Chapter 3

MOTION COORDINATION

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This chapter introduces the robot motion coordination system based on the geometric model of the robot arm. The geometric model and its inverse model are suitable mathematical tools for implementing the motion coordination system based on position control of the robot arm. The geometric model allows mapping trajectories of the robot effector, that are described in the joint space, into the corresponding trajectories in the cartesian space. The robot end effector is represented by a position and an orientation in the three dimensional space. Therefore, a point in the joint space can be converted using the geometric model to a position and an orientation in the cartesian space. The geometric model represents a set of six non-linear equations that give the expression of the effector vector E as function of the joint variables vector θ. The geometric model is denoted by $E = G(\theta)$.

Mechanically, the robot arm can be moved by modifying its position in the joint space. Therefore, we need an operator to transform the desired position and orientation in the cartesian space into position in the joint space. This operator is called the inverse geometric model. To obtain the inverse geometric model, one needs to invert the system equation $E = G(\theta)$, and find the operator $\theta = G^{-1}(E)$. The solution $\theta = G^{-1}(E)$ does not exist for all robot arms and when it exists multiple solutions should be expected in the general case. For example, when the last three degrees of freedom are revolute and their rotation axes are not concurrent, then no closed form solution can be found by inverting the system equation $E = G(\theta)$.

When a closed form does exist, multiple solutions are generally expected. This means that more than one configuration of the joint variables allow setting the robot hand at the same position and orientation. The problems of selecting one solution out of many solutions will be studied on the light of some practical requirements such as global motion continuity. Position control can then be implemented with respect to each joint variable.

The theoretical issues will be followed by examples of robot arms having three and six degrees of freedom.

3.1 Introduction to the geometric method

The principle of this method is to express the position and orientation of the robot hand as function of the joint variables. means of an absolute representation of the end effector. A simple representation of the robot hand consists of considering the frame of reference which is attached to last link of the chain. The effector of a robot arm having n degrees of freedom can be geometrically represented by means the coordinate of the origin of frame R_n and its orientation matrix R_n with respect to a fixed frame R_0. The frame R_n is called the effector frame of reference. The position and orientation of the effector frame (See Figure 2.1-A) can be totally determined by means of the following information:

1. The robot hand center or vector $O_0O_{n,0}$ which references the origin of R_n with respect to R_0.
2. The robot hand orientation matrix $M_0^n = [X_{n,0}, Y_{n,0}, Z_{n,0}]$ which determines the orientation of frame R_n with respect to frame R_0.

These parameters are function of the joint variables $\theta_1, \theta_2, ..., \theta_n$. For this we note them as follows $O_0O_{n,0}(\theta)$ and $M_0^n(\theta)$. Figure 2.1-B shows the relation between the effector frame and the joint variables for a 6 d.o.f. robot arm. The basic geometrical representation of the arm is reduced to the short expression:

$$\begin{aligned} G(\theta) &= \{O_0O_{n,0}(\theta), M_0^n(\theta)\} && (3.1) \\ \text{Geometrical} & \quad \text{Basic representation} && (3.2) \\ \text{representation} & \quad \text{of the end effector} && (3.3) \end{aligned}$$

The direct geometric problem is to compute the basic representation with respect to a reference cartesian coordinate R_0, given the joint variables $\theta_1, \theta_2, ..., \theta_{n-1}$ and θ_n:

$$\begin{aligned} \theta &= (\theta_1, \theta_2, ..., \theta_n)^t \rightarrow \{O_0O_n(\theta), M_0^n(\theta)\} && (3.4) \\ & \quad \text{associated a set of geometric equations} \end{aligned}$$

The inverse geometric problem is to compute the joint variables $\theta_1, ..., \theta_n$, given the basic representation of the end effector:

$$\{O_0O_n, M_o^n\} \rightarrow \theta = (\theta_1, ..., \theta_n)^t \tag{3.5}$$

The inverse geometric system allows assigning the position and orientation of the end effector by acting on the joint variables. At the lowest level, the robot arm can only be controlled by moving the joint positions which can be rovolute or prismatic degrees of freedom. The control system allows generating torques to move and maintain the robot arm according to a prescribed trajectory in the joint space. The prescribed trajectory, or desired trajectory, is generated by the inverse geometric transform as image of a trajectory described in the effector space. Therefore, the inverse geometric system is one basic concept to robot motion coordination because it allows controlling the robot arm at the level of the robot end effector and the cartesian space.

3.2 Elaboration of the geometric model

This problem consists of finding the expression of the basic effector position and orientation, with respect to a fixed frame of reference R_0, as a function of the joint variables vector θ.

Consider an n degrees of freedom arm as shown in Figure 2.2. A frame of reference is attached to every link and these frames are all observed with respect to a fixed frame R_0. Every link L_i is represented by means of a vector $O_{i-1}O_i$. The basic Effector parameters are the position vector $O_0O_{n,0}(\theta)$ and the orientation matrix $M_0^n(\theta)$. The position vector $O_0O_{n,0}$ can be decomposed as the sum of the link vectors:

$$O_0O_{n,0} = \sum_{i=1}^{n} O_{i-1,0}O_{i,0} \tag{3.6}$$

Using the translation matrix between frames R_0 and R_1, each link vector can be expressed with respect to its own frame of reference:

$$O_{i-1}O_{i,0} = M_0^i.O_{i-1}O_{i,i} \tag{3.7}$$

Vector $O_{i-1}O_{i,0}$ can then be expressed as:

$$O_0O_{n,0} = \sum_{i=1}^{n} M_0^i.O_{i-1}O_{i,i} \tag{3.8}$$

Vector $O_{i-1}O_{i,i}$ has simple expression because it is represented with respect to its own frame of reference R_i . Obviously, both vector $O_{i-1}O_{i,0}$ and $\sum_{i=1}^{n} M_0^i$ can be expressed in a recursive form, we obtain:

$$\begin{aligned} M_0^i &= M_0^{i-1}.M_{i-1}^i \\ O_0O_{i,0} &= O_0O_{i-1,0} + M_0^i.O_{i-1}O_{i,i} \end{aligned} \tag{3.9}$$

where M_0^{i-1} is the transfer matrix between frames R_{i-1} and R_0, and M_{i-1}^i is the transfer matrix between frames R_i and R_{i-1} .

