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Inverse Kinematics

What is IK?

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[Fanuc]

[Ronan Boulic]

What is IK ?

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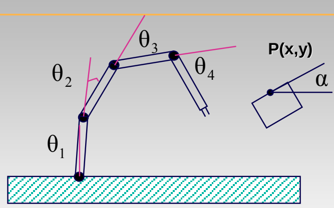

2D robot DOFs ?

3D robot DOFs ?

forward kinematics

joint angles θ_i → end effector position and orientation x,y α

inverse kinematics

A Simple Example

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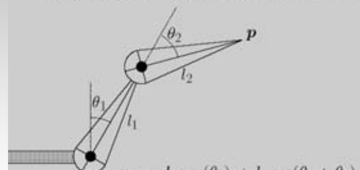
Two link robot

Simple System: A Two Segment Arm



$$p_z = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

$$p_x = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

[James O'Brien]

A Simple Example

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Direct IK Solution

Direct IK: Solve for θ_1 and θ_2

$$\theta_2 = \cos^{-1}\left(\frac{p_z^2 + p_x^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$$

$$\theta_1 = \frac{-p_z l_2 \sin(\theta_2) + p_x(l_1 + l_2\cos(\theta_2))}{p_x l_2 \sin(\theta_2) + p_z(l_1 + l_2\cos(\theta_2))}$$

[James O'Brien]

Problems

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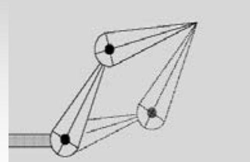
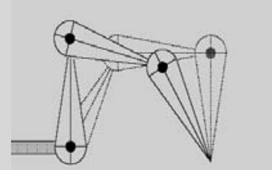
Multiple solutions separated in configuration space



Multiple solutions connected in configuration space



[James O'Brien]

- + unreachable goals

Solving for Constraints

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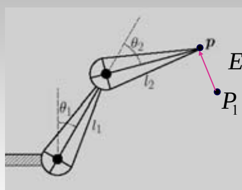

$$E = P - P_1$$

Newton's Method

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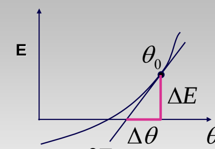

$$\frac{\partial E}{\partial \theta}$$

$$\frac{\partial E}{\partial \theta} = \frac{\Delta E}{\Delta \theta}$$

$$\Delta \theta = \left(\frac{\partial E}{\partial \theta}\right)^{-1} \Delta E$$

$$\theta' = \theta - \Delta \theta$$

$$\frac{\partial E}{\partial \theta} = \frac{\partial P}{\partial \theta}$$

Jacobian

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$$\Delta \theta = \left(\frac{\partial P}{\partial \theta}\right)^{-1} \Delta P \qquad \Delta \theta = J^{-1} \Delta P$$

Jacobian is given by $J = \frac{\partial P}{\partial \theta} \approx \frac{\Delta P}{\Delta \theta}$

$$J \cdot \Delta \theta = \Delta P$$

velocities: $J \cdot \frac{\Delta \theta}{\Delta t} = \frac{\Delta P}{\Delta t}$

$$J \cdot \dot{q} = \dot{x}$$

Jacobian Example

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$$p_z = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$p_x = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

Simple System: A Two Segment Arm

$$\frac{\partial p_z}{\partial \theta_1} = -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2)$$
$$\frac{\partial p_x}{\partial \theta_1} = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$\frac{\partial p_z}{\partial \theta_2} = - l_2 \sin(\theta_1 + \theta_2)$$
$$\frac{\partial p_x}{\partial \theta_2} = + l_2 \cos(\theta_1 + \theta_2)$$

[James O'Brien]

Jacobian Example

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Example for two segment arm

$$J = \begin{bmatrix} \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} \\ \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} \end{bmatrix}$$

[James O'Brien]

Inverting the Jacobian

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- m constraints, n DOF

Inversion of the Jacobian matrix

- If $J_{(m,n)}$ is not square, use the pseudoinverse
  - full rank matrices:

$m>n$: $\mathbf{J}^+ = (\mathbf{J}^T\mathbf{J})^{-1}\mathbf{J}^T$ — overconstrained, minimizes $\|J \cdot \dot{q} - \dot{x}\|$

