $ROTW$, we obtain:
\begin{equation}
[ROTW(\alpha)]^{-1} = [Co(ROTW)]^t
\end{equation}
Note that the determinant $det(ROTW(\alpha))=1$ and
the co-factor matrix associated with any
matrix of the set $S$ is identical to that matrix.  The
inverse matrix can then be obtained by means of
a transposition:
\begin{equation}
[ROTW(\alpha)]^{-1} = [Co(ROTW)]^t
\end{equation}
Assume a rotation matrix $ROTW(\alpha)$ represents
the transfer matrix between frame $R_1$ can be expressed
with
respect to $R_0$ by the equation:
\begin{equation}
U_{R0} = ROTW (\alpha).U_{R1}
\end{equation}
Figure 2.9 shows the correspondence between
frames $R_0$ and $R_1$.  The
inverse rotation matrix $[ROTW(\alpha)]^t$ is the
transfer matrix from frame $R_{R0}$ to
frame $R_{R1}$, we
have:
\[
R_{R1} = [ROTW(\alpha)]^t.U_{R0}
\]
Consider a rotation matrix $ROTW(\alpha)$, we note that\
the extra-diagonal terms of that
matrix depend on the anti-symmetric function $sin(\alpha)$.  The
transpose matrix of an arbitrary rotation matrix $ROTW(\alpha)$ is
then equal to $ROTW(-\alpha)$, for $W=X,Y$, or $Z$ and
for arbitrary value of $\alpha$.  Figure 2.9 shows
the correspondence between frames  $R_0$ and $R_1$ in
case of a single rotation $\alpha$ about axis $W=Z$.  Therefore, we have:
\begin{equation}
[ROTW(\alpha)]^{-t} = [ROTW(\alpha)]^t = ROTW(-\alpha)
\end{equation}
The ordered {\em product of rotation matrices }of the
same type is identical to one single rotation whose
magnitude is the sum of that of the
rotation matrices.  In the following
we develop this observation.  Assume
the frames $R_0$ and $R_1$ initially coincide
with respect to their origin and their orthonormal
vectors.  As shown in Figure 2.10, the operation of
rotating the frame $R_1$ by
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successive angles $\alpha_1,...,\alpha_k$ using
the same type of rotation is equivalent to
a product of rotation matrices.  The resulting
rotation matrix $M$, which represents the
transfer matrix from frame $R_1$ to
frame $R_0$ , is then given by
the product of rotation matrices:
\begin{equation}
M = \prod_{i=1}^k ROTW(\alpha_i)
\end{equation}
The successive rotations of frame $R_1$ is
associative operation with respect to
the angles $\alpha_1,...,\alpha_k$.  This
results in a single rotation matrix whose angle is
the sum $\sum_{i=1}^{i=k}\alpha_i$,
therefore we have:
\begin{equation}
M=\prod_{i=y}^k ROTW(\alpha_i)=ROTW \left( \sum_{i=t}^k\alpha_i\right)
\end{equation}
Figure 2.11 shows the successive transformation of frame $R_1$ and
their resulting overall rotation matrix
with respect to $R_0$.  This result is
valid only in case of a single type of
rotation, i.e. the parameter $W$ is
the same for all the rotations.  Note that the
order of these successive rotation does not affect the
resulting rotation matrix provided that all the
rotations belong to the same type.\\

When successive rotations are performed but with different
type, then the order of rotations become primordial.  This is
because the product of two rotation matrices is {\em not
commutative law} when different type of
rotations are considered.  For example, consider the
product of two
matrices from the set $S$ such as $ROTX(\theta_1).ROTY(\theta_2)$:
\begin{eqnarray}
ROTX(\theta_1).ROTY(\theta_2) & = & \left[
\begin{array}{ccc}
1&0&0 \\
0&C1&-S1 \\
0&S1&C1
\end{array}
\right]
.