The basic geometric model consists of computing the position and orientation of the robot end effector:

$$\begin{aligned} O_oO_{n,0} &= \sum_{i=1}^{n} M_o^i.O_{i-1}O_{i,i} \\ M_0^n &= \prod_{i-1}^{n} M_{i-1}^i \end{aligned} \tag{3.10}$$

The system equation 2.6 allows forward, recursive computation of the basic geometric model computation.

3.3 Case of a three-revolute arm

Let us consider the transporter part of a robot arm that is defined by three revolute joints as shown in Figure 2.3. We use the following topological form in order to describe the geometric structure of the arm:

$$\begin{aligned} &Link\ 1\ \ (R(Z), Z(L_1)) \\ &Link\ 2\ \ (R(X), Z(L_2)) \\ &Link\ 3\ \ (R(X), Z(L_3)) \end{aligned} \tag{3.11}$$

Consider link L_1 which is revolute and defined as a rotation about Z_0 axis $(R(Z))$, and the link body L_1 is along vector Z_1 of frame R_1 . The other links are defined in a similar manner.

For the first end 0_1 we have:

$$O_0O_{1,0} = M_0^1.O_0O_{1,1} \tag{3.12}$$

As link L_1 is defined by a rotation about axis Z_0, then the transfer matrix between frames R_1 and R_0 is a $ROTZ(\theta_1)$, we have:

$$M_0^1 = \begin{bmatrix} C1 & -S1 & 0 \\ -S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [X_{1,0}, Y_{1,0}, Z_{1,0}] \tag{3.13}$$

The frame R_1 is defined such that the link vector O_0O_1 is along vector Z_1, then we have:

$$O_0O_1 = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$$

Therefore, vector

$$O_0O_{1,0} = \begin{bmatrix} C1 & -S1 & 0 \\ -S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$$

For the second end point O_2, we have:

$$\begin{aligned} O_0O_{2,0} &= O_0O_{1,0} + M_0^2.O_1O_{2,2} \\ M_0^2 &= M_0^1.M_1^2 \end{aligned} \tag{3.14}$$

Where M_1^2 is a rotation matrix axis X_1, we have:

$$M_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C2 & -S2 \\ 0 & -S2 & C2 \end{bmatrix} \tag{3.15}$$

Let us evaluate the orientation matrix M_0^2 of frame R_2, we have:

$$\begin{aligned} M_0^2 = M_0^1.M_1^2 &= \begin{bmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & C2 & -S2 \\ 0 & S2 & C2 \end{bmatrix} \\ &= \begin{bmatrix} C1 & -S1C2 & S1S2 \\ S1 & C1C2 & -C1S2 \\ 0 & S2 & C2 \end{bmatrix} \end{aligned} \tag{3.16}$$

Vector $O_0O_{2,0}$ is the sum of two terms:

$$O_0O_{2,0} = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + M_0^2 . \begin{bmatrix} 0 \\ 0 \\ L_2 \end{bmatrix} = \begin{bmatrix} S1S2L_2 \\ -C1S2L_2 \\ L_1 + C2L_2 \end{bmatrix} \tag{3.17}$$

Finally, vector O_0O_3 is given by:

$$O_0O_{3,0} = O_0O_{2,0} + M_0^3.O_2O_{3,3} \tag{3.18}$$

and the orientation matrix is:

$$\begin{aligned} M_0^3 = M_0^2.M_2^3 &= M_0^2 . \begin{bmatrix} 1 & 0 & 0 \\ 0 & C3 & -S3 \\ 0 & -S3 & C3 \end{bmatrix} \\ &= \begin{bmatrix} C1 & -S1C23 & S1S23 \\ S1 & C1C23 & -C1S23 \\ 0 & S23 & C23 \end{bmatrix} \end{aligned} \tag{3.19}$$

where $S23 = SIN(\theta_2 + \theta_3)$ and $C23 = COS(\theta_2 + \theta_3)$. Vector $O_0O_{3,3}$ is the sum:

$$\begin{aligned} O_0O_{3,0} &= \begin{bmatrix} S1S2L_2 \\ -C1S2L_2 \\ L_1 + C2L_2 \end{bmatrix} + M_0^3 \begin{bmatrix} 0 \\ 0 \\ L_3 \end{bmatrix} \\ &= \begin{bmatrix} S1(S2L_2 + S23L_3) \\ -C1(S2L_2 + S23L_3) \\ L_1 + C2L_2 + C23L_3 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \end{aligned} \tag{3.20}$$

The geometric system for the transporter part is defined by:

$$(\theta_1, \theta_2, \theta_3) \to \left\{O_0O_3(\theta), M_0^3(\theta)\right\}$$

The inverse geometric model of the transporter consists of finding closed form solutions for θ_1, θ_2, and θ_3 as functions of the transporter end point coordinates X, Y, and Z. Evaluation of the geometric model of the transporter allows writing the coordinate of vector $O_0O_{3,0}(\theta)$ as follows:

$$\begin{aligned} X &= S1(S2L_2 + S23L_3) \\ Y &= -C1(S2L_2 + S23L_3) \\ Z &= L_1 + C2L_2 + C23L_3 \end{aligned} \tag{3.21}$$

To solve the system we first consider the expressions of X and Y in order to evaluate $(S2L_2+S23L_3)$, we can easily obtain:

$$S2L_2 + S23L_3 = \pm\sqrt{x^2 + Y^2} \tag{3.22}$$

When the point O_3 is not on the Z_0 axis, i.e., $X^2 + Y^2 \neq 0$, then the sine and cosine of θ_1 can be evaluated as follows:

$$S1 = \pm\frac{X}{\sqrt{X^2+Y^2}} \quad and \quad C1 = \pm\frac{-Y}{\sqrt{X^2+Y^2}} \tag{3.23}$$

Depending on the sign, we have two solutions for the angle θ_1 . When the sign $(+)$ is selected, we obtain the following solution:

$$S1^+ = \pm\frac{X}{\sqrt{X^2+Y^2}} \quad and \quad C1^+ = \frac{-Y}{\sqrt{X^2+Y^2}} \tag{3.24}$$

