# Chapter 3

# MOTION COORDINATION

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This chapter introduces the robot motion coordination system based on the geometric model of the robot arm. The geometric model and its inverse model are suitable mathematical tools for implementing the motion coordination system based on position control of the robot arm. The geometric model allows mapping trajectories of the robot effector, that are described in the joint space, into the corresponding trajectories in the cartesian space. The robot end effector is represented by a position and an orientation in the three dimensional space. Therefore, a point in the joint space can be converted using the geometric model to a position and an orientation in the cartesian space. The geometric model represents a set of six non-linear equations that give the expression of the effector vector E as function of the joint variables vector $\theta$. The geometric model is denoted by $E = G(\theta)$ .

Mechanically, the robot arm can be moved by modifying its position in the joint space. Therefore, we need an operator to transform the desired position and orientation in the cartesian space into position in the joint space. This operator is called the inverse geometric model. To obtain the inverse geometric model, one needs to invert the system equation $E = G(\theta)$, and find the operator $\theta = G^{-1}(E)$ . The solution $\theta = G^{-1}(E)$ does not exist for all robot arms and when it exists multiple solutions should be expected in the general case. For example, when the last three degrees of freedom are revolute and their rotation axes are not concurrent, then no closed form solution can be found by inverting the system equation $E = G(\theta)$ .

When a closed form does exist, multiple solutions are generally expected. This means that more than one configuration of the joint variables allow setting the robot hand at the same position and orientation. The problems of selecting one solution out of many solutions will be studied on the light of some practical requirements such as global motion continuity. Position control can then be implemented with respect to each joint variable.

The theoretical issues will be followed by examples of robot arms having three and six degrees of freedom.

## 3.1 Introduction to the geometric method

The principle of this method is to express the position and orientation of the robot hand as function of the joint variables. means of an absolute representation of the end effector. A simple representation of the robot hand consists of considering the frame of reference which is attached to last link of the chain. The effector of a robot arm having n degrees of freedom can be geometrically represented by means the coordinate of the origin of frame $R_n$ and its orientation matrix $R_n$ with respect to a fixed frame $R_0$. The frame $R_n$ is called the effector frame of reference. The position and orientation of the effector frame (See Figure 2.1-A) can be totally determined by means of the following information:

1. The robot hand center or vector $O_0O_{n,0}$ which references the origin of $R_n$ with respect to $R_0$.
2. The robot hand orientation matrix $M_0^n = [X_{n,0}, Y_{n,0}, Z_{n,0}]$ which determines the orientation of frame $R_n$ with respect to frame $R_0$.

These parameters are function of the joint variables $\theta_1, \theta_2, ..., \theta_n$. For this we note them as follows $O_0O_{n,0}(\theta)$ and $M_0^n(\theta)$ . Figure 2.1-B shows the relation between the effector frame and the joint variables for a 6 d.o.f. robot arm. The basic geometrical representation of the arm is reduced to the short expression:

$$G(\theta) = \{O_0O_{n,0}(\theta), M_0^n(\theta)\} \tag{3.1}$$
$$\text{Geometrical} \quad \text{Basic representation} \tag{3.2}$$
$$\text{representation} \quad \text{of the end effector} \tag{3.3}$$

The direct geometric problem is to compute the basic representation with respect to a reference cartesian coordinate $R_0$, given the joint variables $\theta_1, \theta_2, ..., \theta_{n-1}$ and $\theta_n$:

$$\theta = (\theta_1, \theta_2, ..., \theta_n)^t \rightarrow \{O_0O_n(\theta), M_0^n(\theta)\} \tag{3.4}$$
$$\text{associated a set of geometric equations}$$

The inverse geometric problem is to compute the joint variables $\theta_1, ..., \theta_n$, given the basic representation of the end effector:

$$\{O_0O_n, M_o^n\} \rightarrow \theta = (\theta_1, ..., \theta_n)^t \tag{3.5}$$

The inverse geometric system allows assigning the position and orientation of the end effector by acting on the joint variables. At the lowest level, the robot arm can only be controlled by moving the joint positions which can be rovolute or prismatic degrees of freedom. The control system allows generating torques to move and maintain the robot arm according to a prescribed trajectory in the joint space. The prescribed trajectory, or desired trajectory, is generated by the inverse geometric transform as image of a trajectory described in the effector space. Therefore, the inverse geometric system is one basic concept to robot motion coordination because it allows controlling the robot arm at the level of the robot end effector and the cartesian space.

## 3.2 Elaboration of the geometric model

This problem consists of finding the expression of the basic effector position and orientation, with respect to a fixed frame of reference $R_0$, as a function of the joint variables vector $\theta$.

Consider a serial robot arm consisting of an $n$ degrees of freedom arm as shown in Figure 2.2. A frame of reference is attached to every link and these frames are all observed with respect to a fixed frame $R_0$. Every link $L_i$ is represented by means of a vector $O_{i-1}O_i$ . The basic Effector parameters are the position vector $O_0O_{n,0}(\theta)$ and the orientation matrix $M_0^n(\theta)$. The position vector $O_0O_{n,0}$ can be decomposed as the sum of the link vectors:

$$O_0O_{n,0} = \sum_{i=1}^{n} O_{i-1,0}O_{i,0} \tag{3.6}$$

Using the translation matrix between frames $R_0$ and $R_1$, each link vector can be expressed with respect to its own frame of reference:

$$O_{i-1}O_{i,0} = M_0^i.O_{i-1}O_{i,i} \tag{3.7}$$

Vector $O_{i-1}O_{i,0}$ can then be expressed as:

$$O_0O_{n,0} = \sum_{i=1}^{n} M_0^i.O_{i-1}O_{i,i} \tag{3.8}$$

Vector $O_{i-1}O_{i,i}$ has simple expression because it is represented with respect to its own frame of reference $R_i$ . Obviously, both vector $O_{i-1}O_{i,0}$ and $\sum_{i=1}^{n} M_0^i$ can be expressed in a recursive form, at step i (link i) we obtain:

$$\begin{aligned} M_0^i &= M_0^{i-1}.M_{i-1}^i \\ O_0O_{i,0} &= O_0O_{i-1,0} + M_0^i.O_{i-1}O_{i,i} \end{aligned} \tag{3.9}$$

where $M_0^{i-1}$ is the transfer matrix between frames $R_{i-1}$ and $R_0$, and $M_{i-1}^i$ is the transfer matrix between frames $R_i$ and $R_{i-1}$ .

The basic geometric model consists of computing the position and orientation of the robot end effector:

$$\begin{aligned} O_oO_{n,0} &= \sum_{i=1}^{n} M_o^i.O_{i-1}O_{i,i} \\ M_0^n &= \prod_{i-1}^{n} M_{i-1}^i \end{aligned} \tag{3.10}$$

The system equation 2.6 allows forward, recursive computation of the basic geometric model computation. It is interesting to find ways of efficiently computing of the above system equations based on the knowledge of motion type and geometric length of each link. One may derive procedural approaches for efficient real-time computation of the above geometric model.

## 3.3 Case of a three-revolute arm

Let us consider the transporter part of a robot arm that is defined by three revolute joints as shown in Figure 2.3. We use the following topological form in order to describe the geometric structure of the arm:

$$\begin{aligned} &Link\ 1\ \ (R(Z), Z(L_1)) \\ &Link\ 2\ \ (R(X), Z(L_2)) \\ &Link\ 3\ \ (R(X), Z(L_3)) \end{aligned} \tag{3.11}$$

Consider link $L_1$ which is revolute and defined as a rotation about $Z_0$ axis $(R(Z))$, and the link body $L_1$ is along vector $Z_1$ of frame $R_1$ $(Z(L_1))$. The other links are defined in a similar manner.

