Chapter 2

VECTORS AND FRAMES

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This chapter introduces the basic concepts of vectors and frames in the three dimensional space R^3. A frame of reference is defined by using three orthonormal unit vectors. This allows deriving the elementary rotation matrices that define the relation between two frame of references. These rotation matrices have interesting properties with regards to the evaluation of their determinant and their inverse transformation. The properties of these rotation matrices will be outlined. Further, we obtain the geometric vector relations between a moving frame R_1 and a fixed R_0.

Next, we introduce the notion of robot arm at the topological and mathematical levels. Both levels will be dealing with the geometical description. A robot arm is intrdouced as a series of links that are indterconnected by means of joints. Two types of joints are used for interconnecting the robot links. The topology of robot arms is decomposed into two subsystems: the transporter and the effector. The transporter is the robot positioning subsystem, while the effector is the orienting subsystem. Several types of transporter geometry are studied by the well known coordinate systems.

A second look on the orientation matrices is presented with the objective to investigate the mathematical transformations that are required for each type of interconnection topology. Finally, we study the properties of the global (composite) rotation matrix of a moving frame. This presents an extension to the properties of the set of rotation matrices which has been studied earlier. Throughout this book the sentence "frame of reference", or simply "frame" designate the same meaning.

2.1 Three dimensional space vectors

A cartesian coordinate system is defined, in the three dimensional space, by introducing three orthonormal vectors X, Y, and Z. Figure 2.1 shows a cartesian coordinate system which is defined using three orthonormal vectors X, Y, and Z. The set of vectors $\{X, Y, Z\}$ is referred to as orthonormal when the vectors are orthogonal and each has unit of length. These unit vectors can be expressed respectively as:

$$\begin{aligned} X &= 1.X + 0.Y + 0.Z \\ Y &= 0.X + 1.Y + 0.Z \\ Z &= 0.X + 0.Y + 1.Z \end{aligned} \tag{2.1}$$

To simplify this notation, we can write them in the following form:

$$X = (1, 0, 0)^t$$

$$Y = (0,1,0)^t \tag{2.2}$$
$$Z = (0,0,1)^t$$

where (t) denotes the transposition. A cartesian frame of reference R is defined by using the three orthonormal vectors $\{X, Y, Z\}$ and their intersecting origin O:

$$R = \{X, Y, Z \text{ and } 0\} \tag{2.3}$$

Figure 2.2 shows a cartesian frame of reference defined by three orthonormal vectors $\{X, Y, Z\}$ and O as origin. Observed in frame R, a point A can be associated a vector OA_R, where the subscript θ_R denotes that the vector is observed in frame R. Let the scalars A_x, A_y, and A_z denotes the projection of vector OA_R on the axis X, Y, and Z of frame R, respectively. Therefore we can express the vector as follows:

$$OA_R = A_x.X + A_y.Y + A_z.Z \tag{2.4}$$

Figure 2.3 shows point A which is observed in frame R. This notation can be shortly reduced to:

$$OA_R = (A_x A_y A_z)^t \tag{2.5}$$

More generally, a vector U whose components are U_x, U_y and U_z with respect to axes X, Y, and Z of frame R can be expressed as follows:

$$U = (U_x U_y U_z)^t$$

Definition: The norm or length of a vector U is denoted as $|U|$ and is defined by:

$$|U| = \sqrt{U_x^2 + U_y^2 + U_z^2} \tag{2.6}$$

Definition: The dot product or scalar product of two vectors U and V which are observed in frame R, is a scalar defined by:

$$U^t.V = U_x.V_x + U_y.V_y + U_z.V_z$$

The dot product is proportional to the cosine of the angle between the two vectors and their lengths. Let θ be the angle between the vectors U and V, the dot product can then be expressed as follows:

$$U^t.V = |U|.|V|.cos(\theta) \tag{2.7}$$

The dot product $U^t.V$ is commutative law because:

$$U^t.V = V^t.U = |U|.|V|cos(\theta) = |V||U|.cos(-\theta) \tag{2.8}$$

The dot product $U^t.U$ is nil if the vector U is nil. When the vector U is not nil, then the dot product is always a positive scalar. The dot product of two non nil and non-colinear vectors U and V is nil if and only vectors U and V are orthogonal. In this case the angle θ between these vectors is equal to $\pm \prod /2$, i.e. $cos(\theta) = 0$.

The dot product is linear function because we have:

$$(\alpha.U + \beta.V).W = \alpha(U.W) + \beta.(V.W)$$

where α and β are two arbitrary scalars.

Definition: The cross product of two vectors U and V , that are observed in framed R, is a vector $U \times V$ which is orthogonal to U and V in the right-handed sense. The cross product vector is defined as follows:

$$U \times V = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} \times \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} U_y.V_z - U_z.V_y \\ U_z.V_x - U_x.V_z \\ U_x.V_y - U_y.V_x \end{bmatrix} \tag{2.9}$$

The cross product $U \times V$ can also be expressed as function off the product of the vector lengths and the sine of the angle from U to V:

$$|U \times V| = |U|.|V|.sin(\theta) \tag{2.10}$$

Figure 2.4 shows vectors U and V and the vector that defines their cross product. The cross product $U \times V$ defines a vector $W = U \times V$ which is orthogonal to both vectors U and V, we can easily verify that:

$$W^t.U = W^t.V = 0$$

2.2 Frame rotation

The prime interest of finding the rotation matrix between two frame of references R_1 and R_0 is to determine a transfer matrix which allows converting the coordinate of a vector, that is observed in a frame, into the coordinate of the same vector with respect to the other frame. We assume the frame $R_1 = \{X_1.Y_1.Z_1$ and $0_1\}$ is subject to some motion with respect to a fixed reference coordinate system $R_0 = \{X_0.Y_0.Z_0$ and $\theta_0\}$

Initially we assume the origin θ_1 coincides with θ_0 and vectors X_1, Y_1 and Z_1 coincide with vectors X_0, Y_0 and Z_0, respectively. Figure 2.5 shows two initially coinciding frame of references.