Note that the knowledge of both sine and cosine of an angle allows finding a unique solution for that angle. The other solution will be obtained by using sign $(-)$:

$$S1^- = -S1^+ \text{ and } C1^- = -C1^+ \tag{3.25}$$

Consequently two solutions are expected for the angle θ_1:

$$\begin{aligned} (S1^+, C1^+) &\to \theta_1^+ \\ (S1^-, C1^-) &\to \theta_1^- \end{aligned} \tag{3.26}$$

The solution θ_1^+ can be evaluated as follows:

$$\theta_1^+ = TAN^{-1}(S1^+, C1^+)$$

Solutions θ_1^+ and θ_1^- differ by Π, in the domain $[-\Pi, +\Pi]$, Therefore, we have:

$$\theta_1^- = \theta_1^+ - Sign(\theta_1^+).\Pi \tag{3.27}$$

When the robot is following a trajectory in the cartesian space, the previous solution that corresponds to the previous point of the trajectory can be used in order to select one solution out of two. Therefore, to determine a unique solution θ one may compare θ_1^+ and θ_1^- to the previous value of θ . Clearly, the closest solution to the previous one allows satisfying a continuity criteria on the cartesian trajectory. In addition to finding a solution θ_1, this operation allows finding the sign of $S1$, $C1$, and $S2L_2 + S23L_3$. To determine θ_3, we consider the expression of X, Y, and Z. We have:

$$\begin{aligned} X^2 + Y^2 &= (S2L_2 + S23L_3)^2 \\ (Z - L_1)^2 &= (C2L_2 + C23L_3)^2 \end{aligned} \tag{3.28}$$

After developing the above relations, we obtain:

$$\begin{aligned} X^2 + Y^2 &= (S2L_2)^2 + (S23L_3)^2 + 2S2S23L_2L_3 \\ (Z - L_1)^2 &= (C2L_2)^2 + (C23L_3)^2 + 2C2C23L_2L_3 \end{aligned} \tag{3.29}$$

and adding:

$$X^2 + Y^2 + (Z - L_1)^2 = L_2^2 + L_3^2 + 2C3L_2L_3 \tag{3.30}$$

We obtain $C(\theta_3)$ and $S(\theta_3)$:

$$\begin{aligned} C3 = (X^2 + Y^2 + (Z - L_1)^2 - L_2^2 - -L_3^2)/2L_2L_3 \\ S3 = \pm\sqrt{1 - C3^2} \end{aligned} \tag{3.31}$$

Two symmetric solutions for θ_3 are expected as shown in Figure 2.4. Naturally these solutions correspond to two different configurations but both allows the transporter end point being set at the coordinates specified by X, Y, and Z. Obviously, this arm can reach all the position of its work space by specifying the angle θ_3 in one of the intervals $[0, +\Pi]$ and $[-\Pi, 0]$. Depending on which interval is selected, the sign of $S3$ can then be determined. A criteria on space occupancy of the arm can be used in order to chose one of these intervals. Once the term $S3$ is sign of θ_3 is found, angle θ_3 can then be evaluated as follows:

$$\theta_3 = TAN^{-1}(S3, C3)$$

Finally, to determine angle θ_2, we consider the following equations:

$$\begin{aligned} S1X - C1Y &= S2L_2 + S23L_3 \\ Z - L_1 &= C2L_2 + C23L_3 \end{aligned} \tag{3.32}$$

After developing $S23$ and $C23$, we can write these equations in a matrix form:

$$\begin{bmatrix} S1X - C1Y \\ Z - L_1 \end{bmatrix} = \begin{bmatrix} S3L_3 & L_2 + C3L_3 \\ L_2 + C3L_3 & -S3L_3 \end{bmatrix} \cdot \begin{bmatrix} C2 \\ S2 \end{bmatrix} \tag{3.33}$$

The determinant of this matrix is given by:

$$\Delta = (S3L_3)^2 + (L_2 + C3L_3)^2 = -(L_2^2 + L_3^2 + 2L_2L_3C3) \tag{3.34}$$

In general, Δ is not nil except when $L_2 = L_3$ and θ_3 is equal $\pm\Pi$. This configuration of θ_3 cannot occur in a mechanical robot arm. The solution $C2$ and $S2$ can always be obtained as follows:

$$\begin{aligned} S2 &= \frac{(XS1 - YC1)(L_2 + C3L_3) - (Z - L_1)S3L_3}{L_2^2 + L_3^2 + 2L_2L_3C3} \\ C2 &= \frac{(XS1 - YC1)C3L_3 + (Z - L_1)(L_2 + C3L_3)}{L_2^2 + L_3^2 + 2L_2L_3C3} \end{aligned} \tag{3.35}$$

The solution for the angle θ_2 can then be obtained as follows:

$$\theta_2 = TAN^{-1}(S2, C2)$$

This solution depends on the selected values of θ_1, θ_3, and their respective signs.

3.4 Multiple solutions and singularities

Let us consider the transporter defined in Section 2.3. When only considering the links L_2 and L_3, two solution are generally expected when the transporter end point is set to any position specified by X, Y, and Z. Mathematically, this is because $S3$ cannot be determined by using the system $O_0O_3, 0(\theta) = (X, Y, Z)^t$. On the other hand, the mechanical structure of this arm indicates clearly that two configurations for (θ_2, θ_3) exist while the end point O_3 is fixed. These configurations are:

$$\{\theta_1, \theta_3^+, \theta_2(\theta_1, \theta_3^+)\} \ and \ \{\theta_1, \theta_3^+, \theta_2(\theta_1, \theta_3^-)\} \tag{3.36}$$

To make decision about which solution should be kept, one needs to assign use one of the following methods:

1. Use of a continuity criteria on the cartesian trajectory so that decision will be made by comparing the solutions θ_3^+ and θ_3^- to the previous solution. This method allows maintaining the sign of angle θ_3 fixed during the motion of the arm. To initialize the motion, the starting configuration should implicitly include this information about the selected sign of θ_3.