For the first end $0_1$ we have:

$$O_0O_{1,0} = M_0^1.O_0O_{1,1} \tag{3.12}$$

As link $L_1$ is defined by a rotation about axis $Z_0$, then the transfer matrix between frames $R_1$ and $R_0$ is a $ROTZ(\theta_1)$, we have:

$$M_0^1 = \begin{bmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [X_{1,0}, Y_{1,0}, Z_{1,0}] \tag{3.13}$$

The frame $R_1$ is defined such that the link vector $O_0O_1$ is along vector $Z_1$, then we have:

$$O_0O_1 = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$$

Therefore, vector

$$O_0O_{1,0} = \begin{bmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$$

For the second end point $O_2$, we have:

$$\begin{aligned} O_0O_{2,0} &= O_0O_{1,0} + M_0^2.O_1O_{2,2} \\ M_0^2 &= M_0^1.M_1^2 \end{aligned} \tag{3.14}$$

Where $M_1^2$ is a rotation matrix axis $X_1$, we have:

$$M_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C2 & -S2 \\ 0 & S2 & C2 \end{bmatrix} \tag{3.15}$$

Let us evaluate the orientation matrix $M_0^2$ of frame $R_2$, we have:

$$\begin{aligned} M_0^2 = M_0^1.M_1^2 &= \begin{bmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & C2 & -S2 \\ 0 & S2 & C2 \end{bmatrix} \\ &= \begin{bmatrix} C1 & -S1C2 & S1S2 \\ S1 & C1C2 & -C1S2 \\ 0 & S2 & C2 \end{bmatrix} \end{aligned} \tag{3.16}$$

Vector $O_0O_{2,0}$ is the sum of two terms:

$$O_0O_{2,0} = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + M_0^2. \begin{bmatrix} 0 \\ 0 \\ L_2 \end{bmatrix} = \begin{bmatrix} S1S2L_2 \\ -C1S2L_2 \\ L_1 + C2L_2 \end{bmatrix} \tag{3.17}$$

Finally, vector $O_0O_3$ is given by:

$$O_0O_{3,0} = O_0O_{2,0} + M_0^3.O_2O_{3,3} \tag{3.18}$$

and the orientation matrix is:

$$\begin{aligned} M_0^3 = M_0^2.M_2^3 &= M_0^2. \begin{bmatrix} 1 & 0 & 0 \\ 0 & C3 & S3 \\ 0 & S3 & C3 \end{bmatrix} \\ &= \begin{bmatrix} C1 & -S1C23 & S1S23 \\ S1 & C1C23 & -C1S23 \\ 0 & S23 & C23 \end{bmatrix} \end{aligned} \tag{3.19}$$

where $S23 = SIN(\theta_2 + \theta_3)$ and $C23 = COS(\theta_2 + \theta_3)$ . Vector $O_0O_{3,3}$ is the sum:

$$\begin{aligned} O_0O_{3,0} &= \begin{bmatrix} S1S2L_2 \\ -C1S2L_2 \\ L_1 + C2L_2 \end{bmatrix} + M_0^3 \begin{bmatrix} 0 \\ 0 \\ L_3 \end{bmatrix} \\ &= \begin{bmatrix} S1(S2L_2 + S23L_3) \\ -C1(S2L_2 + S23L_3) \\ L_1 + C2L_2 + C23L_3 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \end{aligned} \tag{3.20}$$

The geometric system for the transporter part is defined by:

$$(\theta_1, \theta_2, \theta_3) \rightarrow \left\{ O_0O_3(\theta), M_0^3(\theta) \right\}$$

The inverse geometric model of the transporter consists of finding closed form solutions for $\theta_1, \theta_2$, and $\theta_3$ as functions of the transporter end point coordinates $X$, $Y$, and $Z$. Evaluation of the geometric model of the transporter allows writing the coordinate of vector $O_0O_{3,0}(\theta)$ as follows:

$$\begin{aligned} X &= S1(S2L_2 + S23L_3) \\ Y &= -C1(S2L_2 + S23L_3) \\ Z &= L_1 + C2L_2 + C23L_3 \end{aligned} \tag{3.21}$$

To solve the system we first consider the expressions of $X$ and $Y$ in order to evaluate $(S2L_2+S23L_3)$, we can easily obtain:

$$S2L_2 + S23L_3 = \pm\sqrt{x^2 + Y^2} \tag{3.22}$$

When the point $O_3$ is not on the $Z_0$ axis, i.e., $X^2 + Y^2 \neq 0$ , then the sine and cosine of $\theta_1$ can be evaluated as follows:

$$S1 = \pm\frac{X}{\sqrt{X^2 + Y^2}} \ \ and \ \ C1 = \pm\frac{-Y}{\sqrt{X^2 + Y^2}} \tag{3.23}$$

Depending on the sign, we have two solutions for the angle $\theta_1$ . When the sign $(+)$ is selected, we obtain the following solution:

$$S1^+ = \frac{X}{\sqrt{X^2 + Y^2}} \ \ and \ \ C1^+ = \frac{-Y}{\sqrt{X^2 + Y^2}} \tag{3.24}$$

Note that the knowledge of both sine and cosine of an angle allows finding a unique solution for that angle. The other solution will be obtained by using sign $(-)$:

$$S1^- = -S1^+ \text{ and } C1^- = -C1^+ \tag{3.25}$$

Consequently two solutions are expected for the angle $\theta_1$:

$$\begin{aligned} (S1^+, C1^+) &\rightarrow \theta_1^+ \\ (S1^-, C1^-) &\rightarrow \theta_1^- \end{aligned} \tag{3.26}$$

The solution $\theta_1^+$ can be evaluated as follows:

$$\theta_1^+ = TAN^{-1}(S1^+, C1^+)$$

Solutions $\theta_1^+$ and $\theta_1^-$ differ by $\pi$, in the domain $[-\pi, +\pi]$, Therefore, we have:

$$\theta_1^- = \theta_1^+ - Sign(\theta_1^+).\pi \tag{3.27}$$

When the robot is following a trajectory in the cartesian space, the previous solution that corresponds to the previous point of the trajectory can be used in order to select one solution out of two. Therefore, to determine a unique solution $\theta$ one may compare $\theta_1^+$ and $\theta_1^-$ to the previous value of $\theta$ . Clearly, the closest solution to the previous one allows satisfying a continuity criteria on the cartesian trajectory. In addition to finding a solution $\theta_1$, this operation allows finding the sign of $S1$, $C1$, and $S2L_2 + S23L_3$. To determine $\theta_3$, we consider the expression of $X$, $Y$, and $Z$. We have:

$$\begin{aligned} X^2 + Y^2 &= (S2L_2 + S23L_3)^2 \\ (Z - L_1)^2 &= (C2L_2 + C23L_3)^2 \end{aligned} \tag{3.28}$$

After developing the above relations, we obtain:

$$\begin{aligned} X^2 + Y^2 &= (S2L_2)^2 + (S23L_3)^2 + 2S2S23L_2L_3 \\ (Z - L_1)^2 &= (C2L_2)^2 + (C23L_3)^2 + 2C2C23L_2L_3 \end{aligned} \tag{3.29}$$

and adding:

$$X^2 + Y^2 + (Z - L_1)^2 = L_2^2 + L_3^2 + 2C3L_2L_3 \tag{3.30}$$

We obtain $C(\theta_3)$ and $S(\theta_3)$:

$$\begin{aligned} C3 &= (X^2 + Y^2 + (Z - L_1)^2 - L_2^2 - L_3^2)/2L_2L_3 \\ S3 &= \pm\sqrt{1 - C3^2} \end{aligned} \tag{3.31}$$