Assume the frame R_1 is rotated by an angle θ about the Z_0 axis. As frame R_1 is only rotated, both origin still coincide but the orientation of R_1 is now changing as a result of the rotation. Figure 2.6 shows the frame R_1 after rotating this frame by an angle θ about axis Z can be expressed with respect to frame R_0 as follows:

$$\begin{aligned} X_{1,0} &= (cos(\theta)\ sin(\theta)\ 0)^t \\ Y_{1,0} &= (-sin(\theta)\, cos(\theta)\ 0)^t \\ X_{1,0} &= (0\ 0\ 1)^t \end{aligned}$$

where $X_{1,0}$, $Y_{1,0}$, and $Z_{1,0}$ denote the orthonormal vectors of frame R_1. Note that the first index is used as the vector identifier and the second index is used to state that these vectors are observed with respect to frame R_0.

Considering now a vector O_1A whose components are A_{x1}, A_{y1}, and A_{z1} with respect to frame R_1. By definition we have:

$$O_1A_{R1} = A_{x1}.X_{1,0} + A_{y1}.Y_{1,0} + A_{z1}.Z_{1,0} \tag{2.11}$$

To express this vector in R_0 we can substitute the vectors $X_{1,0}$, $Y_{1,0}$, and $Z_{1,0}$ with their expressions that give the coordinate of that vector with respect to R_0:

$$\begin{aligned} O_0A_{R0} &= A_{x1}.(cos(\theta).X_{0,0} + sin(\theta).Y_{0,0}) + \\ &\quad A_{y1}.(-sin(\theta).X_{0,0} + cos(\theta).Y_{0,0}) + A_{z1}.Z_{0,0} \end{aligned} \tag{2.12}$$

where vectors $X_{0,0}$, $Y_{0,0}$, and $Z_{0,0}$ are the orthonormal vectors of frame R_0 which are observed with respect to the same frame. By grouping the terms with respect to vectors $Z_{0,0}$, $Y_{0,0}$ and $Z_{0,0}$ ob obtain:

$$\begin{aligned} O_0A_{R0} &= (A_{x1}.cos(\theta) - A_{y1}.sin(\theta)).X_{0,0} + \\ &\quad (A_{x1}.sin(\theta) + A_{y1}.cos(\theta)).Y_{0,0} + A_{z1}.Z_{0,0} \end{aligned} \tag{2.13}$$

Vector O_oA is represented in R_0 by the general rleation:

$$O_0A_{R0} = A_{x0}.X_0 + A_{y0}.Y_0 + A_{z0}.Z_0$$

This vector can be expressed in matrix form:

$$O_0A_{R0} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} A_{x1} \\ A_{y1} \\ A_{z1} \end{bmatrix} = \begin{bmatrix} A_{x0} \\ A_{y0} \\ A_{z0} \end{bmatrix}$$

The transfer matrix from R_1 into frame R_0 represents the rotation matrix about the Z_0 axis by an angle θ. The rotation matrix is denoted by $ROTZ(\theta)$, we have:

$$O_0A_{R0} = ROTZ(\theta).O_1A_{R1}$$

This matrix can be derived from the expression of vectors $X_{1,0}, Y_{1,0}$, and $Z_{1,0}$. It is important to note that the columns of $ROTZ(\theta)$ are the vectors $X_{1,0}, Y_{1,0}$ and $Z_{1,0}$, respectively.

$$ROTZ(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C(\theta) & -S(\theta) & 0 \\ S(\theta) & C(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $C(\theta)$ denotes $cos(\theta)$ and $S(\theta)$ is $sin(\theta)$, respectively. Similarly, we denote by $ROTX(\alpha)$ and $ROTY(\beta)$ the rotation matrices about the X_0 by an angle α and Y_0 by an angle β, respectively. Figures 2.7 and 2.8 show the effect on frame R_1 of these two rotations. These figures assume that in both cases the frame R_1 was initially coinciding with the fixed frame of reference, we have therefore:

$$ROTX(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C(\alpha) & -S(\alpha) \\ 0 & S(\alpha) & C(\alpha) \end{bmatrix} \tag{2.14}$$

The rotation matrix $ROTY(\beta)$ is defined by:

$$ROTY(\beta) = \begin{bmatrix} C(\beta) & 0 & S(\beta) \\ 0 & 1 & 0 \\ -S(0) & 0 & C(\beta) \end{bmatrix} \tag{2.15}$$

The three rotation matrices $ROTX(\alpha)$, $ROTY(\beta)$, and $ROTZ(\theta)$ define the set of fundamental rotations in the cartesian coordinate system. Note that every rotation matrix becomes the identity matrix whenever the rotation angle becomes nil.

2.3 Rotation matrices

For ease of discussion we denote by $ROTW$ an arbitrary rotation matrix, where W may be X, Y or Z. The rotation matrices define the set $S = \{ROTW : W = X, Y, Z\}$ of fundamental rotations in the three dimensional space. There are three fundamental properties for the set S of rotation matrices:

- The determinant
- The inverse
- The product

The **determinant** of any rotation matrix $ROTW(\alpha)$, of the set S, is equal to the unit:

$$det(ROTW(\alpha)) = 1 \quad \text{for } W : X, Y \text{ or } Z$$

We can easily see that for an arbitrary angle α and; for $W = X, Y$, or Z, we have:

$$det(ROTW(\alpha)) = S(\alpha).S(\alpha) + C(\alpha).C(\alpha) = 1 \tag{2.16}$$

The *inverse matrix* of every rotation matrix $ROTW(\alpha)$, of the set S, is a matrix whose transpose is identical to $ROTW(\alpha)$,

$$[ROTW(\alpha)]^{-1} = [ROTW(\alpha)]^t \tag{2.17}$$

where θ^t denotes the transpose of a matrix. To prove this relation, consider a matrix M and let M^{-1} denotes its inverse matrix, we have:

$$M^{-1} = \frac{1}{det(M)}[Co(M)]^t \tag{2.18}$$

where $Co(M)$ denotes the co-factor matrix associated to M. Using this formula for a rotation matrix $ROTW$, we obtain:

$$[ROTW(\alpha)]^{-1} = [Co(ROTW)]^t \tag{2.19}$$

Note that the determinant $det(ROTW(\alpha)) = 1$ and the co-factor matrix associated with any matrix of the set S is identical to that matrix. The inverse matrix can then be obtained by means of a transposition:

$$[ROTW(\alpha)]^{-1} = [Co(ROTW)]^t \tag{2.20}$$

Assume a rotation matrix $ROTW(\alpha)$ represents the transfer matrix between frame R_1 can be expressed with respect to R_0 by the equation:

$$U_{R0} = ROTW(\alpha).U_{R1} \tag{2.21}$$

Figure 2.9 shows the correspondence between frames R_0 and R_1. The inverse rotation matrix $[ROTW(\alpha)]^t$ is the transfer matrix from frame R_{R0} to frame R_{R1}, we have:

$$R_{R1} = [ROTW(\alpha)]^t.U_{R0}$$

Consider a rotation matrix $ROTW(\alpha)$, we note that the extra-diagonal terms of that matrix depend on the anti-symmetric function $sin(\alpha)$. The transpose matrix of an arbitrary rotation matrix $ROTW(\alpha)$ is then equal to $ROTW(-\alpha)$, for $W = X, Y$, or Z and for arbitrary value of α. Figure 2.9 shows the correspondence between frames R_0 and R_1 in case of a single rotation α about axis $W = Z$. Therefore, we have:

$$[ROTW(\alpha)]^{-t} = [ROTW(\alpha)]^t = ROTW(-\alpha) \tag{2.22}$$

The ordered *product of rotation matrices* of the same type is identical to one single rotation whose magnitude is the sum of that of the rotation matrices. In the following we develop this observation. Assume the frames R_0 and R_1 initially coincide with respect to their origin and their orthonormal vectors. As shown in Figure 2.10, the operation of rotating the frame R_1 by successive angles $\alpha_1, ..., \alpha_k$ using the same type of rotation is equivalent to a product of rotation matrices. The resulting rotation matrix M, which represents the transfer matrix from frame R_1 to frame R_0 , is then given by the product of rotation matrices:

$$M = \prod_{i=1}^{k} ROTW(\alpha_i) \tag{2.23}$$

The successive rotations of frame R_1 is associative operation with respect to the angles $\alpha_1, ..., \alpha_k$. This results in a single rotation matrix whose angle is the sum $\sum_{i=1}^{i=k} \alpha_i$, therefore we have:

$$M = \prod_{i=y}^{k} ROTW(\alpha_i) = ROTW\left(\sum_{i=t}^{k} \alpha_i\right) \tag{2.24}$$

Figure 2.11 shows the successive transformation of frame R_1 and their resulting overall rotation matrix with respect to R_0. This result is valid only in case of a single type of rotation, i.e. the parameter W is the same for all the rotations. Note that the order of these successive rotation does not affect the resulting rotation matrix provided that all the rotations belong to the same type.

When successive rotations are performed but with different type, then the order of rotations become primordial. This is because the product of two rotation matrices is *not commutative law* when different type of rotations are considered. For example, consider the product of two matrices from the set S such as $ROTX(\theta_1).ROTY(\theta_2)$:

$$\begin{aligned} ROTX(\theta_1).ROTY(\theta_2) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C1 & -S1 \\ 0 & S1 & C1 \end{bmatrix} . \begin{bmatrix} C2 & 0 & S2 \\ 0 & 1 & 0 \\ -S2 & 0 & C2 \end{bmatrix} \\ &= \begin{bmatrix} C2 & 0 & S2 \\ S1S2 & C1 & -S1C2 \\ -C1S2 & S1 & C1C2 \end{bmatrix} \end{aligned} \tag{2.25}$$

where $C1 = cos(\theta_1), S1 = sin(\theta_1), C2 = cos(\theta_2)$, and $S2 = sin(\theta_2)$. Next , we compute the product $ROTY(\theta_2).ROTX(\theta_1)$ as follows:

$$\begin{aligned} ROTY(\theta_2).ROTX(\theta_1) &= \begin{bmatrix} C2 & 0 & S2 \\ 0 & 1 & 0 \\ -S2 & 0 & C2 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & C1 & -S10 \\ 0 & S1 & C1 \end{bmatrix} \\ &= \begin{bmatrix} C1 & S1S2 & C1S2 \\ 0 & C1 & -S1 \\ -S2 & S1C2 & C1C2 \end{bmatrix} \end{aligned} \tag{2.26}$$

For this example, we can see that the matrix product is not commutative law for the set of fundamental rotations.

The geometric interpretation of this property may be illustrated as follows. First the frame R_1 is rotated by an angle 01 about the X_1 axis, and then the frame R_1 is rotated by an angle θ_2 about the Y_1 is axis. The second rotation is performed with respect to Y_1 axis which has been affected by the first rotation $ROTX(\theta_1)$. Therefore, the order of rotations is crucial in findings the overall rotation matrix.

2.4 Relation between cartesian frames

In general the origin of frame R_1 may be placed anywhere in the three dimensional space. Let us consider point A which is observed in frame R_1. The point A_1 can be associated a vector O_1A_{R1}. Let M_0^1 denotes the rotation matrix of frame R_1 relative to R_0 This orientation matrix converts a vector observed in frame R_1, referenced by the upper subscript, to a vector which means that this matrix can be product of three, the product of two, or a single rotation matrix from the set S of fundamental rotation matrices.