\left[
\begin{array}{ccc}
C2&0&S2 \\
0&1&0 \\
-S2 & 0 & C2
\end{array}
\right]  \\
& = &
\left[
\begin{array}{ccc}
C2&0&S2 \\
S1S2&C1&-S1C2 \\
```

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-C1S2&S1&C1C2
\end{array}
\right]\nonumber
\end{eqnarray}
where $C1=cos(\theta_1),S1=sin(\theta_1),C2=cos(\theta_2)$, and
$S2=sin(\theta_2)$.
Next , we compute the product $ROTY(\theta_2).ROTX(\theta_1)$ as follows:
\begin{eqnarray}
ROTY(\theta_2).ROTX(\theta_1) & = & \left[
\begin{array}{ccc}
C2&0&S2 \\
0&1&0 \\
-S2&0&C2
\end{array}
\right]
.
\left[
\begin{array}{ccc}
1&0&0 \\
0&C1&-S10 \\
0 & S1 & C1
\end{array}
\right]  \\
& =&
\left[
\begin{array}{ccc}
C1&S1S2&C1S2 \\
0&C1&-S1 \\
-S2&S1C2&C1C2
\end{array} \nonumber
\right]
\end{eqnarray}
For this example, we can see that
the matrix product is not commutative law
for the set of fundamental rotations. \\

The geometric interpretation of this
property may be illustrated as follows.  First the
frame $R_1$ is rotated by an
angle 01 about the $X_1$ axis, and
then the frame $R_1$ is rotated by an
angle $\theta_2$ about the $Y_1$ is axis.  The
second rotation is performed with respect to $Y_1$
axis which has been affected by the first
rotation $ROTX(\theta_1)$.  Therefore,
the order of rotations is crucial in
findings the overall rotation matrix.

\section{Relation between cartesian frames}
In general the origin of frame $R_1$ may be placed
anywhere in the three dimensional space.  Let us
consider point $A$ which is observed in frame
$R_1$.  The point $A_1$ can be associated a
vector $O_1A_{R1}$.  Let $M_0^1$ denotes
the rotation matrix of frame $R_1$ relative
to $R_0$  This
orientation matrix converts a vector observed
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in frame R_1, referenced by the upper subscript, to
a vector which means that this matrix can
be product of three, the product of two, or a single
rotation matrix from the set S of fundamental
rotation matrices. \\

The point A can also be observed in frame
R_0 the
vector associated to point A is O_0A_{R0}. To
find the relation between vectors O_0A_{R1} and O_0A_{R0}, one
may express the vector O_0A_{R0} as
the sum of two vectors:
\begin{itemize}
\item
The translation of the origins O_0 and O_1 with
reference to frame R_0. This
vector is denoted by $O_0 O_{1,0}$.
\item
The vector $O_1 A_{R0}$ which
is observed in frame R_0. This
vector can be obtained from
vector $O_1 A_{R1}$ by using the orientation matrix
M, we have: $O_1A_{R0}=M_0^1O_1A_{R1}$ \\
Therefore, the vector $O_0 A_{R0}$ can be
expressed as:
\end{itemize}
\begin{equation}
O_0A_{R0}=O_0O_{1.0}+M_0^1.O_1A_{R1}
\end{equation}
Figure 2.11 shows a point A which
is observed with respect to frame R_0 and
R_1.
\section{Definition of a robot arm}
A robot arm is, in general, a manipulator that consists
of several rigid bodies, called links or
articulators, connected in series by revolute or
prismatic joints. Figures 2.12 shows
a revolute and a prismatic joint, respectively. \\

A revolute joint allows link L_{i+1} to rotate with
respect to the previous link L_i. A
rotation angle θ_{i+1} can be sued to define the
angular position of L_{i+1} relative to L_i. \\

A prismatic joint allows link L_{i+1} to
translate with respect to the previous link L_i. A
translation variable θ_{i+1} can also be
used to define the linear position of L_{i+1} relative
to L_i.
\subsection{Description of an articulated robot arm}
The first end of the articulated chain, of a robot
arm, is attached to a supporting base while
the other end is free and equipped with a special tool which is
called the grasping system. Obviously , the
grasping system is designed for grasping and manipulating objects.