Two symmetric solutions for $\theta_3$ are expected as shown in Figure 2.4. Naturally these solutions correspond to two different configurations but both allows the transporter end point being set at the coordinates specified by $X$, $Y$, and $Z$. Obviously, this arm can reach all the positions of its work space by specifying the angle $\theta_3$ in one of the intervals $[0, +\pi]$ and $[-\pi, 0]$ . Depending on which interval is selected, the sign of $S3$ can then be determined. A criteria on space occupancy of the arm can be used in order to chose one of these intervals. Once the term $S3$ is found, angle $\theta_3$ can then be evaluated as follows:

$$\theta_3 = TAN^{-1}(S3, C3)$$

Finally, to determine angle $\theta_2$, we consider the following equations:

$$\begin{aligned} S1X - C1Y &= S2L_2 + S23L_3 \\ Z - L_1 &= C2L_2 + C23L_3 \end{aligned} \tag{3.32}$$

After developing $S23$ and $C23$, we can write these equations in a matrix form:

$$\begin{bmatrix} S1X - C1Y \\ Z - L_1 \end{bmatrix} = \begin{bmatrix} S3L_3 & L_2 + C3L_3 \\ L_2 + C3L_3 & -S3L_3 \end{bmatrix} \cdot \begin{bmatrix} C2 \\ S2 \end{bmatrix} \tag{3.33}$$

The determinant of this matrix is given by:

$$\Delta = (S3L_3)^2 + (L_2 + C3L_3)^2 = -(L_2^2 + L_3^2 + 2L_2L_3C3) \tag{3.34}$$

In general, $\Delta$ is not nil except when $L_2 = L_3$ and $\theta_3$ is equal $\pm\pi$ . This configuration of $\theta_3$ cannot occur in a mechanical robot arm. The solution $C2$ and $S2$ can always be obtained as follows:

$$\begin{aligned} S2 &= \frac{(XS1 - YC1)(L_2 + C3L_3) - (Z - L_1)S3L_3}{L_2^2 + L_3^2 + 2L_2L_3C3} \\ C2 &= \frac{(XS1 - YC1)S3L_3 + (Z - L_1)(L_2 + C3L_3)}{L_2^2 + L_3^2 + 2L_2L_3C3} \end{aligned} \tag{3.35}$$

The solution for the angle $\theta_2$ can then be obtained as follows:

$$\theta_2 = TAN^{-1}(S2, C2)$$

This solution depends on the selected values of $\theta_1$, $\theta_3$, and their respective signs.

## 3.4 Multiple solutions and singularities

Let us consider the transporter defined in Section 2.3. When only considering the links $L_2$ and $L_3$, two solution are generally expected when the transporter end point is set to any position specified by $X$, $Y$, and $Z$. Mathematically, this is because $S3$ cannot be determined by using the system $O_0O_3, 0(\theta) = (X, Y, Z)^t$ . On the other hand, the mechanical structure of this arm indicates clearly that two configurations for $(\theta_2, \theta_3)$ exist while the end point $O_3$ is fixed. These configurations are:

$$\{\theta_1, \theta_3^+, \theta_2(\theta_1, \theta_3^+)\} \ and \ \{\theta_1, \theta_3^-, \theta_2(\theta_1, \theta_3^-)\} \tag{3.36}$$

To make a decision about which solution should be kept, one needs to use one of the following methods:

1. Use of a continuity criteria on the cartesian trajectory so that decision will be made by comparing the solutions $\theta_3^+$ and $\theta_3^-$ to the previous solution. This method allows maintaining the sign of angle $\theta_3$ fixed during the motion of the arm. To initialize the motion, the starting configuration should implicitly include this information about the selected sign of $\theta_3$.

2. Use of a flag to indicate the current value of the sign of $\theta_3$ . In this case, no comparison will be made but rather the sign of $\theta_3$ will be selected according to the value of the flag which should be appropriately initialized by the system. This solution can be augmented with a function that allows switching the value of the flag and generation of smooth motion between the two configurations.

Regarding the angle $\theta_1$, we always have two solutions $\theta_1^+$ and $\theta_1^-$ for each of which there exists two possible configuration for angles $\theta_2$ and $\theta_3$ . Figure 2.5 shows that the arm admits four solutions when solving the system $O_0O_{3,0}(\theta) = (X, Y, Z)^t$:

$$\theta_1^- = \theta_1^+ - sign(\theta_1^+).\pi \tag{3.37}$$

Because $\theta_1$ is in $[-\pi, +\pi]$, the solutions are:

$$\left[\theta_1^+, \theta_3^+, \theta_2(\theta_1^+, \theta_3^+)\right]$$

$$\left[\theta_1^+, \theta_3^-, \theta_2(\theta_1^+, \theta_3^-)\right]$$

$$\left[\theta_1^-, \theta_3^+, \theta_2(\theta_1^-, \theta_3^+)\right]$$

$$\left[\theta_1^-, \theta_3^-, \theta_2(\theta_1^-, \theta_3^-)\right]$$

The selection of one solution for angle $\theta_1$ is easier because the values of $\theta_1^+$ and $\theta_1^-$ always differ by $\pi$ . Therefore, one needs to compare with the previous value of $\theta_1$ in order to select either $\theta_1^+$ or $\theta_1^-$ . As angle $\theta_1$ is generally defined in $[-\pi, +\pi]$, no pre-selection can be made with respect to the solutions $\theta_1^+$ and $\theta_1^-$. Let us consider the case when ($X = 0$ and $Y = 0$), i.e., the end point $0_3$ is on the $Z_1$ axis as shown in Figure 2.6. The system equation becomes:

$$\begin{aligned} X &= S1(S2L_2 + S23L_3) = 0 \\ Y &= -C1(S2L_2 + S23L_3) = 0 \\ Z &= L_1 + C2L_2 + C23L_3 \end{aligned} \tag{3.38}$$

As $S1$ and $C1$ cannot be equal to zero simultaneously, then the term $S2L_2 + S23L_3$ is nil. The system equation becomes:

$$\begin{aligned} S2L_2 + S23L_3 &= 0 \\ C2L_2 + C23L_3 &= Z - L_1 \end{aligned} \tag{3.39}$$

After developing the terms $S23$ and $C23$, the solution for $\theta_2$ can then be obtained as follows:

$$\begin{aligned} C2 &= -\frac{(z - L_1)(L_2 + C3L_3)}{L_2^2 + L_3^2 + 2C3L_2L_3} \\ S2 &= \frac{S3L_3(Z - L_1)}{L_2^2 + L_3^2 + 2C3L_2L_3} \end{aligned} \tag{3.40}$$

On the other hand, the solution for $\theta_3$ is obtained by using the precedent equation of $C3$:

$$\begin{aligned} C3 &= \frac{(Z - L_1)^2 - L_2^2 - L_3^2}{2L_2L_3} \\ S3 &= \pm\sqrt{1 - C3^2} \end{aligned} \tag{3.41}$$

Suppose we determine the solution for $\theta_2$ and $\theta_3$ as previously discussed, the remaining angle $\theta_1$ is undetermined because no information is available about this angle. The system equation $O_0O_{3,0}(\theta) = (X, Y, Z)^t$ does not give any information regarding angle $\theta_1$ when $X = 0$ and $Y = 0$. This situation is called a singularity case for $\theta_1$ because there exists an infinite number of values for $\theta_1$ for which the system's equations (2.38) is satisfied. One may keep angle $\theta_1$ unchanged until the arm moves to a new point in which the condition ($X = 0$ and $Y = 0$) is no more satisfied. Another method consists of using the criteria on trajectory continuity which can be helpful in this case. The application of this criteria consists of extrapolating the trajectory of angle $\theta_1(t)$ by using time polynomial approximation. A discrete time polynomial $\theta_1(nT)$ can be used to approximate the time function $\theta_1(t)$ over a finite number of points $\theta_1(n-1), ..., \theta_1(n-k)$ . The values of $\theta_1(n-i)$ represent the ith previous solution of the system equations. Using $k$ previous solutions, the associated polynomial can be used to predict the current value of $\theta_1$:

$$\theta_1(n) = F\left[\theta_1(n-1), ..., \theta_1(n-k)\right]$$

If one assumes constant sampling period of the system, a second order polynomial approximation can then give the following solution:

$$\theta_1(n) = 3\theta(n-1) - 3\theta(n-2) + \theta(n-3) \tag{3.42}$$

## 3.5 Case of a two-revolute and one-prismatic arm

Now consider a robot arm whose transporter part has two revolute and one prismatic joints. Figure 2.7 shows this structure that can be defined by using the following geometric topology:

$$\begin{array}{c} Link\ 1\ \ (R(Z), Z(L_1)) \\ Link\ 2\ \ (R(X), Z(L_2)) \\ Link\ 3\ \ (P(X), Z(L_3)) \end{array} \tag{3.43}$$

Note here that the length of link $L_2$ is nil which means that no link translation will be generated by angle $\theta_2$. The degree of freedom $\theta_2$ is only used to orient the next link $L_3$ .