2. Use of a flag to indicate the current value of the sign of θ_3 . In this case, no comparison will be made but rather the sign of θ_3 will be selected according to the value of the flag which should be appropriately initialized by the system. This solution can be augmented with a function that allows switching the value of the flag and generation of smooth motion between the two configurations.

Regarding the angle θ_1, we always have two solutions θ_1^+ and θ_1^- for each of which there exists two possible configuration for angles θ_2 and θ_3 . Figure 2.5 shows that the arm admits four solutions when solving the system $O_0O_{3,0}(\theta) = (X, Y, Z)^t$:

$$\theta_1^- = \theta_1 - sign(\theta_1^+).\pi \tag{3.37}$$

Because θ_1 is in $[-\pi, +\pi]$, the solutions are:

$$\left[\theta_1^+, \theta_3^+, \theta_2(\theta_1^+, \theta_3^+)\right]$$

$$\left[\theta_1^+, \theta_3^-, \theta_2(\theta_1^+, \theta_3^-)\right]$$

$$\left[\theta_1^-, \theta_3^+, \theta_2(\theta_1^-, \theta_3^+)\right]$$

$$\left[\theta_1^-, \theta_3^-, \theta_2(\theta_1^-, \theta_3^-)\right]$$

The selection of one solution for angle θ_1 is easier because the values of θ_1^+ and θ_1^- always differ by Π . Therefore, one needs to compare with the previous value of θ_1 in order to select either θ_1^+ or θ_1^- . As angle θ_1 is generally defined in $[-\Pi, +\Pi]$, no pre-selection can be made with respect to the solutions θ_1^+ and θ_1^-. Let us consider the case when ($X = 0$ and $Y = 0$), i.e., the end point 0_3 is on the Z_1 axis as shown in Figure 2.6. The system equation becomes:

$$\begin{aligned} X &= S1(S2L_2 + S23L_3) = 0 \\ Y &= -C1(S2L_2 + S23L_3) = 0 \\ Z &= L_1 + C2L_2 + C23L_3 \end{aligned} \tag{3.38}$$

As $S1$ and $C1$ cannot be equal to zero simultaneously, then the term $S2L_2 + S23L_3$ is nil. The system equation becomes:

$$\begin{aligned} S2L_2 + S23L_3 &= 0 \\ C2L_2 + C23L_3 &= Z - L_1 \end{aligned} \tag{3.39}$$

After developing the terms $S23$ and $C23$, the solution for θ_2 can then be obtained as follows:

$$\begin{aligned} C2 &= -\frac{(z - L_1)(L_2 + C3L_3)}{L_2^2 + L_3^2 + 2C3L_2L_3} \\ S2 &= \frac{S3L_3(Z - L_1)}{L_2^2 + L_3^2 + 2C3L_2L_3} \end{aligned} \tag{3.40}$$

On the other hand, the solution for θ_3 is obtained by using the precedent equation of $C3$:

$$\begin{aligned} C3 &= \frac{(Z - L_1)^2 - L_2^2 - L_3^2}{2L_2L_3} \\ S3 &= \pm\sqrt{1 - C3^2} \end{aligned} \tag{3.41}$$

Suppose we determine the solution for θ_2 and θ_3 as previously discussed, the remaining angle θ_1 is undetermined because no information is available about this angle. The system equation $O_0O_{3,0}(\theta) = (X, Y, Z)^t$ does not give any information regarding angle θ_1 when $X = 0$ and $Y = 0$. This situation is called a singularity case for θ_1 because there exists an infinite number of values for θ_1 for which the system's equations (2.38) is satisfied. One may keep angle θ_1 unchanged until the arm moves to a new point in which the condition ($X = 0$ and $Y = 0$) is no more satisfied. Another method consists of using the criteria on trajectory continuity which can be helpful in this case. The application of this criteria consists of extrapolating the trajectory of angle $\theta_1(t)$ by using time polynomial approximation. A discrete time polynomial $\theta_1(nT)$ can be used to approximate the time function $\theta_1(t)$ over a finite number of points $\theta_1(n-1), ..., \theta_1(n-k)$. The values of $\theta_1(n-i)$ represent the ith previous solution of the system equations. Using k previous solutions, the associated polynomial can be used to predict the current value of θ_1:

$$\theta_1(n) = F\left[\theta_1(n-1), ..., \theta_1(n-k)\right]$$

If one assumes constant sampling period of the system, a second order polynomial approximation can then give the following solution:

$$\theta_1(n) = 3\theta(n-1) - 3\theta(n-2) + \theta(n-3) \tag{3.42}$$

3.5 Case of a two-revolute and one-prismatic arm

Now consider a robot arm whose transporter part has two revolute and one prismatic joints. Figure 2.7 shows this structure that can be defined by using the following geometric topology:

$$\begin{array}{l} \textit{Link } 1 \;\; (R(Z), Z(L_1)) \\ \textit{Link } 2 \;\; (R(X), Z(L_2)) \\ \textit{Link } 3 \;\; (P(X), Z(L_3)) \end{array} \tag{3.43}$$

Note here that the length of link L_2 is nil which means that no link translation will be generated by angle θ_2. The degree of freedom θ_2 is only used to orient the next link L_3 .

Let us evaluate the geometric model of this transporter. For this, we first evaluate the vector $O_0O_{1,0}$. As the rotation matrix of frame R_1 is defined by $ROTZ(\theta_1)$ with respect to frame R_0, then vector $O_0O_1, 0$ can be expressed as:

$$M_0O_{1,0} \;=\; M_0^1 \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} \tag{3.44}$$

To evaluate vector $O_0O_{2,0}$, we first need to evaluate the rotation matrix M_0^2 of frame R_2 with respect to frame R_0:

$$\begin{aligned} M_0^2 = M_0^1.M_1^2 = ROTZ(\theta_1).ROTX(\theta_2) &= \begin{bmatrix} C1 & S1 & 0 \\ -S1 & C1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & C2 & -S2 \\ 0 & S2 & C2 \end{bmatrix} \\ &= \begin{bmatrix} C1 & S1C2 & -S1S2 \\ -S1 & C1C2 & -C1S2 \\ 0 & S2 & C2 \end{bmatrix} \end{aligned} \tag{3.45}$$