The grasping system allows rigid attachment of the manipulated
object with the robot arm. This operation is

required prior to moving the object. Functionally a robot arm having six degrees of freedom (d.o.f) can be divided into two substructures which are the "Transporter" and the "Effector" parts. Figure 2.13 shows the transporter and effector parts. The transporter is responsible of transferring and positioning the effector which includes the grasping system and the work piece. Usually the end part utilizes three rotary motions called pitch, yaw and roll, and their combination orients the grasping system according to the desired orientation of the manipulated object. The end part, with these three typical motions, allows orienting the effector part of the arm. Figure 2.13 shows a typical combination of rotary motion for the grasping system.\\

The work volume is a sphere of influence for the robot arm which can position the transporter unit to any position and orientation within the sphere. The Transporter places or positions the effector which can orient itself in a convenient way according to the desired position and orientation of the manipulated object. The transporter is the positioning mechanism, while the effector is the orienting mechanism for the arm. In general, the transporter includes three links that are the shoulder, elbow, and forearm while the effector part includes the three known orientations that are the yaw, pitch, and roll. Figure 2.13 shows the characteristics degrees of freedom for a six-revolute robot arm. \\

Clearly, to position and orient the work piece both transporter and effector subsystems should coordinate their motion in a convenient way. As both transporter and effector have each a number of degrees of freedom, it becomes important to find a solution for these degrees of freedom such that to satisfy the requirements on the position and orientation of the work piece. The system that solves this problem is called the motion coordination system. The design of motion coordination systems will be discussed in Chapter 3.

\subsection{Types of Transporters}
There are four basic categories of construction which define the architecture of the transporter part of a robot. These are

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defined by using several mathematical coordinate
systems that are the cartesian, cylindrical, spherical, and
revolute structures.  In the following
sub-section
we study these structures.

%\subsubsection{The cartesian structure}
A {\em cartesian transporter} consists of
three mutually orthogonal linear axes.  The three
degrees of freedom are all prismatic
translations.  Figure 2.14 shows a cartesian robot arm.
The cartesian structure is the simplest structure
because it does not require any mathematical
transformation in order to convert the
coordinate of the transporter end point into the link
coordinates. \\

Figure 2.14 shows a cartesian transporter which is
composed of three prismatic joints.  The
position of the joint gives directly the
coordinates of the transporter end
point.

%\subsubsection{The cylindrical structure}
A {\em cylindrical transporter} consists of
two prismatic and one
revolute joints.  Figure 2.15
shows a cylindrical transporter whose
two prismatic degrees of freedom are
denoted by 1 and $r$, respectively
and $\theta$ denotes its revolute
degree of freedom.  The coordinates $X,Y,$ and
$Z$ of
the
transporter end point are given as follows:
\begin{eqnarray}
X & = & r.cos(\theta) \\
Y & = & r.sin(\theta) \\
Z & = & 1
\end{eqnarray}
Generally, the cylindrical
transporter, as any robotic structure,
can be moved by controlling
its three degrees of freedom $r,1$ and $\theta$.  The
operation of positioning the transporter end point
at a location ;defined by $X,Y,$ and $Z$ is equivalent
to solving the system equation
previously
defined.  One needs to find a closed
form formula in order to evaluate the variables $r,1$, and
$\theta$ which correspond to the
transporter end point $X,Y$, and $Z$.  When $r$ is not
nil, one can find
the solution:
\begin{eqnarray}
r & = & \sqrt{X^2+Y^2}\nonumber \\
cos(\theta) & = & \frac{X}{\sqrt{X^2 + Y^2}}\\
```

```
sin(\theta)&=& \frac{Y}{\sqrt { X^2 + Y^2}}\nonumber \\
1 & = & Z
\end{eqnarray}
The knowledge of $cos(\theta)$ and $sin(\theta)$ allows
finding a unique solution $\theta$.