The point A can also be observed in frame R_0 the vector associated to point A is O_0A_{R0}. To find the relation between vectors O_0A_{R1} and O_0A_{R0}, one may express the vector O_0A_{R0} as the sum of two vectors:

- The translation of the origins O_0 and O_1 with reference to frame R_0. This vector is denoted by $O_0O_{1,0}$.
- The vector O_1A_{R0} which is observed in frame R_0. This vector can be obtained from vector O_1A_{R1} by using the orientation matrix M, we have: $O_1A_{R0} = M_0^1 O_1A_{R1}$
  Therefore, the vector O_0A_{R0} can be expressed as:

$$O_0A_{R0} = O_0O_{1.0} + M_0^1.O_1A_{R1} \tag{2.27}$$

Figure 2.11 shows a point A which is observed with respect to frame R_0 and R_1.

2.5 Definition of a robot arm

A robot arm is, in general, a manipulator that consists of several rigid bodies, called links or articulators, connected in series by revolute or prismatic joints. Figures 2.12 shows a revolute and a prismatic joint, respectively.

A revolute joint allows link L_{i+1} to rotate with respect to the previous link L_i. A rotation angle θ_{i+1} can be sued to define the angular position of L_{i+1} relative to L_i.

A prismatic joint allows link L_{i+1} to translate with respect to the previous link L_i. A translation variable θ_{i+1} can also be used to define the linear position of L_{i+1} relative to L_i.

2.5.1 Description of an articulated robot arm

The first end of the articulated chain, of a robot arm, is attached to a supporting base while the other end is free and equipped with a special tool which is called the grasping system. Obviously , the grasping system is designed for grasping and manipulating objects. The grasping system allows rigid attachment of the manipulated object with the robot arm. This operation is required prior to moving the object. Functionally a robot arm having six degrees of freedom (d.o.f) can be divided into two substructures which are the "Transporter" and the "Effector" parts. Figure 2.13 shows the transporter and effector parts. The transporter is responsible of transferring and positioning the effector which includes the grasping system and the work piece. Usually the end part utilizes three rotary motions called pitch, yaw and roll, and their combination orients the grasping system according to the desired orientation of the manipulated object. The end part, with these three typical motions, allows orienting the effector part of the arm. Figure 2.13 shows a typical combination of rotary motion for the grasping system.

The work volume is a sphere of influence for the robot arm which can position the transporter unit to any position and orientation within the sphere. The Transporter places or positions the effector which can orient itself in a convenient way according to the desired position and orientation of the manipulated object. The transporter is the positioning mechanism, while the effector is the orienting mechanism for the arm. In general, the transporter includes three links that are the shoulder, elbow, and forearm while the effector part includes the three known orientations that are the yaw, pitch, and roll. Figure 2.13 shows the characteristics degrees of freedom for a six-revolute robot arm.

Clearly, to position and orient the work piece both transporter and effector subsystems should coordinate their motion in a convenient way. As both transporter and effector have each a number of degrees of freedom, it becomes important to find a solution for these degrees of freedom such that to satisfy the requirements on the position and orientation of the work piece. The system that solves this problem is called the motion coordination system. The design of motion coordination systems will be discussed in Chapter 3.

2.5.2 Types of Transporters

There are four basic categories of construction which define the architecture of the transporter part of a robot. These are defined by using several mathematical coordinate systems that are the cartesian, cylindrical, spherical, and revolute structures. In the following sub-section we study these structures.

A *cartesian transporter* consists of three mutually orthogonal linear axes. The three degrees of freedom are all prismatic translations. Figure 2.14 shows a cartesian robot arm. The cartesian structure is the simplest structure because it does not require any mathematical transformation in order to convert the coordinate of the transporter end point into the link coordinates.

Figure 2.14 shows a cartesian transporter which is composed of three prismatic joints. The position of the joint gives directly the coordinates of the transporter end point.

A *cylindrical transporter* consists of two prismatic and one revolute joints. Figure 2.15 shows a cylindrical transporter whose two prismatic degrees of freedom are denoted by 1 and r, respectively and θ denotes its revolute degree of freedom. The coordinates X, Y, and Z of the transporter end point are given as follows:

$$X = r.cos(\theta) \tag{2.28}$$
$$Y = r.sin(\theta) \tag{2.29}$$
$$Z = 1 \tag{2.30}$$

Generally, the cylindrical transporter, as any robotic structure, can be moved by controlling its three degrees of freedom $r, 1$ and θ. The operation of positioning the transporter end point at a location ;defined by X, Y, and Z is equivalent to solving the system equation previously defined. One needs to find a closed form formula in order to evaluate the variables $r, 1$, and θ which correspond to the transporter end point X, Y, and Z. When r is not nil, one can find the solution:

$$\begin{aligned} r &= \sqrt{X^2+Y^2} \\ cos(\theta) &= \frac{X}{\sqrt{X^2+Y^2}} \qquad (2.31) \\ sin(\theta) &= \frac{Y}{\sqrt{X^2+Y^2}} \\ 1 &= Z \qquad (2.32) \end{aligned}$$

The knowledge of $cos(\theta)$ and $sin(\theta)$ allows finding a unique solution θ.