Let us evaluate the geometric model of this transporter. For this, we first evaluate the vector $O_0O_{1,0}$ . As the rotation matrix of frame $R_1$ is defined by $ROTZ(\theta_1)$ with respect to frame $R_0$, then vector $O_0O_1, 0$ can be expressed as:

$$M_0O_{1,0} \quad = \quad M_0^1 \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} \tag{3.44}$$

To evaluate vector $O_0O_{2,0}$, we first need to evaluate the rotation matrix $M_0^2$ of frame $R_2$ with respect to frame $R_0$:

$$\begin{aligned} M_0^2 = M_0^1.M_1^2 = ROTZ(\theta_1).ROTX(\theta_2) &= \begin{bmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & C2 & -S2 \\ 0 & S2 & C2 \end{bmatrix} \\ &= \begin{bmatrix} C1 & -S1C2 & S1S2 \\ S1 & C1C2 & -C1S2 \\ 0 & S2 & C2 \end{bmatrix} \end{aligned} \tag{3.45}$$

The expression of vector $O_0O_{2,0}$ is then given by:

$$O_0O_{2,0} \quad = \quad O_0O_{1,0} + M_0^2 . \begin{bmatrix} 0 \\ 0 \\ L_2 \end{bmatrix} \tag{3.46}$$

Evaluation of this vector gives:

$$\begin{aligned} O_0O_{2,0} &= \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + \begin{bmatrix} C1 & -S1C2 & S1S2 \\ S1 & C1C2 & -C1S2 \\ 0 & S2 & C2 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ L_2 \end{bmatrix} \\ &= \begin{bmatrix} S1S2L_2 \\ -C1S2L_2 \\ L_1 + C2L_2 \end{bmatrix} \end{aligned} \tag{3.47}$$

The third degree of freedom is prismatic, therefore the transfer matric between frame $R_3$ and frame $R_2$, is equal to the identity matrix, therefore we have:

$$M_2^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I, \text{ and } M_0^3 = M_0^2 \tag{3.48}$$

$O_0O_{3,0}$ can then be expressed as follows:

$$\begin{aligned} O_0O_{3,0} &= \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + M_0^3 \begin{bmatrix} 0 \\ 0 \\ L_3 + \theta_3 \end{bmatrix} \\ &= \begin{bmatrix} -S1S2(L_2 + L_3 + \theta_3) \\ -C1S2(L_2 + L_3 + \theta_3) \\ L_1 + C2(L_2 + L_3 + \theta_3) \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \end{aligned} \tag{3.49}$$

This equation represents the geometric model of the transporter. As it can be seen, the revolute variables $(\theta_1, \theta_2)$ appears in the trigonometric functions *sin* and *cos* while the prismatic variable $\theta_3$ appears as a linear function that increases or decreases the length of link $L_3$ . to find the inverse geometric transform, we need to invert the system equation previously obtained:

$$\begin{aligned} X &= S1S2(L_2 + L_3 + \theta_3) \\ Y &= -C1S2(L_2 + L_3 + \theta_3) \\ Z &= L_1 + C2(L_2 + L_3 + \theta_3) \end{aligned} \tag{3.50}$$

First, we can eliminate the variable $\theta_1$ from the system equations taking the squares of $X$, $Y$, and $Z$, we have the following two equations:

$$\begin{aligned} X^2 + Y^2 &= S2^2(L_2 + L_3 + \theta_3)^2 \\ (Z - L_1)^2 &= C2^2(L_2 + L_3 + \theta_3)^2 \end{aligned} \tag{3.51}$$

The prismatic variable $\theta_3$ can be evaluated as follows:

$$X^2 + Y^2 + (Z - L_1)^2 = (L_2 + L_3 + \theta_3)^2 \tag{3.52}$$

For a robot arm, it is meaningful to assume that $L_2 + L_3 + \theta_3 \geq 0$ because of mechanical constraints. Therefore, we have the solution for angle $\theta_3$:

$$\theta_3 = \sqrt{X^2 + Y^2 + (Z - L_1)^2} - L_3 - L_2 \tag{3.53}$$

The term $L_2 + L_3 + \theta_3$ is known, we can find two solutions for angle $theta_2$ as follows:

$$\begin{aligned} C2 &= \frac{Z - L_1}{L_2 + L_3 + \theta_3} \\ S2 &= \pm\sqrt{X^2 + Y^2}/(L_2 + L_3 + \theta_3) \quad (L_2 + L_3 + \theta_3 > 0) \end{aligned} \tag{3.54}$$

This gives two solutions $\theta_2^+$ and $\theta_2^-$ as a result of using $(S2^+, C2)$ and $(S2^-, C2)$, respectively. Therefore, we have:

$$\theta_2^+(S2^+, C2) \text{ and } \theta_2^-(S2^-, C2) \tag{3.55}$$

These solutions are symmetrical as shown in Figure 2.8:

$$\theta_2^+ = -\theta_2^- \tag{3.56}$$

To find angle $\theta_1$, we consider the term $S2(L_2 + L_3 + \theta_3)$ which can be expressed by:

$$S2(L_2 + L_3 + \theta_3) = \pm\sqrt{X^2 + Y^2} \tag{3.57}$$

The sign of the square root is identical to that of $\theta_2$, we can then write:

$$S2(L_2 + L_3 + \theta_3) = Sign(\theta_2).\sqrt{X^2 + Y^2} \tag{3.58}$$

Since, angle $\theta_1$ can be found if and only if $X^2 + Y^2 \neq 0$ . We have:

$$S1 = Sign(\theta_2)\frac{X}{\sqrt{X^2 + Y^2}} \text{ and } C1 = -Sign(\theta_2)\frac{Y}{\sqrt{X^2 + Y^2}} \tag{3.59}$$

Two solutions are obtained for angle $\theta_1$ depending on the sign of $\theta_2$:

$$\theta_1^+(S1^+, C1^+) \text{ and } \theta_1^-(S1^-, C1^-) \tag{3.60}$$

Obviously these solutions differ by $\pi$, we therefore have:

$$\theta_1^- = \theta_1^+ - Sign(\theta_1^+).\pi \tag{3.61}$$

To select one solution out of two, we better compare both solutions $\theta_1^+$ and $\theta_1^-$ to the previous value of $\theta_1$ . As $\theta_1^+ +$ and $\theta_1^- +$ differ by $\pi$, then the closest solution to $\theta_1$ will be selected. Following this comparison, the sign of angle $\theta_2$ can then be uniquely determined because of the previous expressions of $S1$ and $C1$. As a result, a unique solution for both angles $\theta_1$ and $\theta_2$ can then be found using the trajectory continuity as described above.