$m<n$: $\mathbf{J}^+ = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}$ — underconstrained, minimizes $\|\dot{q}\|$

  - rank deficient matrices: use SVD or other methods

Pseudoinverse

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Least-Squares Inverse

$$AA^+A = A$$
$$(AA^+)^T = AA^+$$
$$A^+AA^+ = A^+$$
$$(A^+A)^T = A^+A$$
$$\left(A^+\right)^+ = A$$
$$(A^+)^T = (A^T)^+$$

derived

but...

$$(AB)^+ \neq B^+A^+$$

Iterative IK solution

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```
repeat
  E = P - Ptarget
  dX = k*error;  // k<1
  compute J
  compute J*(J)
  compute dQ = J* dX
  Q = Q + dQ     // update joint angles
until |error| < epsilon
```


Pseudoinverse Discussion

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- instable around singularities
- can weight the joint movement

Cost function with a weighting matrix: $C(dQ) = dQ^T W dQ$ ← nxn weighting matrix positive definite

$dQ = W^{-1}J^T(JW^{-1}J^T)^{-1}dX$

- damped least squares solution helps avoid singularities

$$(A^+)^\lambda = A^T(AA^T + \lambda^2 I)^{-1}$$

minimizes $\|A\dot{q} - \dot{x}\|^2 + \lambda^2\|\dot{q}\|^2$

Pseudoinverse Discussion

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secondary task

- obstacle avoidance
- joint limit avoidance
- singularity avoidance

$$\dot{q} = J^+\dot{x} + (I - J^+J)z$$

Jacobian Transpose Method

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- Jacobian transpose method uses the transpose of the Jacobian matrix rather than the p-inverse

Find Δq by:

$$\Delta q = J^T \Delta x$$

rather than:

$$\Delta q = J^+ \Delta x$$

- Avoids expensive inversion
- Avoids singularity problems

But why is this a reasonable thing to do? ....

[Baxter, UNC]

Jacobian Transpose Method

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Principal of Virtual Work

- "Virtual" because amount is infinitessimal
- Work = force x distance. Work = torque x angle

$F \cdot \Delta x = \tau \cdot \Delta q$ (energy equal in any coordinates)

$F^T \Delta x = \tau^T \Delta q$

$\Delta x = J \Delta q$ (forward kinematics)

$F^T J \Delta q = \tau^T \Delta q$ (substitution)

$F^T J = \tau^T$ (transpose both sides)

$\tau = J^T F$

[Baxter, UNC]

Jacobian Transpose Method

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Good and Bad of J^T

+ Cheaper evaluation step than pseudoinverse
+ No singularities
- Scaling problems
  - J^+ has nice property that solution has minimal norm at every step.
  - J^T doesn't have this property. Joints far from end effector experience larger torques, hence take disproportionately large steps.
  - Can throw in a constant diagonal scaling matrix to counteract some scaling probs

    $\dot{q} = K J^T F(q)$ where each K_{ii} set apropriately
- Slower to converge than J^+
  - (2x slower according to Das, Slotine & Sheridan)

[Baxter, UNC]

Cyclic Coordinate Descent

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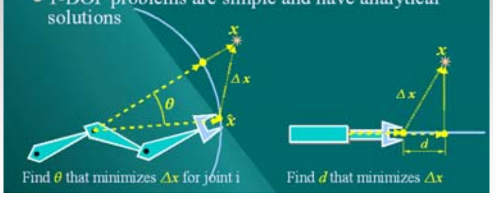
- Actually a much simpler idea
  - Just solve 1 DOF IK problems repeatedly up chain
- 1-DOF problems are simple and have analytical solutions



[Baxter, UNC]

Cyclic Coordinate Descent

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Good and Bad of CCD

+ Simple to implement
+ Often effective
+ Stable around singular configuration
+ Computationally cheap
+ Can combine with other more accurate optimization method like BFS when close enough

BUT

- Can lead to odd solutions if per step deltas not limited, making method slower
- Doesn't necessarily lead to smooth motion

[Baxter, UNC]

Machine Learning

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$$f : \vec{x} \rightarrow \vec{q}$$

problem: one-to-many mapping

$$f : \vec{x}, \vec{q}_0 \rightarrow \vec{q}$$

Human Motion

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- hands typically travel in straight-line paths
- strength influences trajectory of some motions
- course project ?