%\subsubsection{the spherical structure}
A {\em spherical transporter} consists of one prismatic and
two revolute joints.  Figure 2.16 shows a
spherical transporter whose prismatic degree of
freedom is denoted by $r$ and its
two revolute degrees of freedom are
denoted by $\theta$, and $/varphi$, respectively.  The
coordinates $X,Y$, and $Z$ of a
spherical transporter end point are given by:
\begin{eqnarray}
X & = & r.cos(\theta).cos(\varphi) \nonumber \\
Y & = & r.cos(\theta).sin(\varphi)  \\
Z & = & r.sin(\theta)\nonumber
\end{eqnarray}
To position the spherical transporter end  point at
a location defined by $X,Y$, and $Z$ one
can find two solutions provided that $r$ is not nil:
\[
r = \sqrt{X^2 + Y^2 + Z^2}
\]
when $\theta$ is defined in the interval $[-\prod,+\prod)$, then
two solutions $\theta_1$ and $\theta_2$ are
expected.  These are given by:
\begin{eqnarray}
\theta_1 & = & sin^{-1} \left( \frac{ Z}{\sqrt { X^2 + Y^2 + Z^2 }}\right)
\nonumber \\
\theta_2 & = & Sign(\theta_1).\prod-\theta_1
\end{eqnarray}
The angle $\varphi$ can only be uniquely found if
and only if $cos(\theta)/neq \theta$, we have:
\[
cos(\varphi) = \frac{X}{cos(\theta)} \; \; \; \; sin(\varphi ) =
\frac{Y}{cos(\theta)}
\]
When both cosine and sine functions are defined, then
one can find a unique solution $\varphi$.  Note
here that both solutions satisfy the requirement of
having the transporter end point located at the
desired cartesian point $(X \; Y \; Z)^t$.

%\subsubsection{The general structure}
A {\em general revolute structure} consists of arbitrary combination of
prismatic and revolute joints.  Figure 2.17 shows an example of
a three-revolute transporter.  The  three degrees of freedom
are denoted by $\theta_1,\theta_2$, and $\theta_3$, respectively.  Chapter 3
deals with the analysis of general
structures.  The method to obtain the
coordinate of the end point
will be explained in that
chapter.  The following expressions represent these coordinates
which are given here to
```

illustrate this commonly used general structure:
\begin{eqnarray}
X & = & S1.(S2L_2 + S23L_3) \nonumber \\
Y & = & -C1.(S2L_2 + S23L_3) \\
Z & = & L_1 + C2L_2+C23L_3 \nonumber
\end{eqnarray}
where L_1, L_2, and L_3 denote the
link lengths, respectively, and $S1=sin(\theta_1)$, $C1=cos(\theta_1)$,
$S2=sin(\theta_2)$, $C2=cos(\theta_2)$,
$S23=sin(\theta_2 + \theta_3)$, and
$C23=cos(\theta_2+\theta_3)$.

In chapter 3 we will extensively analyze this structure and
develop a methodology for deriving the coordinate equations
together with the solution $\theta_i = F_i(X,Y,Z)$, where
$i=1,2,3$.

\subsection{Algebraic relationships of articulated structures}
In this section we study the relation between two consecutive links
interconnected by means of a joint. Two types of
joints will be investigated: the revolute and prismatic
types. The resulting relations will be used, in
chapter 3, to develop a complete method for
deriving the
geometric model of a robot arm. Generally, a geometric
model is useful to obtain the coordinate of
the robot end point with respect to a fixed frame of
reference. The geometric model is one important method
for designing the motion coordination system of a robot arm.\\

Figure 2.18 shows the assignment of frame R_i to link L_i.
Consider two successive links L_i and L_{i+1} of an
articulated system which is shown on Figure 2.19. The link L_i
is geometrically formed by the vector $O_{i-1}O_i$ and
supports its frame of reference $R_i=\{X_i, Y_iZ_i \; \mbox{and} \; O_i\}$.