Consider the case of point $O_3$ when it is located on the $Z_0$ axis, this situation corresponds to $X = 0$ and $Y = 0$. As the first and second equation are nil, then no information is provided regarding angle $\theta_1$ . Clearly, this case presents a singularity point for the transporter. Any value of the angle $\theta_1$ satisfies the system equations which becomes:

$$\begin{aligned} \theta_2 &= 0 \\ Z - L_1 &= L_2 + L_3 + \theta_3 \end{aligned} \tag{3.62}$$

Solving this system gives:

$$\begin{aligned} \theta_3 &= Z - L_1 - L_2 - L_3 \\ \theta_2 &= 0 \\ \theta_1 & \quad \text{is undetermined} \end{aligned} \tag{3.63}$$

As discussed previously, no mathematical solution can provided by the knowledge of the current point. To ensure continuity of the trajectory, the control program should generate solutions that satisfy the continuity criteria with respect to the trajectory in the joint space. As proposed in the previous example, a time polynomial of $\theta_1(t)$ can be used in order to predict the next point of angle $\theta_1$ .

## 3.6 Case of a one-prismatic and two-revolute arm

Consider the three d.o.f. robot arm which is shown in Figure 2.9. The arm architecture has the following geometric topology:

$$\begin{array}{c} Link\ 1\ (P(Z),\ Z(L_1)) \\ Link\ 2\ (R(Z),\ Z(L_2)) \\ Link\ 3\ (R(X),\ Z(L_3)) \end{array} \tag{3.64}$$

This robot arm is observed relative to a fixed frame of reference $R_0$. The link $L_1$ is defined along the $Z_0$ axis and the first degree of freedom $\theta_1$ is prismatic, the coordinates of point $O_{1,0}$ becomes:

$$O_0O_{1,0} = M_0^1 O_0O_{1,1} = \begin{bmatrix} 0 \\ 0 \\ L_1 + \theta_1 \end{bmatrix} \tag{3.65}$$

where the matrix $M_0^1$ is the identity matrix because link $L_1$ is prismatic. Now we can express the coordinates of point $O_2$ as follows:

$$O_0O_{2,0} = O_0O_{1,0} + M_0^2.O_1O_{2,2} \tag{3.66}$$

The link $L_2$ is defined along the $Y_2$ axis and its motion is revolute about the $Z_1$ axis, we have:

$$M_0^2 = M_0^1.M_1^2 = M_1^2 = \begin{bmatrix} C2 & -S2 & 0 \\ S2 & C2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.67}$$

The orientation matrix of Link $L_2$ can simply be expressed as:

$$O_2O_{2,0} = \begin{bmatrix} 0 \\ 0 \\ L_1 + \theta_1 \end{bmatrix} + \begin{bmatrix} C2 & -S2 & 0 \\ S2 & C2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ L_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L_1 + L_2 + \theta_1 \end{bmatrix} \tag{3.68}$$

The link $L_3$ is on the $Y_3$ axis and has a revolute motion about the $X_2$ axis:

$$M_0^3 = M_0^1.M_1^2.M_2^3 = \begin{bmatrix} C2 & -S2 & 0 \\ S2 & C2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & C3 & -S3 \\ 0 & S3 & C3 \end{bmatrix} = \begin{bmatrix} C2 & -S2C3 & S2S3 \\ S2 & C2C3 & -C2S3 \\ 0 & S3 & C3 \end{bmatrix} \tag{3.69}$$

The coordinate of point $O_3$ can then be expressed as:

$$O_0O_{3,0} = O_0O_{2,2} + M_0^3.O_2O_{3,3} = \begin{bmatrix} S2S3L_3 \\ -C2S3L_3 \\ L_1 + L_2 + \theta_1 + C3L_3 \end{bmatrix} \tag{3.70}$$

The inverse geometrical problem is to determine the joint variables $\theta_1$, $\theta_2$ and $\theta_3$ given the parameters $X$, $Y$ and $Z$. We determine $\theta_2$ from the equations of $X$ and $Y$:
Let us eliminate $\theta_2$ by squaring and adding the first two equations, we obtain:

$$X^2 + Y^2 = (S3L_3)^2 \tag{3.71}$$

For simplicity, we assume the term $S3$ is always positive or nil:

$$s3L_3 = \sqrt{X^2 + Y^2} \tag{3.72}$$

The term $S3$ can then be uniquely determined

$$S3 = \frac{1}{L_3}\sqrt{X^2 + Y^2} \tag{3.73}$$

However, two solutions are expected for $\theta_3$ as it can be seen from Figure. This means that S3 cannot be uniquely determined:

$$C3^+ = \sqrt{1 - S3^2} \text{ and } C3^- = -C3^+ \tag{3.74}$$

Two solutions are expected for $\theta_3$:

$$\theta_3^+(C3^+, S3) \text{ and } \theta_3^-(C3^-, S3) \tag{3.75}$$

In any case, if $X^2 + Y^2 \neq 0$ we can determine a unique solution for $\theta_2$:

$$\begin{aligned} S2 &= X/\sqrt{X^2 + Y^2} \\ C2 &= -Y/\sqrt{X^2 + Y^2} \end{aligned} \tag{3.76}$$

Angle $\theta_2$ is obtained as follows:

$$\theta_2 = Tan^{-1}(S2, C2)$$

As the sign of $S3$ cannot be uniquely determined, the third Equation leads to finding two solutions $\theta_1^+$ and $\theta_1^-$ depending on the sign of $S3$:

$$\begin{aligned} \theta_1^+ &= Z - L_1 - L_2 - C3^+.L_3 \\ \theta_1^- &= Z - L_1 - L_2 - C3^-.L_3 \end{aligned} \tag{3.77}$$

Figure 2.10 summarizes the obtained solutions, for each point $O_3$, two configurations of this robot arm can then be found:

$$(\theta_1^+, \theta_2, \theta_3^+) \text{ and } (\theta_1^-, \theta_2, \theta_3^-) \tag{3.78}$$

Note here that none of the angles, in the first (+) and second (−) configurations, differ by $\pi$ . To retain one configuration out of two, one could compare $\theta_1^+$ and $\theta_1^-$ to the previous value of $\theta_1$, i.e., $\theta_1(n-1)$ . The retained solution should be the closest among $\theta_1^+$ and $\theta_1^-$ to $\theta_1(n-1)$ .

**Discussion**

In the case we have $L_3 < L_2$, we obtain two symmetrical solutions for $\theta_3$, since we can eliminate one of them by tacking for $\theta_2$ the domain $[0, -\pi]$.
When the point $O_3$ is on the $Z_0$ axis we have:

$$X = Y = 0, \quad and \quad L_2 + C3L_3 = 0 \rightarrow C3 = -L_2/L_3 \tag{3.79}$$

We can determine $\theta_1$ from the $Z$'s equation. But $\theta_2$ cannot be determine uniquely. There exists an infinite number of solution for $\theta_2$. The continuity criteria can then be used to select one appropriate solution. Finally, we note that the domain of variation of variable $\theta_2$ can be $[-\pi, +\pi]$, and the domain of $\theta_1$ has practical limitation.

## 3.7 Study of a six degree-of-freedom arm

Consider a robot arm composed of six degree-of-freedom. The Transporter part is defined by three revolute joints:

$$\begin{aligned} &Link_1(R(Z), Z(L_1)) \\ &Link_2(R(X), Z(L_2)) \\ &Link_3(R(X), Z(L_3)) \end{aligned} \tag{3.80}$$

The effector part is defined also by three revolute joints:

$$\begin{aligned} &Link_4(R(Z), Z(L_4)) \\ &Link_5(R(X), Z(L_5)) \\ &Link_6(R(Z), Z(L_6)) \end{aligned} \tag{3.81}$$

The whole robot arm is shown on Figure 2.11. To obtain the basic geometric system for the arm, we have to express vectors $O_0O_{6,0}, X_6, Y_6$ and $Z_6$ with respect to the absolute coordinate system $R_0$ . The transporter part of this arm is the same as in Section 2.3. For this we can use results concerning $O_0O_{3,0}$ and $M_0^3$ which have been computed and discussed in this Chapter.