The expression of vector $O_0O_{2,0}$ is then given by:

$$O_0O_{2,0} \;=\; O_0O_{1,0} + M_0^2. \begin{bmatrix} 0 \\ 0 \\ L_2 \end{bmatrix} \tag{3.46}$$

Evaluation of this vector gives:

$$\begin{aligned} O_0O_{2,0} &= \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + \begin{bmatrix} C1 & S1C2 & -S1S2 \\ -S1 & C1C2 & -C1S2 \\ 0 & S2 & C2 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ L_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} \end{aligned} \tag{3.47}$$

The third degree of freedom is prismatic, therefore the transfer matric between frame R_3 and frame R_2, is equal to the identity matrix, therefore we have:

$$M_2^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I, \text{ and } M_0^3 = M_0^2 \tag{3.48}$$

$O_0O_{3,0}$ can then be expressed as follows:

$$\begin{aligned} O_0O_{3,0} &= \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + M_0^3 \begin{bmatrix} 0 \\ 0 \\ L_3 + \theta_3 \end{bmatrix} \\ &= \begin{bmatrix} -S1S2(L_3+\theta_3) \\ -C1S2(L_3+\theta_3) \\ L_1 + C2(L_3+\theta_3) \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \end{aligned} \tag{3.49}$$

This equation represents the geometric model of the transporter. As it can be seen, the revolute variables (θ_1, θ_2) appears in the trigonometric functions SIN and COS while the prismatic variable θ_3 appears as a linear function that increases or decreases the length of link L_3 . to find the inverse geometric transform, we need to invert the system equation previously obtained:

$$\begin{aligned} X &= S1S2(L_3+\theta_3) \\ Y &= -C1S2(L_3+\theta_3) \\ Z &= L_1 + C2(L_3+\theta_3) \end{aligned} \tag{3.50}$$

First, we can eliminate the variable θ_1 from the system equations taking the squares of X, Y, and Z, we have the following two equations:

$$\begin{aligned} X^2 + Y^2 &= S2^2(L_3+\theta_3)^2 \\ (Z-L_1)^2 &= C2^2(L_3+\theta_3)^2 \end{aligned} \tag{3.51}$$

The prismatic variable θ_3 can be evaluated as follows:

$$X^2 + Y^2 + (Z-L_1)^2 = (L_3+\theta_3)^2 \tag{3.52}$$

For a robot arm, it is meaningful to assume that $L_3 + \theta_3 \leq 0$ because of mechanical constraints. Therefore, we have the solution for angle θ_3:

$$\theta_3 = \sqrt{X^2 + Y^2 + (Z-L_1)^2} - L_3 \tag{3.53}$$

The term $L_3 + \theta_3$ is known, we can find two solution for angle $theta_2$ as follows:

$$\begin{aligned} C2 &= \frac{Z-L_1}{L_3+\theta_3} \\ S2 &= \pm\sqrt{X^2+Y^2}/(L_3+\theta_3) \quad (L_3+\theta_3 > 0) \end{aligned} \tag{3.54}$$

This gives two solutions θ_2^+ and θ_2^- as a result of using $(S2^+, C2)$ and $(S2^-, C2)$, respectively. Therefore, we have:

$$\theta_2^+(S2^+, C2) \text{ and } \theta_2^-(S2^-, C2) \tag{3.55}$$

These solutions are symmetrical as shown in Figure 2.8:

$$\theta_2^+ = -\theta_2^- \tag{3.56}$$

To find angle θ_1, we consider the term $S2(L_3 + \theta_3)$ which can be expressed by:

$$S2(L_3 + \theta_3) = \pm\sqrt{X^2 + Y^2} \tag{3.57}$$

The sign of the square root is identical to that of θ_2, we can then write:

$$S2(L_3 + \theta_3) = Sign(\theta_2).\sqrt{X^2 + Y^2} \tag{3.58}$$

Since, angle θ_1 can be found if and only if $X^2 + Y^2 \neq 0$. We have:

$$S1 = Sign(\theta_2)\frac{X}{\sqrt{X^2 + Y^2}} \text{ and } C1 = -Sign(\theta_2)\frac{Y}{\sqrt{X^2 + Y^2}} \tag{3.59}$$

Two solutions are obtained for angle θ_1 depending on the sign of θ_2:

$$\theta_1^+(S1^+, C1^+) \text{ and } \theta_1^-(S1^-, C1^-) \tag{3.60}$$

Obviously these solutions differ by Π, we therefore have:

$$\theta_1^- = \theta_1^+ - Sign(\theta_1^+).\Pi \tag{3.61}$$

To select one solution out of two, we better compare both solutions θ_1^+ and θ_1^- to the previous value of θ_1 . As $\theta_1^+ +$ and $\theta_1^- +$ differ by Π, then the closest solution to θ_1 will be selected. Following this comparison, the sign of angle θ_2 can then be uniquely determined because of the previous expressions of $S1$ and $C1$. As a result, a unique solution for both angles θ_1 and θ_2 can then be found using the trajectory continuity as described above.

Consider the case of point O_3 when it is located on the Z_0 axis, this situation corresponds to $X = 0$ and $Y = 0$. As the first and second equation are nil, then no information is provided regarding angle θ_1 . Clearly, this case presents a singularity point for the transporter. Any value of the angle θ_1 satisfies the system equations which becomes:

$$\begin{aligned} \theta_2 &= 0 \\ Z - L_1 &= L_3 + \theta_3 \end{aligned} \tag{3.62}$$

Solving this system gives:

$$\begin{aligned} \theta_3 &= Z - L_1 - L_3 \\ \theta_2 &= 0 \\ \theta_1 & \quad \text{is undetermined} \end{aligned} \tag{3.63}$$

As discussed previously, no mathematical solution can provided by the knowledge of the current point. To ensure continuity of the trajectory, the control program should generate solutions that satisfy the continuity criteria with respect to the trajectory in the joint space. As proposed in the previous example, a time polynomial of $\theta_1(t)$ can be used in order to predict the next point of angle θ_1 .