This
frame is chosen such that vector $O_{i-1}O_i$ is
parallel to axis Z_i. since,
the link axis is supported by vector Z_i. Similarly, link
L_{i+1} is geometrically formed by
the vector O_iO_{i+1} and supports
its frame of reference $R_{i+1}=\{ X_{i+1}, Y_{i+1},Z_{i+1}, \; \mbox{and} \;
O_{i+1}\}$.
Frame R_{i+1} is chosen such that vector
O_iO_{i+1} is
parallel to axis Z_i. \\

Initially , both frames R_i and R_{i+1} are
parallel. This means that all three
orthonormal vectors of R_i and R_{i+1} are
respectively parallel to
each another. In this case both links will be
aligned. In the
following we study the case of revolute and prismatic joints..

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\subsection{Case of a revolute joint}
Link $L_{i+1}$ is said to be revolute with respect to
link $L_i$ when
frame $R_{i+1}$ can rotate relative
to either axes $X_i, Y_i$, or $Z_i$.  When
observed in frame $R_i$, the end point
$O_{i+1}$ of $L_{i+1}$ can
be associated a vector
$O_iO_{i+1}$
which will be denoted by
$O_iO_{i+1,i}$
to indicate that
the vector is observed in frame $R_i$.  Figure 2.20 shows
the case of a revolute joint that interfaces the
links $L_i$ and $L_{i+1}$. \\

As frame $R_{i+1}$ can rotate with
respect to $R_i$, then a
transfer matrix $M_i^{i+1}$ can be
used to represent the rotation between links $L_i$ and $L_{i+1}$.  Therefore,
vector
$O_iO_{i+1,i}$ can be
expressed as follows:
\begin{equation}
O_iO_{i+1,1} =M_i^{i+1}.O_iO_{i+1,i+1}
\end{equation}
where $O_iO_{i+1,i+1}$  denote the
vector $O_iO_{i+1}$ observed in frame $R_{i+1}$.  Note
here that vector $O_iO_{i+1,i+1}$ has simple expression because
link $L_{i+1}$ is parallel to
axis $Z_{i+1}$.  Therefore, the
vector $O_iO_{i+1,i+1}$  always has the following expression:
\begin{equation}
O_iO_{i+1,i+1} =O_{i-1}O_i+M_i^{i+1}.O_iO_{i+1,i+1}
\end{equation}
This expression gives the coordinate of the end point $O_{i+1}$ of
$L_{i+1}$ with respect to
the origin of the previous link $L_i$. \\

For the case shown in Figure 2.19, link $L_{i+1}$ is animated by
a revolute motion about $X_i$ axis.  Note there that
$X_i$ is parallel to $X_{i+1}$ axis
whatever the value of the
rotation angle.  Let denote by $\theta_{i+1}$ the rotation
angle that define the rotation matrix between frames $R_i$
and $R_{i+1}$.  The
rotation matrix $M_i^{i+1}$ is
equal to $ROTX(\theta_{i+1})$, then
for this example we have:
\begin{eqnarray*}
O_{i-1} O_{i+1,i} & = & \left[
\begin{array}{c}
0 \\
0 \\
L_i
\end{array}
\right]
```

```
+
\left[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & C(i+1) & -S(i+1) \\
0 & S(i+1) & C(i+1)
\end{array}
\right]
.