A *spherical transporter* consists of one prismatic and two revolute joints. Figure 2.16 shows a spherical transporter whose prismatic degree of freedom is denoted by r and its two revolute degrees of freedom are denoted by θ, and $/varphi$, respectively. The coordinates X, Y, and Z of a spherical transporter end point are given by:

$$\begin{aligned} X &= r.cos(\theta).cos(\varphi) \\ Y &= r.cos(\theta).sin(\varphi) \\ Z &= r.sin(\theta) \end{aligned} \tag{2.33}$$

To position the spherical transporter end point at a location defined by X, Y, and Z one can find two solutions provided that r is not nil:

$$r = \sqrt{X^2+Y^2+Z^2}$$

when θ is defined in the interval $[-\prod, +\prod)$, then two solutions θ_1 and θ_2 are expected. These are given by:

$$\begin{aligned} \theta_1 &= sin^{-1}\left(\frac{Z}{\sqrt{X^2+Y^2+Z^2}}\right) \\ \theta_2 &= Sign(\theta_1).\prod -\theta_1 \end{aligned} \tag{2.34}$$

The angle φ can only be uniquely found if and only if $cos(\theta)/neq\theta$, we have:

$$cos(\varphi) = \frac{X}{cos(\theta)} \quad sin(\varphi) = \frac{Y}{cos(\theta)}$$

When both cosine and sine functions are defined, then one can find a unique solution φ. Note here that both solutions satisfy the requirement of having the transporter end point located at the desired cartesian point $(X\ Y\ Z)^t$.

A *general revolute structure* consists of arbitrary combination of prismatic and revolute joints. Figure 2.17 shows an example of a three-revolute transporter. The three degrees of freedom are denoted by θ_1, θ_2, and θ_3, respectively. Chapter 3 deals with the analysis of general structures. The method to obtain the coordinate of the end point will be explained in that chapter. The following expressions represent these coordinates which are given here to illustrate this commonly used general structure:

$$\begin{aligned} X &= S1.(S2L_2 + S23L_3) \\ Y &= -C1.(S2L_2 + S23L_3) \\ Z &= L_1 + C2L_2 + C23L_3 \end{aligned} \tag{2.35}$$

where L_1, L_2, and L_3 denote the link lengths, respectively, and $S1 = sin(\theta_1)$, $C1 = cos(\theta_1)$, $S2 = sin(\theta_2)$, $C2 = cos(\theta_2)$, $S23 = sin(\theta_2 + \theta_3)$, and $C23 = cos(\theta_2 + \theta_3)$.

In chapter 3 we will extensively analyze this structure and develop a methodology for deriving the coordinate equations together with the solution $\theta_i = F_i(X, Y, Z)$, where $i = 1, 2, 3$.

2.5.3 Algebraic relationships of articulated structures

In this section we study the relation between two consecutive links interconnected by means of a joint. Two types of joints will be investigated: the revolute and prismatic types. The resulting relations will be used, in chapter 3, to develop a complete method for deriving the geometric model of a robot arm. Generally, a geometric model is useful to obtain the coordinate of the robot end point with respect to a fixed frame of reference. The geometric model is one important method for designing the motion coordination system of a robot arm.

Figure 2.18 shows the assignment of frame R_i to link L_i. Consider two successive links L_i and L_{i+1} of an articulated system which is shown on Figure 2.19. The link L_i is geometrically formed by the vector $O_{i-1}O_i$ and supports its frame of reference $R_i = \{X_i, Y_iZ_i \text{ and } O_i\}$. This frame is chosen such that vector $O_{i-1}O_i$ is parallel to axis Z_i. since, the link axis is supported by vector Z_i. Similarly, link L_{i+1} is geometrically formed by the vector O_iO_{i+1} and supports its frame of reference $R_{i+1} = \{X_{i+1}, Y_{i+1}, Z_{i+1}, \text{ and } O_{i+1}\}$. Frame R_{i+1} is chosen such that vector O_iO_{i+1} is parallel to axis Z_i.

Initially , both frames R_i and R_{i+1} are parallel. This means that all three orthonormal vectors of R_i and R_{i+1} are respectively parallel to each another. In this case both links will be aligned. In the following we study the case of revolute and prismatic joints..

2.5.4 Case of a revolute joint

Link L_{i+1} is said to be revolute with respect to link L_i when frame R_{i+1} can rotate relative to either axes X_i, Y_i, or Z_i. When observed in frame R_i, the end point O_{i+1} of L_{i+1} can be associated a vector O_iO_{i+1} which will be denoted by $O_iO_{i+1,i}$ to indicate that the vector is observed in frame R_i. Figure 2.20 shows the case of a revolute joint that interfaces the links L_i and L_{i+1}.

As frame R_{i+1} can rotate with respect to R_i, then a transfer matrix M_i^{i+1} can be used to represent the rotation between links L_i and L_{i+1}. Therefore, vector $O_iO_{i+1,i}$ can be expressed as follows:

$$O_iO_{i+1,1} = M_i^{i+1}.O_iO_{i+1,i+1} \tag{2.36}$$

where $O_iO_{i+1,i+1}$ denote the vector O_iO_{i+1} observed in frame R_{i+1}. Note here that vector $O_iO_{i+1,i+1}$ has simple expression because link L_{i+1} is parallel to axis Z_{i+1}. Therefore, the vector $O_iO_{i+1,i+1}$

always has the following expression:

$$O_iO_{i+1,i+1} = O_{i-1}O_i + M_i^{i+1}.O_iO_{i+1,i+1} \tag{2.37}$$

This expression gives the coordinate of the end point O_{i+1} of L_{i+1} with respect to the origin of the previous link L_i.