3.6 Case of a one-prismatic and two-revolute arm

Consider the three d.o.f. robot arm which is shown in Figure 2.9. The arm architecture has the following geometric topology:

$$\begin{array}{l} \textit{Link } 1\ (P(Z),\ Z(L_1)) \\ \textit{Link } 2\ (R(Z),\ Z(L_2)) \\ \textit{Link } 3\ (R(Z),\ Z(L_3)) \end{array} \tag{3.64}$$

This robot arm is observed relative to a fixed frame of reference R_0. The link L_1 is defined along the Z_0 axis and the first degree of freedom θ_1 is prismatic, the coordinates of point $O_{1,0}$ becomes:

$$O_0O_{1,0} = M_0^1.O_0O_{1,1} = \begin{bmatrix} 0 \\ 0 \\ L_1 + \theta_1 \end{bmatrix} \tag{3.65}$$

where the matrix M_0^1 is the identity matrix because link L_1 is prismatic. Now we can express the coordinates of point O_2 as follows:

$$O_0O_{2,0} = O_0O_{1,0}M_0^2.O_1O_{2,2} \tag{3.66}$$

The link L_2 is defined along the Y_2 axis and its motion is revolute about the Z_1 axis, we have:

$$M_0^2 = M_0^1.M_1^2 = M_1^2 = \begin{bmatrix} C2 & -S2 & 0 \\ S2 & C2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.67}$$

The orientation matrix of Link L_2 can simply be expressed as:

$$O_2O_{2,0} = \begin{bmatrix} 0 \\ 0 \\ L_1 + \theta_1 \end{bmatrix} + \begin{bmatrix} C2 & -S2 & 0 \\ S2 & C2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 \\ L_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -S2L_2 \\ C2L_2 \\ L_1 + \theta_1 \end{bmatrix} \tag{3.68}$$

The link L_3 is on the Y_3 axis and has a revolute motion about the X_2 axis:

$$M_0^3 = M_0^1.M_1^2.M_2^3 = \begin{bmatrix} C2 & -S2 & 0 \\ S2 & C2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & C3 & -S3 \\ 0 & S3 & C3 \end{bmatrix} = \begin{bmatrix} C2 & -S2C3 & S2S3 \\ S2 & C2C3 & -C2S3 \\ 0 & S3 & C3 \end{bmatrix} \tag{3.69}$$

The coordinate of point O_3 can then be expressed as:

$$O_0O_{3,0} = O_0O_{2,2} + M_0^3.O_2O_{3,3} = \begin{bmatrix} -S2(L_2 + C3L_3) \\ C2(L_2 + C3L_3) \\ L_1 + \theta_1 + S3L_3 \end{bmatrix} = [X\ Y\ Z^t] \tag{3.70}$$

The inverse geometrical problem is to determine the joint variables θ_1, θ_2 and θ_3 given the parameters X, Y and Z. We determine θ_2 from the equations of X and Y:
Let us eliminate θ_2 by squaring and adding the first two equations, we obtain:

$$X^2 + Y^2 = (L_2 + C3L_3)^2 \tag{3.71}$$

For simplicity, we assume $L_2 \geq L_3$, the term $L_2 + C3L_3$ is always positive or nil:

$$L_2 + C3L_3 = \sqrt{X^2 + Y^2} \tag{3.72}$$

The term $C3$ can then be uniquely determined

$$C3 = \frac{1}{L_3}\left[\sqrt{X^2 + Y^2} - L_2\right] \tag{3.73}$$

However, two symmetrical solutions are expected for θ_3 as it can be seen from Figure. This means that S3 cannot be uniquely determined:

$$S3^+ = \sqrt{1 - C3^2} \text{ and } S3^- = -S3^+ \tag{3.74}$$

Two solutions are expected for θ_3:

$$\theta_3^+(C3, S3^+) \text{ and } \theta_3^-(C3, S3^-) \tag{3.75}$$

In any case, if $X^2 + Y^2 \neq 0$ we can determine a unique solution for θ_2:

$$\begin{aligned} S2 &= -X/\sqrt{X^2 + Y^2} \\ C2 &= Y/\sqrt{X^2 + Y^2} \end{aligned} \tag{3.76}$$

Angle θ_2 is obtained as follows:

$$\theta_2 = Tan^{-1}(S2, C2)$$

As the sign of $S3$ cannot be uniquely determined, the third Equation leads to finding two solutions θ_1^+ and θ_1^- depending on the sign of $S3$:

$$\begin{aligned} \theta_1^+ &= Z - L_1 - S3^+.L_3 \\ \theta_1^- &= Z - L_1 - S3^-.L_3 \end{aligned} \tag{3.77}$$

Figure 2.10 summarizes the obtained solutions, for each point O_3, two configurations of this robot arm can then be found:

$$(\theta_1^+, \theta_2, \theta_3^+) \text{ and } (\theta_1^-, \theta_2, \theta_3^-) \tag{3.78}$$

Note here that none of the angles, in the first $(+)$ and second $(-)$ configurations, differ by Π . To retain one configuration out of two, one could compare θ_1^+ and θ_1^- to the previous value of θ_1, i.e., $\theta_1(n-1)$. The retained solution should be the closest among θ_1^+ and θ_1^- to $\theta_1(n-1)$.

Discussion

In the case we have $L_3 < L_2$, we obtain two symmetrical solutions for θ_3, since we can eliminate one of them by tacking for θ_2 the domain $[0, -\pi]$.
When the point O_3 is on the Z_0 axis we have:

$$X = Y = 0, \quad and \quad L_2 + C3L_3 = 0 \rightarrow C3 = -L_2/L_3 \tag{3.79}$$

We can determine θ_1 from the Z's equation. But θ_2 cannot be determine uniquely. There exists an infinite number of solution for θ_2. The continuity criteria can then be used to select one appropriate solution. Finally, we note that the domain of variation of variable θ_2 can be $[-\pi, +\pi]$, and the domain of θ_1 has practical limitation.