\left[
\begin{array}{c}
0 \\
0 \\
L_{i+1}
\end{array}
\right] \\
& = &
\left[
\begin{array}{c}
0 \\
-S(i+1).L_{i+1} \\
L_i + C(i+1).L_{i+1}
\end{array}
\right]
\end{eqnarray*}
One can verify that for $\theta_{i+1}=0$, both links
will be aligned and
the coordinate of $O_{i+1}$, with
respect to $R_i$, will be
$[0 \; 0 \; L_i+L_{i+1}]^t$.  We also
note that the vectors $X_{i+1}, Y_{i+1}$,
and $Z_{i+1}$ are
identical to the columns of the
rotation matrix when referenced with
respect to $R_i$, we
have:
\begin{eqnarray}
X_{i+1,i} & = & [1 \; 0 \; 0]^t \nonumber \\
Y_{i+1,i} & = & [0 \; C(\theta_{i+1}) \; S(\theta_{i+1})]^t \\
Z_{i+1,i} & = & [0 \; -S (\theta_{i+1}) \; C(\theta_{i+1})]^t\nonumber
\end{eqnarray}

\subsection{Case of prismatic link}
Link $L_{i+1}$ is said to be
prismatic with respect to link $L_i$ when
frame $R_{i+1}$ can only be
translated relative to some or all axes $X_i,Y_i$, or $Z_i$ of
frame $R_i$.  Figure 2.20 shows a frame
$R_{i+1}$ that
can be translated with respect to axis $Z$ of frame $R_i$. \\

As frame $R_{i+1}$ can only be translated with
respect to $R_i$, then a transfer matrix $M_i^{i+1}$
is the identity matrix.  Therefore, vector $O_i O_{i+1,i}$ can
be expressed as follows:
\begin{equation}
O_i O_{i+1,i}= M_i^{i+1}.O_i O_{i+1,i+1}=O_iO_{i+1,i+1}
```

\end{equation}
Note here that vector
$O_i O_{i+1,i+1}$
has also simple expression because
link L_{i+1} is
parallel to axis
Z_{i+1}. Assume
frame R_{i+1}
can be translated with
respect to axis Z_i, then
the vector $O_iO_{i+1,i+1}$
will be expressed as follows:
\begin{equation}
O_i O_{i+1,i+1}=(O \; O \; L_{i+1}+\theta_{i+1})^t
\end{equation}
where θ_{i+1} is the
linear translation variable which is
defined alaong axis Z_i. Note
here that the variable θ_{i+1} will
appear as a component of the axis X_{i+1},Y_{i+1} or
Z_{i+1} when the
translation of R_{i+1} with respect to R_i
is defined with respect to that axis. Now, we can
express the vector
$O_{i+1}O_{i+1,i}$ as follows:
\begin{equation}
O_{i-1} O_{i+1,i}= O_{i-1}O_{i,i}+O_i O_{i+1,i+1}
\end{equation}
This expression gives the coordinate of the
end point O_{i+1} of L_{i+1} with
respect to the origin of
the previous link L_i.

For the case shown in the Figure 2.20, link L_{i+1}
is animated by a prismatic motion along
axis Z_i. Note here that
R_{i+1} will
remain parallel to R_i whatever
the value of the linear variable
θ_{i+1}.

\section{Properties of rotation matrices}
Consider the links L_{i-1},L_i which are attached to
frames R_{i-1},R_i, and R_{i+1}
respectively. Let M_{i-1}^i and
M_i^{i+1} denote
the rotation matrices between frames
R_i and R_{i-1}, and
between frames R_{i+1} and R_i, respectively. Figure 2.20
shows the three links
L_{i-1},L_i and L_{i+1}. The product
$M_{i-1}^i.M_i^{i+1}$ represents
the translation matrix between frames
R_{i+1} and R_{i-1}, which is
function of two independent variables
θ_i and θ_{i+1}. In other
words the

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product $M_{i-1}^i.M_i^{i+1}$ gives
the coordinates of vectors $X_{i+1},Y_{i+1}$, and $Z_{i+1}$ with
respect to frame
$R_{i-1}$.
\begin{equation}
M_{i-1}^i .M_i^{i+1}= [X_{i+1,i-1} Y_{i+1,i-1}Z_{i+1,i-1}]
\end{equation}
Figure 2.20 shows the correspondence between the
frames $R_{i+1},R_i$ and $R_{i-1}$,
respectively.
To simplify this notation, we take the following contracted form:
\begin{equation}
M_{i-1}^i.M_i^{i+1}=M_{i-1}^{i+1}
\end{equation}
In the general case we also have:
\begin{equation}
M_i^{i+1}.M_{i+1}^{i+2}...M_{j-1}^j=M_i^j = [X_{i,j}Y_{i,j}Z_{i,j}]
\end{equation}
Note that there is a correspondence chain between the
frames $R_i,R_{i-1},....,$ and $R_i$,
respectively.