For the case shown in Figure 2.19, link L_{i+1} is animated by a revolute motion about X_i axis. Note there that X_i is parallel to X_{i+1} axis whatever the value of the rotation angle. Let denote by θ_{i+1} the rotation angle that define the rotation matrix between frames R_i and R_{i+1}. The rotation matrix M_i^{i+1} is equal to $ROTX(\theta_{i+1})$, then for this example we have:

$$\begin{aligned} O_{i-1}O_{i+1,i} &= \begin{bmatrix} 0 \\ 0 \\ L_i \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & C(i+1) & -S(i+1) \\ 0 & S(i+1) & C(i+1) \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ L_{i+1} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -S(i+1).L_{i+1} \\ L_i + C(i+1).L_{i+1} \end{bmatrix} \end{aligned}$$

One can verify that for $\theta_{i+1} = 0$, both links will be aligned and the coordinate of O_{i+1}, with respect to R_i, will be $[0\ 0\ L_i + L_{i+1}]^t$. We also note that the vectors X_{i+1}, Y_{i+1}, and Z_{i+1} are identical to the columns of the rotation matrix when referenced with respect to R_i, we have:

$$\begin{aligned} X_{i+1,i} &= [1\ 0\ 0]^t \\ Y_{i+1,i} &= [0\ C(\theta_{i+1})\ S(\theta_{i+1})]^t \\ Z_{i+1,i} &= [0\ -S(\theta_{i+1})\ C(\theta_{i+1})]^t \end{aligned} \tag{2.38}$$

2.5.5 Case of prismatic link

Link L_{i+1} is said to be prismatic with respect to link L_i when frame R_{i+1} can only be translated relative to some or all axes X_i, Y_i, or Z_i of frame R_i. Figure 2.20 shows a frame R_{i+1} that can be translated with respect to axis Z of frame R_i.

As frame R_{i+1} can only be translated with respect to R_i, then a transfer matrix M_i^{i+1} is the identity matrix. Therefore, vector $O_iO_{i+1,i}$ can be expressed as follows:

$$O_iO_{i+1,i} = M_i^{i+1}.O_iO_{i+1,i+1} = O_iO_{i+1,i+1} \tag{2.39}$$

Note here that vector $O_iO_{i+1,i+1}$ has also simple expression because link L_{i+1} is parallel to axis Z_{i+1}. Assume frame R_{i+1} can be translated with respect to axis Z_i, then the vector $O_iO_{i+1,i+1}$ will be expressed as follows:

$$O_iO_{i+1,i+1} = (O\ O\ L_{i+1} + \theta_{i+1})^t \tag{2.40}$$

where θ_{i+1} is the linear translation variable which is defined alaong axis Z_i. Note here that the variable θ_{i+1} will appear as a component of the axis X_{i+1}, Y_{i+1} or Z_{i+1} when the translation of R_{i+1} with respect to R_i is defined with respect to that axis. Now, we can express the vector $O_{i+1}O_{i+1,i}$ as follows:

$$O_{i-1}O_{i+1,i} = O_{i-1}O_{i,i} + O_iO_{i+1,i+1} \tag{2.41}$$

This expression gives the coordinate of the end point O_{i+1} of L_{i+1} with respect to the origin of the previous link L_i.

For the case shown in the Figure 2.20, link L_{i+1} is animated by a prismatic motion along axis Z_i. Note here that R_{i+1} will remain parallel to R_i whatever the value of the linear variable θ_{i+1}.

2.6 Properties of rotation matrices

Consider the links L_{i-1}, L_i which are attached to frames R_{i-1}, R_i, and R_{i+1} respectively. Let M_{i-1}^{i} and M_i^{i+1} denote the rotation matrices between frames R_i and R_{i-1}, and between frames R_{i+1} and R_i, respectively. Figure 2.20 shows the three links L_{i-1}, L_i and L_{i+1}. The product $M_{i-1}^{i}.M_i^{i+1}$ represents the translation matrix between frames R_{i+1} and R_{i-1}, which is function of two independent variables θ_i and θ_{i+1}. In other words the product $M_{i-1}^{i}.M_i^{i+1}$ gives the coordinates of vectors X_{i+1}, Y_{i+1}, and Z_{i+1} with respect to frame R_{i-1}.

$$M_{i-1}^{i}.M_i^{i+1} = [X_{i+1,i-1} Y_{i+1,i-1} Z_{i+1,i-1}] \tag{2.42}$$

Figure 2.20 shows the correspondence between the frames R_{i+1}, R_i and R_{i-1}, respectively. To simplify this notation, we take the following contracted form:

$$M_{i-1}^{i}.M_i^{i+1} = M_{i-1}^{i+1} \tag{2.43}$$

In the general case we also have:

$$M_i^{i+1}.M_{i+1}^{i+2}...M_{j-1}^{j} = M_i^{j} = [X_{i,j} Y_{i,j} Z_{i,j}] \tag{2.44}$$

Note that there is a correspondence chain between the frames $R_i, R_{i-1}, ...$, and R_i, respectively.

In the following we determine some properties of the translation matrices which are (1) the determinant of the product, (2) the inverse of product, and (3) commutativity of matrix product.

The determinant of the product of rotation matrices is always equal to one:

$$|M_i^j| = |M_i^{i+1}.M_{i+1}^{i+2}...M_{i-1}^{i}| = \prod_{k-i}^{i-1} | = 1 \tag{2.45}$$

The inverse of a product M_i^j of rotation matrices is equal to its transposed matrix $[M_i^j]^t$

$$\begin{aligned} \left[M_i^j\right]^{-1} &= \left[M_i^{i+1}...M_{j-1}\right]^{-1} \\ &= \left[M_{j-1}^{i}\right]^{-1}...\left[M_i^{i+1}\right] \\ &= \left[M_{j-1}^{i}\right]...\left[M_i^{i+1}\right]^t && (2.46) \\ &= M_j^{j-1}...M_{i+1}^{i} \\ &= M_j^{j} && (2.47) \end{aligned}$$

Note that M_i^i is the identity matrix I_3.