We start with the point $O_4$ and we use vector $O_0O_{3,0}$ and matrix $M_0^3$ . The vector $O_0O_4$ is given by the general expression:

$$O_0O_{4,0} = O_0O_{3,0} + M_0^4.O_3O_{4,4} \tag{3.82}$$

and $M_0^4$ is the product:

$$\begin{aligned} M_0^4 = M_0^3.M_3^4 &= \begin{bmatrix} C1 & -S1C23 & S1S23 \\ S1 & C1C23 & -C1S23 \\ 0 & S23 & C23 \end{bmatrix} . \begin{bmatrix} C4 & -S4 & 0 \\ S4 & C4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} C1C4 - S1C23S4 & -C1S4 - S1C23C4 & S1S23 \\ S1C4 + C1C23S4 & -S1S4 + C1C23C4 & -C1S23 \\ S23S4 & S23C4 & C23 \end{bmatrix} \end{aligned} \tag{3.83}$$

From Section 2.3, Vector $O_0O_{3,0}$ has the following expression:

$$O_0O_{3,0} = \begin{bmatrix} S1(S2L_2 + S23L_3) \\ -C1(S2L_2 + S23L_3) \\ L_1 + C2L_2 + C23L_3 \end{bmatrix} \tag{3.84}$$

From equation (2.84) we obtain $O_0O_{4,0}$:

$$O_0O_{4,0} = \begin{bmatrix} S1(S2L_2 + S23(L_3 + L_4)) \\ -C1(S2L_2 + S23(L_3 + L_4)) \\ L_1 + C2L_2 + C23(L_3 + L_4) \end{bmatrix} \tag{3.85}$$

Now, for the point $O_5$ we have:

$$O_0O_{5,0} = O_0O_{4,0} + M_0^5.O_4O_{5,5} \tag{3.86}$$

Link $L_5$ is a revolute link about $X$ axis. Therefore, $M_0^5$ is given by:

$$M_0^5 = M_0^4.M_4^5 = M_0^4. \begin{bmatrix} 1 & 0 & 0 \\ 0 & C5 & -S5 \\ 0 & S5 & C5 \end{bmatrix} = \{X_5, Y_5, Z_5\} \tag{3.87}$$

$$X_5 = \begin{bmatrix} C1C4 - S1C23S4 \\ S1C4 + C1C23S4 \\ S23S4 \end{bmatrix} \tag{3.88}$$

$$Y_5 = \begin{bmatrix} -(C1S4 + S1C23C4)C5 + S1S23S5 \\ -(S1S4 - C1C23C4)C5 - C1S23S5 \\ -S23C4C5 + C23S5 \end{bmatrix} \tag{3.89}$$

$$Z_5 = \begin{bmatrix} (C1S4 + S1C23C4)S5 + S1S23C5 \\ -(C1C23C4 - S1S4)S5 - C1S23C5 \\ -S23C4S5 + C23C5 \end{bmatrix} \tag{3.90}$$

Because Link $L6$ is a revolute link about $Z$ axis we have:

$$Z_{6,0} = Z_5, 0 \tag{3.91}$$

Since Vector $O_0O_{5,0}$ may be obtained simply as follows:

$$O_0O_{5,0} = O_0O_{4,0} + M_0^5.O_4O_{6,6} \tag{3.92}$$

Particularly we have:

$$O_0O_{6,0} = O_0O_{4,0} + (L5 + L6).Z_{6,0} \tag{3.93}$$

The cartesian coordinate of the end Effector is:

$$O_0O_{6,0} = \begin{bmatrix} S1(S2L_2 + S23(L_3 + L_4)) + L5.Z_{x6} \\ -C1(S2L_2 + S23(L_3 + L_4)) + L5.Z_{y6} \\ L_1 + C2L_2 + C23(L_3 + L_4)) + L5.Z_{z6} \end{bmatrix} \tag{3.94}$$

Finally we compute the basic orientation matrix $M_0^6$ using $M_0^5$ and the rotation property of joint $L_6$, we have:

$$M_0^6 = M_0^4.M_5^6 = \{X_6, Y_6, Z_6\} = \begin{bmatrix} X_x & X_y & X_z \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix} \tag{3.95}$$

For each component of this matrix we obtain:

$$\begin{aligned}
X_x &= C1C4C6 - S1C23S4C6 - C1S4C5S6 - S1C23C4C5S6 \\
&\quad +S1S23S5S6 && (3.96) \\
X_y &= S1C4C6 + C1C23S4C6 - S1S4C5S6 + C1C23C4C5S6 \\
&\quad -C1S23S5S6 && (3.97) \\
X_z &= S23S4C6 + S23C4C5S6 + C23S5S6 && (3.98) \\
Y_x &= C1C4S6 + S1C23S4S6 - C1S4C5C6 - S1C23C4C5C6 \\
&\quad +S1S23S5C6 && (3.99) \\
Y_y &= -S1C4S6 - C1C23S4S6 - S1S4C5C6 + C1C23C4C5C6 \\
&\quad -C1S23S5C6 && (3.100) \\
Y_z &= S23S4S6 + S23C4C5C6 + C23S5C6 && (3.101) \\
Z_x &= C1S4S5 + S1C23C4S5 + S1S23C5 && (3.102) \\
Z_y &= S1S4S5 - C1C23C4S5 - C1S23C5 && (3.103) \\
Z_z &= -S23C4S5 + C23C5 && (3.104)
\end{aligned}$$

## 3.8 The inverse geometric transform

The inverse geometrical problem of the arm, defined in Section 2.3, is to compute the joint variables $\theta_1, ..., \theta 6$, given the basic representation of the End-effector:

$$\underbrace{O_0O_{6,0}}_{\text{Hand Center}} \text{ and } \underbrace{M_0^6 = \{X_6, Y_6, Z_6\}}_{\text{Hand Orientation Matrix}} \tag{3.105}$$

This system equations consists of twelve nonlinear, redundant equations with respect to $\theta_1, ..., \theta 6$ . The first three equations concern the three cartesian coordinates of the hand center $O_0O_{6,0}$ . These equations may be written such that the unknown terms appear on the right-hand of the equal sign:

$$\begin{aligned}
X_6 - L5.Z_{x6} &= X_4 = S1(S2L_2 + S23(L_3 + L4)) \\
Y_6 - L5.Z_{y6} &= Y_4 = -C1(S2L_2 + S23(L_3 + L4)) \\
Z_6 - L5.Z_{z6} &= Z_4 = L_1 + C2L_2 + C23(L_3 + L4)
\end{aligned} \tag{3.106}$$

Vector $O_0O_5$ can be expressed as follows:

$$O_0O_{4,0} = O_0O_{6,0} - L5.Z_{6,0} \tag{3.107}$$

Equations of $X_4, Y_4$ and $Z_4$ are similar to the three revolutes transporter developed in Section 2.4. The solutions for $\theta_1, \theta_2$ and $\theta_3$ have the same form as in Section 2.3.