3.7 Study of a six degree-of-freedom arm

Consider a robot arm composed of six degree-of-freedom. The Transporter part is defined by three revolute joints:

$$\begin{aligned} &Link_1(R(Z), Z(L_1)) \\ &Link_2(R(X), Z(L_2)) \\ &Link_3(R(X), Z(L_3)) \end{aligned} \tag{3.80}$$

The effector part is defined also by three revolute joints:

$$\begin{array}{c} Link_4(R(Z), Z(L_4)) \\ Link_5(R(X), Z(L_5)) \\ Link_6(R(Z), Z(L_6)) \end{array} \quad (3.81)$$

The whole robot arm is shown on Figure 2.11. To obtain the basic geometric system for the arm, we have to express vectors $O_0O_{6,0}, X_6, Y_6$ and Z_6 with respect to the absolute coordinate system R_0 . The transporter part of this arm is the same as in Section 2.3. For this we can use results concerning $O_0O_{3,0}$ and M_0^3 which have been computed and discussed in this Chapter.

We start with the point O_4 and we use vector $O_0O_{3,0}$ and matrix M_0^3 . The vector O_0O_4 is given by the general expression:

$$O_0O_{4,0} = O_0O_{3,0} + M_0^4.O_3O_{4,4} \quad (3.82)$$

and M_0^4 is the product:

$$\begin{aligned} M_0^4 = M_0^3.M_3^4 &= \begin{bmatrix} C1 & -S1C23 & S1S23 \\ S1 & C1C23 & -C1S23 \\ 0 & S23 & C23 \end{bmatrix} . \begin{bmatrix} C4 & -S4 & 0 \\ S4 & C4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} C1C4 - S1C23S4 & -C1S4 - S1C23C4 & S1S23 \\ S1C4 + C1C23S4 & -S1S4 + C1C23C4 & -C1S23 \\ S23S4 & S23C4 & C23 \end{bmatrix} \end{aligned} \quad (3.83)$$

From Section 2.3, Vector $O_0O_{3,0}$ has the following expression:

$$O_0O_{3,0} = \begin{bmatrix} S1(S2L_2 + S23L_3) \\ -C1(S2L_2 + S23L_3) \\ L_1 + C2L_2 + C23L_3) \end{bmatrix} \quad (3.84)$$

From equation (2.84) we obtain $O_0O_{4,0}$:

$$O_0O_{4,0} = \begin{bmatrix} S1(S2L_2 + S23(L_3 + L_4)) \\ -C1(S2L_2 + S23(L_3 + L_4)) \\ L_1 + C2L_2 + C23(L_3 + L_4) \end{bmatrix} \quad (3.85)$$

Now, for the point O_5 we have:

$$O_0O_{5,0} = O_0O_{4,0} + M_0^5.O_4O_{5,5} \quad (3.86)$$

Link L_5 is a revolute link about X axis. Therefore, M_0^5 is given by:

$$M_0^5 = M_0^4.M_4^5 = M_0^4. \begin{bmatrix} 1 & 0 & 0 \\ 0 & C5 & -S5 \\ 0 & S5 & C5 \end{bmatrix} = \{X_5, Y_5, Z_5\} \quad (3.87)$$

$$X_5 = \begin{bmatrix} C1C4 - S1C23S4 \\ S1C4 + C1C23S4 \\ S23S4 \end{bmatrix} \quad (3.88)$$

$$Y_5 = \begin{bmatrix} -(C1S4 + S1C23C4)C5 + S1S23S5 \\ -(S1S4 - C1C23C4)C5 - C1S23S5 \\ -S23C4C5 + C23S5 \end{bmatrix} \quad (3.89)$$

$$Z_5 = \begin{bmatrix} (C1S4 + S1C23C4)S5 + S1S23C5 \\ -(C1C23C4 - S1S4)S5 - C1S23C5 \\ -S23C4S5 + C23C5 \end{bmatrix} \quad (3.90)$$

Because Link $L6$ is a revolute link about Z axis we have:

$$Z_{6,0} = Z_5, 0 \tag{3.91}$$

Since Vector $O_0O_{5,0}$ may be obtained simply as follows:

$$O_0O_{5,0} = O_0O_{4,0} + M_0^5.O_4O_{6,6} \tag{3.92}$$

Particularly we have:

$$O_0O_{6,0} = O_0O_{4,0} + (L5 + L6).Z_{6,0} \tag{3.93}$$

The cartesian coordinate of the end Effector is:

$$O_0O_{6,0} = \begin{bmatrix} S1(S2L_2 + S23(L_3 + L_4)) + L5.Z_{x6} \\ -C1(S2L_2 + S23(L_3 + L_4)) + L5.Z_{y6} \\ L_1 + C2L_2 + C23(L_3 + L_4)) + L5.Z_{z6} \end{bmatrix} \tag{3.94}$$

Finally we compute the basic orientation matrix M_0^6 using M_0^5 and the rotation property of joint L_6, we have:

$$M_0^6 = M_0^4.M_5^6 = \{X_6, Y_6, Z_6\} = \begin{bmatrix} X_x & X_y & X_z \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix} \tag{3.95}$$

For each component of this matrix we obtain:

$$\begin{aligned}
X_x &= C1C4C6 - S1C23S4C6 - C1S4C5S6 - S1C23C4C5S6 \\
&\quad +S1S23S5S6 && (3.96)\\
X_y &= S1C4C6 + C1C23S4C6 - S1S4C5S6 - C1C23C4C5S6 \\
&\quad -C1S23S5S6 && (3.97)\\
X_z &= S23S4C6 + S23C4C5S6 + C23S5S6 && (3.98)\\
Y_x &= C1C4S6 + S1C23S4S6 - C1S4C5C6 - S1C23C4C5C6 \\
&\quad +S1S23S5C6 && (3.99)\\
Y_y &= -S1C4S6 + C1C23S4S6 - S1S4C5C6 + S1C23C4C5C6 \\
&\quad -C1S23S5C6 && (3.100)\\
Y_z &= S23S4S6 + S23C4C5C6 + C23S5C6 && (3.101)\\
Z_x &= C1S4S5 + S1C23S4C5 + S1S23C5 && (3.102)\\
Z_y &= S1S4S5 - C1C23C4C5 - C1S23C5 && (3.103)\\
Z_z &= -S23C4S5C23C5 && (3.104)
\end{aligned}$$