In general, the matrix product is not commutative for the set S of rotation matrices:

$$M_i^{i+1}.M_{i+1}^{i+2} \neq M_{i+1}^{i+2}.M_i^{i+1} \tag{2.48}$$

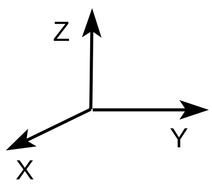

Fig 2.1: Frame of reference

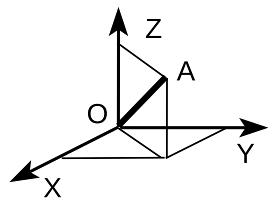

Fig 2.2: A point in a frame

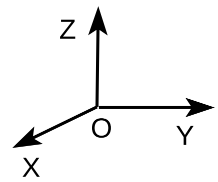

Fig 2.3: A frame and its origin

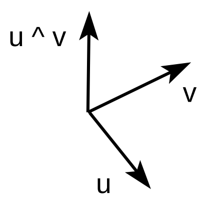

Fig 2.4: Vector product of u and v

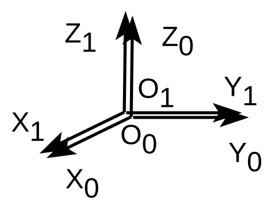

Fig 2.5: Two coinciding frames

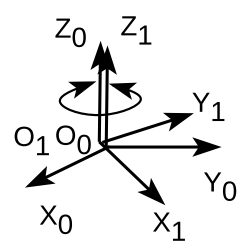

Fig 2.6: R1 rotates arround its Z axis.

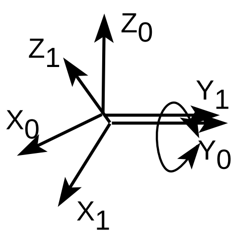

Fig 2.7: R1 rotates arround its Y axis.

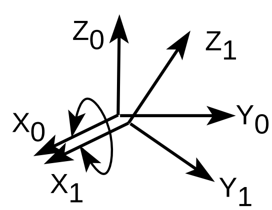

Fig 2.8: R1 rotates arround its Y axis.

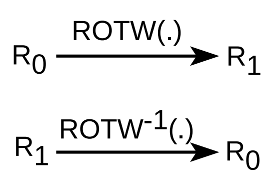

Fig 2.9: Direct and inverse frame transformations

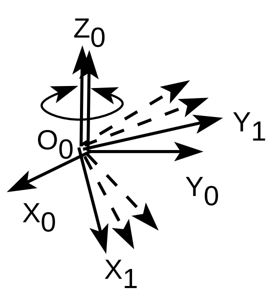

Fig 2.10: Successive rotations of R1

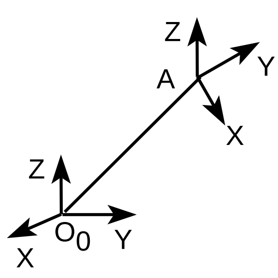

Fig 2.11: R1 is rotated and translated w.r.t. R0

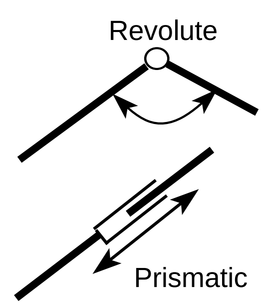

Fig 2.13: A revolute joint and a prismatic joint.