In the following we determine some properties of the translation
matrices which are
(1) the determinant of the product,
(2) the inverse of product, and
(3) commutativity of matrix product.

The determinant of the product of rotation matrices is always
equal to one:

\begin{equation}
|M_i^j|=|M_i^{i+1}.M_{i+1}^{i+2}...M_{i-1}^i|=\prod_{k-i}^{i-1}|=1
\end{equation}

The inverse of a product $M_i^j$ of rotation
matrices is equal to its transposed matrix
$[M_i^j]^t$
\begin{eqnarray}
\left[ M_i^j \right]^{-1}& = & \left[ M_i^{i+1}...M_{j-1} \right ]^{-1}
\nonumber \\
      & = & \left[M_{j-1}^i\right]^{-1}...\left[M_i^{i+1}\right] \nonumber \\
& = & \left[ M_{j-1}^i \right]...\left[ M_i^{i+1} \right]^t \\
& = & M^{j-1}_j...M_{i+1}^i \nonumber \\
& = & M_j^j
\end{eqnarray}
Note that $M_i^i $ is the identity matrix $I_3$.

In general, the matrix product is not commutative
for the set $S$ of rotation matrices:
\begin{equation}
M_i^{i+1}.M_{i+1}^{i+2} \neq M_{i+1}^{i+2}.M_i^{i+1}
\end{equation}

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\begin{center}
{\bf Exercises}
\end{center}
\begin{enumerate}
\item
Consider two vectors $U=(1 \; 5 \; -2)^t$ and $V=(3 \; -2 \; 1)^t$
\begin{enumerate}
\item
Determine their scalar product $U.V$.
\item
Determine their length
\item
Determine two vectors $W_1$ and $W_2$ which
are orthogonal to vectors $U$ and $V$,
respectively.
\item
Determine a unit length vector $W$ which
is orthogonal to both vectors $U$ and $V$.
\end{enumerate}
\item
Consider two vectors $U$ and $V$ as defined
in problem 1.1, and consider the
vector $W=(-3 \; 5 \; 2)^t$.  Find
the angle between vector $U$ and $W$
and the angle between $V$ and $W$.
\item
Consider the vectors $U$ and $V$.  Let $\alpha$ be
their scalar product $\alpha =U^t.V$
\begin{enumerate}
\item
Determine in which geometrical
condition $\alpha$ is minimum or maximum.
\item
Assume the vector $W=U\times V$, where $U=(2 \; 6 \; 3)^t \; V=(3 \; -X \; Y)^t$.
For
what values of $X$ and $Y$ the length of
vector $W$ is minimum or maximum.
\end{enumerate}
\item
The orthonormal vectors $X$ and $Y$ of
a frame of reference $R$ are
defined by $X=(-1 \; 0 \; 0)^t$
and $Y=0 \; -1 \; 0)^t$.
\begin{enumerate}
\item
Find a vector $Z$ that defines a right-handed
frame $R)X,Y,Z)$.
\item
Find a vector $Z$ that defines a left-handed frame $R(X,Y,Z)$.
\item
Find the rotation matrix between the two frames.
```

```latex
\end{enumerate}
\item
A fixed frame $R_0$ is given by:
\begin{itemize}
\item
Its origin $O_0$
\item
Its orthonormal vectors $X_0,Y_0$ and $Z_0$.
\end{itemize}
Another frame $R_1$, in motion
relative to $R_0$.  The frame $R_1$ is
defined by its origin $\theta_0\theta_{1,0}=(1,3,2)^t$ and
its orthonormal vectors $X_{1,0}=(a \; b \; c)^t$ and
$Y_{1,0}=(u \; v \; w)^t$ which
are given with reference to $R_0$.
\begin{enumerate}
\item
Determine the vector $Z_{1,0}$
\item
Determine the transfer matrix between the
frames $R_1$ and $R_0$.