The second step is to determine solutions for $\theta 4, \theta 5$ and $\theta 6$ . For that we observe that the rotation axes of links $L_4$ and $L_5$ are orthogonal. Since the terms $C_4, 4\ S_4, C_5$ and $S_5$ should appear in the rotation matrix $M_3^5$:

$$M_3^5 = M_3^4.M_4^5 = ROTZ(\theta 4).ROTX(\theta 5) \tag{3.108}$$

$$M_3^5 = \begin{bmatrix} C4 & -S4 & 0 \\ S4 & C4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C5 & -S5 \\ 0 & S5 & C5 \end{bmatrix} = \begin{bmatrix} C4 & -S4C5 & S4S5 \\ S4 & C4C5 & -C4S5 \\ 0 & S5 & C5 \end{bmatrix} \tag{3.109}$$

The rotation matrix of the robot hand is known and is given by:

$$M_0^6 = M_0^3.M_3^5.M_5^6 = \{X_6, Y_6, Z_6\} \tag{3.110}$$

In addition, the last three rotation axes are concurrent, and we have:

$$M_6^5 = \begin{bmatrix} C6 & S6 & 0 \\ -S6 & C6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.111}$$

To identify $C_4, S_4, C_5$ and $S_5$ we may use the following equations:

$$M_3^5 = (M_0^3)^{-1}.M_0^6.(M_3^5)^{-1} = M_3^0.M_0^6.M_6^5 \tag{3.112}$$

The system $L_1, L_2$, and $L_3$ has been solved and we suppose a solution $(\theta_1, \theta_2, \theta_3)$ is found. Since the matrix $M_3^0.M_0^6$ is known. As $\theta 6$ is defined as a rotation about $Z5$ axis, then $Z6$ will not be affected by $\theta 6$ . For this, we start by expressing the product $M_3^0.M_0^6$ . For $M_3^0$ we have:

$$M_3^0 = \left[M_0^3\right]^t = [ROTZ(\theta_1).ROTX(\theta_2).ROTX(\theta_3)]^t \tag{3.113}$$

Since the rotation axes of $\theta_2$ and $\theta_3$ are parallel:

$$\begin{aligned} M_0^3 &= ROTZ(\theta_1).ROTX(\theta_2 + \theta_3) \\ M_3^0 &= \begin{bmatrix} C1 & S1 & 0 \\ -S1C23 & C1C23 & S23 \\ S1S23 & -C1S23 & C23 \end{bmatrix} \end{aligned} \tag{3.114}$$

Any $M_0^6$ is given by:

$$M_0^6 = \begin{bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix} \tag{3.115}$$

Since we have:

$$M_3^0.M_0^6 = (V1V2V3)$$

$$V1 = \begin{bmatrix} C1X_x + S1X_y \\ -S1C23X_x + C1C23X_y + S23X_z \\ S1S23X_x - C1S23X_y + C23X_z \end{bmatrix}$$

$$V2 = \begin{bmatrix} C1Y_x + S1Y_y \\ -S1C23Y_x + C1C23Y_y + S23Y_z \\ S1S23Y_x - C1S23Y_y + C23Y_z \end{bmatrix}$$

$$V3 = \begin{bmatrix} C1Z_x + S1Z_y \\ -S1C23Z_x + C1C23Z_y + S23Z_Z \\ S1S23Z_X - C1S23Z_Y + C23Z_Z \end{bmatrix}$$

As $\theta 6$ is a rotation about $Z$ axis, then the third column (or vector) of the produce $M_3^0.M_0^6$ will not be affected by $\theta 6$ . Recall the relation (3-2) and (3-3) we can express the third column of $M_3^5(3-1)$ **independently from** $\theta 6$:

$$\begin{aligned} S4S5 &= C1Z_x + S1Zy & (3.116) \\ C4S5 &= C23(S1Z_x - C1Z_y) - S23Z_z & (3.117) \\ C5 &= S23(S1Z_x - C1Z_y) + C23Z_z & (3.118) \end{aligned}$$

We note that it is out of interest to express $S5$, $C4$, and $S4$ from the same Equations 2.116 because of their dependency of the unknown angle $\theta 6$ . Certainly a similar equations relating $\theta 6$ to $\theta 5$ can be obtained while $\theta 4$ is unknown. As this is completely equivalent to the system 2.116, then it is of no interest. If we examine the mechanical system, it will clearly indicate the impossibility to uniquely determine solution for $\theta 4$ and $\theta 6$ independently from $\theta 5$ . In particular, when $\theta 5 = 0$ the hand orientation will only depend on $\theta 4 + \theta 6$ . Since $\theta 4$ and $\theta 6$ cannot be determined independently from $\theta 5$ . In fact, the mathematical system 2.106 does not give more information than the system 2.116. Using the system equations 2.116 we have:

$$\begin{aligned} C5 &= C23(S1Z_x - C1Z_y) - C23Z_z \\ S5 &= \pm\sqrt{1 - C5^2} & (3.119) \end{aligned}$$

Two symmetrical solutions are then possible for angle $\theta 5$ within $[-\pi, +\pi]$: Angle $\theta 4$ can be determined from the system (3-4) only when $S \neq 0$ (see next pages), we have:

$$\begin{aligned} S4 &= \frac{Z_xC1 + Z_yS1}{S5} \\ C4 &= \frac{C23(S1Z_x - C1Z_y) - S23Z_z}{S5} & (3.120) \end{aligned}$$

As $C4$ and $S4$ depend on the sign of $S5$, then two solutions are expected within $[-\pi, +\pi]$:

$$\{C4^+, S4^+\} \text{ and } \{C4^- = -C4^+ \ \ and \ \ S4^- = -S4^+\} \quad (3.121)$$

The solutions $\theta 4^+$ and $\theta 4^-$ differ by $\pi$ . In the domain $[-\pi, +\pi]$ we have: $\theta 4^+(C4^+, S4^+)$

$$\theta 4^- = \theta 4^+ - sign(\theta 4^+).\pi \quad (3.122)$$

To determine $\theta 6$ when $\theta 5 \neq 0$ we assume the value of $C4, S4, C5$ and $S5$ have been computed according to equations 2.117 and 2.118. Let us express the rotation matrix $M_5^6$, we have:

$$M_5^6 = M_5^3.M_3^0.M_0^6 \quad (3.123)$$

Matrices $M_5^3(\theta 4, \theta 5)$, $M_3^0(\theta_1, \theta_2, \theta_3)$, and $M_0^6$ are given. The matrix $M_5^6$ is a $ROTZ(\theta 6)$:

$$M_5^6 = \begin{bmatrix} C6 & -S6 & 0 \\ S6 & C6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.124)$$

Recall the matrices $M_3^0.M_0^6(3-3)$ and $M_3^5(3-1)$ that have been previously evaluated, we have:

$$M_5^3 = \begin{bmatrix} C4 & S4 & 0 \\ -S4C5 & C4C5 & S5 \\ S4S5 & -C4S5 & C5 \end{bmatrix} \quad (3.125)$$

Expression of $C6$ and $S6$ are then obtained from the product $M_5^3.(M_3^0.M_0^6)$ as follows:

$$\begin{aligned} C6 &= C4(C1X_x + S1X_y) + S4(-S1C23X_x + C1C23X_y + S23X_z) \\ S6 &= -C4(C1Y_x + S1Y_y) + S4(S1C23Y_x - C1C23Y_y - S23Y_z) & (3.126) \end{aligned}$$

Depending on the sign of $S5$, i.e. $C4$ and $S4$, we determine two solution for $\theta 6$:

$$\begin{aligned} S5^+ &= +\sqrt{1-C5^2} \rightarrow (C4^+, S4^+) \rightarrow (C6^+, S6^+) \\ \text{And } S5^- &= -S5^+ \rightarrow (C4^-, S4^-) \rightarrow (C6^-, S6^-) \end{aligned} \tag{3.127}$$

Since, two solutions are also expected for $\theta 6$:

$$\theta 6^+(C6^+, S6^+)$$

$$\theta 6^- = \theta 6^+ - Sign(\theta 6^+).\pi \tag{3.128}$$

Figure 2.13 shows how these solution can be obtained: consider one initial solution (Figure A) and consider the operations $\theta_5 \leftarrow -\theta_5$ (Figure B), $\theta_4 \leftarrow -\theta_4 + \pi$ (Figure C), and $\theta_6 \leftarrow -\theta_6 + \pi$ (Figure D). Obviously, this lead to obtain another possible solution for the position and the orientation of the arm.