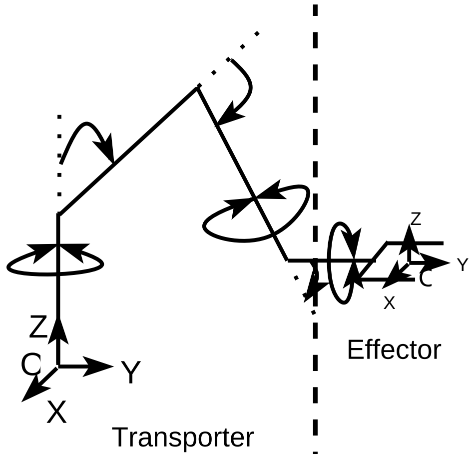

Fig 2.13: Decomposition of an arm

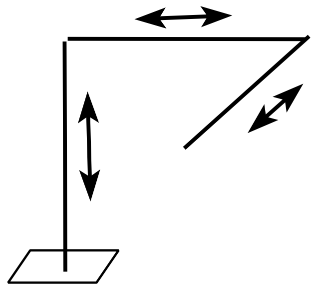

Fig 2.14: A cartesian arm

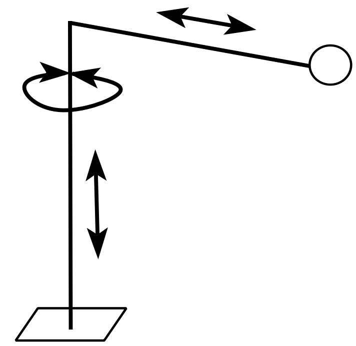

Fig 2.15: A cartesian arm

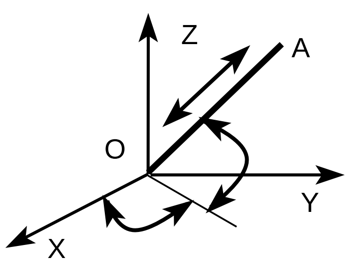

Fig 2.16: A spherical arm

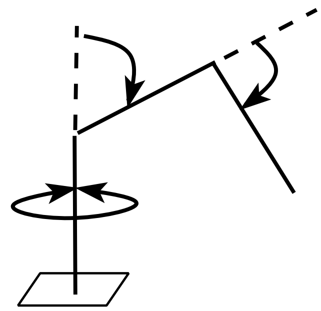

Fig 2.17: A general arm

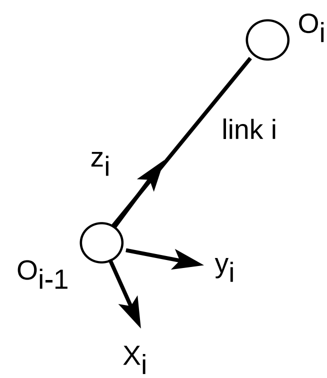

Fig 2.18: A frame attached to link

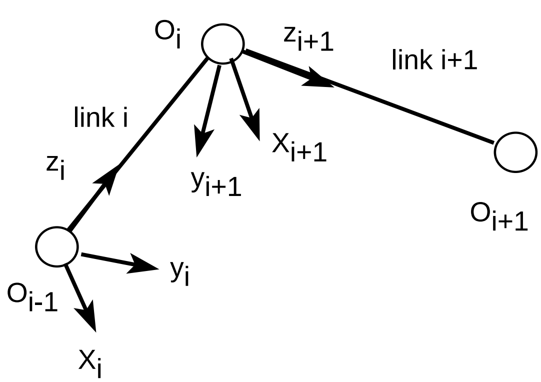

Fig 2.19: Two links with a revolute joint

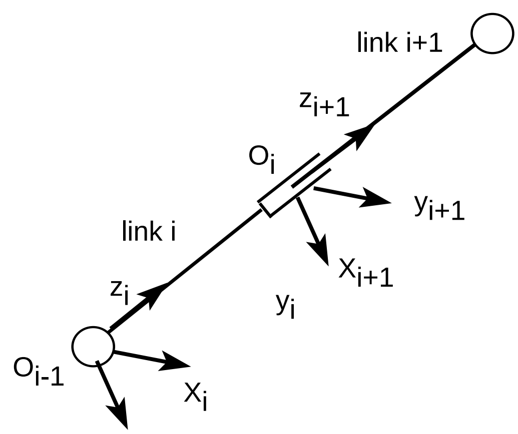

Fig 2.20: Two links with a prismatic joint

Exercises

1. Consider two vectors $U = (1\ 5\ \ -2)^t$ and $V = (3\ \ -2\ 1)^t$

   (a) Determine their scalar product $U.V$.

   (b) Determine their length

   (c) Determine two vectors W_1 and W_2 which are orthogonal to vectors U and V, respectively.

   (d) Determine a unit length vector W which is orthogonal to both vectors U and V.

2. Consider two vectors U and V as defined in problem 1.1, and consider the vector $W = (-3\ 5\ 2)^t$. Find the angle between vector U and W and the angle between V and W.

3. Consider the vectors U and V. Let α be their scalar product $\alpha = U^t.V$

   (a) Determine in which geometrical condition α is minimum or maximum.

   (b) Assume the vector $W = U \times V$, where $U = (2\ 6\ 3)^t$ $V = (3\ \ -X\ Y)^t$. For what values of X and Y the length of vector W is minimum or maximum.

4. The orthonormal vectors X and Y of a frame of reference R are defined by $X = (-1\ 0\ 0)^t$ and $Y = 0\ \ -1\ 0)^t$.

   (a) Find a vector Z that defines a right-handed frame $R)X, Y, Z)$.

   (b) Find a vector Z that defines a left-handed frame $R(X, Y, Z)$.

   (c) Find the rotation matrix between the two frames.

5. A fixed frame R_0 is given by:

   - Its origin O_0
   - Its orthonormal vectors X_0, Y_0 and Z_0.

   Another frame R_1, in motion relative to R_0. The frame R_1 is defined by its origin $\theta_0\theta_{1,0} = (1, 3, 2)^t$ and its orthonormal vectors $X_{1,0} = (a\ b\ c)^t$ and $Y_{1,0} = (u\ v\ w)^t$ which are given with reference to R_0.

   (a) Determine the vector $Z_{1,0}$

   (b) Determine the transfer matrix between the frames R_1 and R_0.

6. A frame R_1, which originally coincide with a fixed frame R_0, is rotated by an angle $\theta = 30^o$ about the X_0's axis.

   (a) Determine the rotation matrix $ROTX(\theta)$.

   (b) Determine the vectors $X_{1,0}, Y_{1,0}$, and $Z_{1,0}$ of frame R_0.

   (c) Suppose the frame R_1 is translated relative to R_0 by the vector $(1\ 2\ 1)^t$. A point A, which observed in frame R_0, is given by: $O_0A = (1\ \ -2\ 2)^t$. Determine the coordinates of A in R_1.

7. Consider the product of two rotation matrices: $ROTX(\theta).ROTY(\alpha) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Determine the angles θ and α and discuss the solutions.

8. Demonstrate that the orientation of an object can be fully described by means of three independent angles in the 3-dimensional space. Define these angles and discuss their relation and their order.

9. Give example of six degrees of freedom robot arms having transporter and effector parts.

   (a) How do you partition these d.o.f. among the transporter and effector substructures.

   (b) Show that the orientation of the effector is generally dependent on the first 3d.o.f. i.e., the interface between the transporter and the effector is designed such that the effector base depend on the orientation of the end segment of the Transporter.

   (c) Four types of transporters have been defined in Chapter 1, what is the advantage of each in terms of:

      i. Accessibility of the end, point.

      ii. Complexity of the relatioins between the coordinate of the end point and the coordinate of each degree of freedom.

10. Consider the four types of transporters that have been defined in Chapter 1. Using the coordinate system in the cartesian, cylindrical, spherical, and revolute spaces, find the expression of the transporter end point coordinate as function of the three degrees of freedom.

11. Consider a set of rotation matrices $\{M_i^{i+1} : i = \theta, ..., n\}$, where M_i^{i+1} denotes the transfer matrix between frames R_{i+1} and R_i.

   (a) What is the geometrical interpretation of the matrix $M_0^n = \{X_n, Y_n, Z_n\}$, where X_n, Y_n, and Z_n are column vectors.

   (b) Express the vectors X_n, Yn, and Z_n in a frame R_k where $O \leq k \leq n$.

   (c) What is the geometrical interpretation of the inverse matrix $[M_0^n]^{-1}$.

   (d) What should be the value of n such that the equation $M = M_0^n$ admits one and only solution. The matrix M is a constant and given matrix.

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Trignometric Formulas

$\sin(-\theta) = -\sin(\theta)$
$\cos(-\theta) = -\cos(\theta)$
$\sin^2(\theta) + \cos\cos 2(\theta) = 1$
$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha).\tan(\beta)}$