\end{enumerate}
\item
A frame $R_1$, which originally coincide with a fixed frame $R_0$, is
rotated by an angle $\theta = 30^o$ about the $X_0$'s axis.
\begin{enumerate}
\item
Determine the rotation matrix $ROTX(\theta)$.
\item
Determine the vectors
$X_{1,0}, Y_{1,0}$, and $Z_{1,0}$ of
frame $R_0$.
\item
Suppose the frame $R_1$ is translated
relative to $R_0$ by the
vector $(1 \; 2 \; 1)^t$.  A
point $A$, which
observed in frame $R_0$, is given by: $O_0A=(1\; -2 \; 2)^t$.
Determine the coordinates of $A$ in $R_1$.
\end{enumerate}
\item
Consider the product of two rotation matrices:
$ROTX(\theta).ROTY(\alpha)=\left[ \begin{array} {ccc} 0&0&1 \\ 1&0&0 \\0&1&0
\end{array} \right]$
Determine the angles $\theta$ and $\alpha$ and discuss the solutions.
\item
Demonstrate that the orientation of an object can
be fully described by means of three
independent angles in the 3-dimensional space.  Define
these angles and discuss their relation and
their order.
\item
Give example of six degrees of freedom robot arms having
transporter and effector parts.
\begin{enumerate}
\item
How do you partition these d.o.f. among
```

```
the transporter and effector
substructures.
\item
Show that the orientation of the effector is generally dependent on
the first 3d.o.f. i.e., the interface between
the transporter and the effector is designed such that the
effector base depend on the orientation of the end segment of
the Transporter.
\item
Four types of transporters have been defined in Chapter 1, what
is the advantage of each in terms of:
\begin{enumerate}
\item
Accessibility of the end, point.
\item
Complexity of the relatioins between
the coordinate of the end point and
the coordinate of each degree of freedom.
\end{enumerate}
\end{enumerate}
\item
Consider the four types of
transporters that have been defined in Chapter 1.  Using
the coordinate system in the cartesian, cylindrical, spherical,
and revolute spaces, find the
expression of the transporter end point coordinate as
function of the three degrees of
freedom.
\item
Consider a set of rotation matrices
$\{M_i^{i+1}:i=\theta,...,n\}$,
where $M_i^{i+1}$ denotes
the transfer matrix between frames $R_{i+1}$ and $R_i$.
\begin{enumerate}
\item
What is the geometrical interpretation of
the matrix $M_0^n=\{X_n,Y_n,Z_n\}$, where
$X_n, Y_n$, and $Z_n$ are column vectors.
\item
Express the vectors $X_n,Yn$, and $Z_n$ in a frame
$R_k$ where $O\leq k \leq n$.
\item
What is the
geometrical interpretation of the inverse
matrix $\left[ M_0^n\right]^{-1}$.
\item
What should be the value of $n$ such
that the equation $M=M_0^n$ admits
one and only solution.  The matrix $M$ is
a constant and given matrix.
\end{enumerate}
\end{enumerate}
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\begin{center}
{\bf Trignometric Formulas}
\end{center}
$\mbox{sin}(-\theta)= -\mbox{sin}(\theta) $ \\
$\mbox{cos}(-\theta) = - \mbox{cos} (\theta)$ \\
$\mbox{sin}^2(\theta) + \mbox{coscos2}(\theta)=1 $\\
$\mbox{sin}(\alpha + \beta)=\mbox{sin} (\alpha) \mbox{cos}
(\beta)+\mbox{cos}(\alpha) \mbox{sin} (\beta) $ \\
$\mbox{cos}(2\theta) = \mbox{cos}^2 (\theta) - \mbox{sin}^2 (\theta) $\\
$\mbox{tan} (\alpha + \beta) = \frac{\mbox{tan}(\alpha)+\mbox{tan}(\beta)}
{1-\mbox{tan}(\alpha).\mbox{tan}(\beta)}$

%\end{document}
```