**Conclusion on the case $\theta 5 \neq 0$**

Two set of solutions are expected:

$$(\theta 5^+, \theta 4^+, \theta 6^+) \text{ and } (\theta 5^+, \theta 4^+, \theta 6^+)$$

where

$$\begin{aligned} \theta 5^+ &\quad \text{and} \quad \theta 5^- \text{are symmetrical} \\ \theta 4^+ &\quad \text{and} \quad \theta 4^- \text{Differ by} \pi \\ \theta 6^+ &\quad \text{and} \quad \theta 6^- \text{Differ by} \pi \end{aligned} \tag{3.129}$$

By trajectory continuity, we may identify a solution. For example, by comparing $(\theta 4^+, \theta 4^-)$ to the previous value of $\theta 4$ (Initial):
**Example:**

$$\begin{aligned} \text{If } |\theta 4_I - \theta 4^+| &> |\theta 4_I - \theta 4^-| \; THEN \\ \theta 4 &= \theta 4^+; \theta 5 = \theta^+; \theta 6 = \theta 6^+ && (3.130) \\ ELSE & && (3.131) \\ \theta 4 &= \theta 4^-; \theta 5^+; \theta 6 = -\theta 6^+ && (3.132) \\ END & && (3.133) \end{aligned}$$

Given the hand center $O_0O_{6,0}$ and the orientation matrix $M_0^6$, it is possible to find at least two configurations for the effector part $(\theta 4, \theta 5, \theta 6)$:

$$(\theta 5, \theta 4^+, \theta 6^+) \text{ and } (-\theta 5, \theta 4^-, \theta 6^-) \tag{3.134}$$

**Case where $\theta 5 = 0$**
This case corresponds to a singular configuration as shown on Figure 2.14.In this situation, the rotation axes of $\theta 4$ and $\theta 6$ are co-linear and concurrent, we have:

$$M_3^6 = M_3^4.M_4^5.M_5^6 = M_3^4.M_5^6 = ROTZ(\theta 4^{\theta} 6) \tag{3.135}$$

And

$$M_3^6 = \begin{bmatrix} C46 & -S46 & 0 \\ C46 & C46 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.136}$$

In order to express C46 and S46, we may use the following equation:

$$M_3^6 = M_3^0.M_0^6 \tag{3.137}$$

The product $M_3^0.M_0^6$ has been previously expressed, we have:

$$\begin{aligned} C46 &= C1X_x + S1X_y \\ S46 &= -(C1Y_X + S1Y_Y) \end{aligned} \tag{3.138}$$

Since $\theta 4$ and $\theta 6$ cannot be expressed independently from each other. In fact the mechanical analysis confirms this mathematical result, and then heuristics can be applied to obtain a possible alternative. For example we may keep unchanged one angle, i.e. $\theta 4$ or $\theta 6$, and determine the other angle from their sum $(\theta 4 + \theta 6)$ as it is identified using $C46$ and $S46$. Another alternative consists of estimating one angle, i.e. $\theta 4$ or $\theta 6$, by extrapolating its time function according to the previous values. Assume the previous values of $\theta 4$ are:

$$\theta 4(t-k), ..., \theta 4(t-1), \theta 4(t) \tag{3.139}$$

And

$$\theta 6(t-k), ..., \theta 6(t-1), \theta 6(t) \tag{3.140}$$

A polynomial approximation with degree $K$ is given by:

$$\theta(t^1) = \sum_{i=0}^{K} a_i \theta(t-i) \tag{3.141}$$

Higher accuracy is obtained for lower degree polynomial. Since we may select to extrapolate either $\theta 4$ or $\theta 6$, according to the lowest polynomial degree. Assume $\theta 4$ is obtained by extrapolation, then $\theta 6$ could now be simply obtained as $\theta 6 = \theta 46 - \theta 4$ .

## Exercises

1. The hand frame origin of a robot arm is given by: $O_0O_{n,0} = \sum_{i=1}^{n} M_o^i.O_{i-1}O_{i,i}$ Determine the coordinate of vector $O_0O_n$ in the frame $R_n$.

2. A 2-degrees of freedom robot arm is defined by:
   Link 1 (Revolute $(Z)$, $L_1$ on $X1$)
   Link 2 (Revolute $(Z)$, $L_2$ on $X2$)
   Where the first row means that the link $L_1$ is defined along $X1$ axis and this link is revolute about $Z$ axis. Link $L_2$ is similarly defined.

   (a) Determine the coordinate of point $O_2$ with respect to frame $R_0$ .

   (b) Obtain the inverse geometric transform: $(X, Y, Z) \rightarrow (\theta_1, \theta_2)$
   where, $\theta_1$ and $\theta_2$ are the angles of links $L_1$ and $L_2$, respectively.

   (c) Discuss the solution of question (b) with regards to multiple solution and possible singularities.

3. The same as in problem 2.2 but with the following robot arm:
   Link 1 (Prismatic $(Z)$, $L_1$ on $Z1$)
   Link 2 (Revolute $(X)$, $L_2$ on $Z2$)

4. A three degrees of freedom arm is defined by:
   Link 1 (Prismatic $(Z)$, $L_1$ on $Z1$)
   Link 2 (Revolute $(Z)$, $L_2$ on $Z2$)
   Link 2 (Revolute $(X)$, $L_3$ on $Z3$)

   (a) Find the geometrical model. For this express the vector $O_0O_3$ as function of the degrees of freedom $\theta_1, \theta_2$, and $\theta_3$ .

   (b) Find the inverse geometrical transform. For this, obtain the expression $\theta_1, \theta_2$, and $\theta_3$ as function of $X, Y$, and $Z$ which are the coordinate of $O_0O_3$ in frame $R_0$ .

   (c) Determine the largest domains for $\theta_1, \theta_2$ and $\theta_3$ for which only one solution $(\theta_1, \theta_2, \theta_3)$ exists.

   (d) Study the case of singularities for this arm and examine the solution the point $O_3$ is on the $Z_0$ axis .

   (e) A singularity in the geometrical model occurs when infinite number of solution for $\theta = G^{-1}(X, Y, Z)$ is observed. Propose method for getting continuous trajectory when passing by a singular point. For this, use information on previous values of $\theta(t-1), \theta(t-2), ..., \theta(t-k)$ in order to find solution for $\theta(t)$ .

5. Give the block diagram of a motion coordination system by using the geometric method. The desired points will be the coordinate of the robot hand center and orientation. In which space correction of the motion is achieved in this method.

6. Explain the principle of motion coordination by refering to the discrete nature of the input points and the transient of the robot controller.

7. A six degrees of freedom robot arm is defined by:

   (a) The first 3 degrees of freedom are those defined in problem 2.4.

   (b) The last 3 degrees of freedom are:
   Link 4 ( Revolute $(Z)$, $L4$ on $Z4$)
   Link 5 ( Revolute $(X)$, $L5$ on $Z5$) Link 6 ( Revolute $(Z)$, $L6$ on $Z6$)

i. The last 3 degrees of freedom have concurrent axes. Prove that the system can be uncoupled as follows: $O_0O_3 = OO_6 - (L5 + L6)Z6$ where $Z6$ is the $Z$ axis of frame $R6$.

ii. Find the geometrical model. For this find the expression of $O_0O_6, 0$ and $M_0^6$ as function of $\theta_1, ..., \theta 6$ .

iii. Find the inverse geometrical transform. For this, find the expression of $\theta_1, ..., \theta 6$ as function of the vector $O_0O_6 = (XYZ)$ and $M_0^6 = \{X_6Y_6Z_6\}$ .

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