3.8 The inverse geometric transform

The inverse geometrical problem of the arm, defined in Section 2.3, is to compute the joint variables $\theta_1, ..., \theta 6$, given the basic representation of the End-effector:

$$\underbrace{O_0O_{6,0}}_{\text{Hand Center}} \text{ and } \underbrace{M_0^6 = \{X_6, Y_6, Z_6\}}_{\text{Hand Orientation Matrix}} \tag{3.105}$$

This system equations consists of twelve nonlinear, redundant equations with respect to $\theta_1, ..., \theta 6$. The first three equations concern the three cartesian coordinates of the hand center $O_0O_{6,0}$. These equations may be written such that the unknown terms appear on the right-hand of the equal sign:

$$\begin{aligned}
X_6 - L5.Z_{x6} &= X_4 = S1(S2L_2 + S23(L_3 + L4)) \\
Y_6 - L5.Z_{y6} &= Y_4 = -C1(S2L_2 + S23(L_3 + L4)) \\
Z_6 - L5.Z_{z6} &= Z_4 = L_1 + C2L_2 + C23(L_3 + L4)
\end{aligned} \tag{3.106}$$

Vector O_0O_5 can be expressed as follows:

$$O_0O_{4,0} = O_0O_{6,0} - L5.Z_{6,0} \tag{3.107}$$

Equations of X_4, Y_4 and Z_4 are similar to the three revolutes transporter developed in Section 2.4. The solutions for θ_1, θ_2 and θ_3 have the same form as in Section 2.3.

The second step is to determine solutions for $\theta 4, \theta 5$ and $\theta 6$. For that we observe that the rotation axes of links L_4 and L_5 are orthogonal. Since the terms $C_4, 4\ S_4, C_5$ and S_5 should appear in the rotation matrix M_3^5:

$$M_3^5 = M_3^4.M_4^5 = ROTZ(\theta 4).ROTX(\theta 5) \tag{3.108}$$

$$M_3^5 = \begin{bmatrix} C4 & -S4 & 0 \\ S4 & C4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C5 & -S5 \\ 0 & S5 & C5 \end{bmatrix} = \begin{bmatrix} C4 & -S4C5 & S4S5 \\ S4 & C4C5 & -C4S5 \\ 0 & S5 & C5 \end{bmatrix} \tag{3.109}$$

The rotation matrix of the robot hand is known and is given by:

$$M_0^6 = M_0^3.M_3^5.M_5^6 = \{X_6, Y_6, Z_6\} \tag{3.110}$$

In addition, the last three rotation axes are concurrent, and we have:

$$M_6^5 = \begin{bmatrix} C6 & S6 & 0 \\ -S6 & C6 & 0 \\ 0 & 01 & \end{bmatrix} \tag{3.111}$$

To identify C_4, S_4, C_5 and S_5 we may use the following equations:

$$M_3^5 = (M_0^3)^{-1}.M_0^6.(M_3^5)^{-1} = M_3^0.M_0^6.M_6^5 \tag{3.112}$$

The system L_1, L_2, and L_3 has been solved and we suppose a solution $(\theta_1, \theta_2, \theta_3)$ is found. Since the matrix $M_3^0.M_0^6$ is known. As $\theta 6$ is defined as a rotation about $Z5$ axis, then $Z6$ will not be affected by $\theta 6$. For this, we start by expressing the product $M_3^0.M_0^6$. For M_3^0 we have:

$$M_3^0 = \left[M_0^3\right]^t = [ROTZ(\theta_1).ROTX(\theta_2).ROTX(\theta_3)]^t \tag{3.113}$$

Since the rotation axes of θ_2 and θ_3 are parallel:

$$\begin{aligned} M_0^3 &= ROTZ(\theta_1).ROTX(\theta_2 + \theta_3) \\ M_3^0 &= \begin{bmatrix} C1 & S1 & 0 \\ -S1C23 & C1C23 & S23 \\ S1S23 & -C1S23 & C23 \end{bmatrix} \end{aligned} \tag{3.114}$$

Any M_0^6 is given by:

$$M_0^6 = \begin{bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix} \tag{3.115}$$

Since we have:

$$M_3^0.M_0^6 = (V1V2V3)$$

$$V1 = \begin{bmatrix} C1X_x + S1X_y \\ -S1C23X_x + C1C23X_y + S23X_z \\ S1S23X_x - C1S23X_y + C23X_z \end{bmatrix}$$

$$V2 = \begin{bmatrix} C1Y_x + S1Y_y \\ -S1C23Y_x + C1C23Y_y + S23Y_z \\ S1S23Y_x - C1S23Y_y + C23Y_z \end{bmatrix}$$

$$V3 = \begin{bmatrix} C1Z_x + S1Z_y \\ -S1C23Z_x + C1C23Z_y + S23Z_Z \\ S1S23Z_X - C1S23Z_Y + C23Z_Z \end{bmatrix}$$

As $\theta 6$ is a rotation about Z axis, then the third column (or vector) of the produce $M_3^0.M_0^6$ will not be affected by $\theta 6$. Recall the relation (3-2) and (3-3) we can express the third column of $M_3^5(3-1)$ **independently from** $\theta 6$:

$$\begin{aligned} S4S5 &= C1Z_x + S1Zy && (3.116) \\ C4S5 &= C23(S1Z_x - C1Z_y) - S23Z_z && (3.117) \\ C5 &= S23(S1Z_x - C1Z_y) + C23Z_z && (3.118) \end{aligned}$$

We note that it is out of interest to express $S5$, $C4$, and $S4$ from the same Equations 2.116 because of their dependency of the unknown angle $\theta 6$. Certainly a similar equations relating $\theta 6$ to $\theta 5$ can be obtained while $\theta 4$ is unknown. As this is completely equivalent to the system 2.116, then it is of no interest. If we examine the mechanical system, it will clearly indicate the impossibility to uniquely determine solution for $\theta 4$ and $\theta 6$ independently from $\theta 5$. In particular, when $\theta 5 = 0$ the hand orientation will only depend on $\theta 4 + \theta 6$. Since $\theta 4$ and $\theta 6$ cannot be determined independently from $\theta 5$. In fact, the mathematical system 2.106 does not give more information than the system 2.116. Using the system equations 2.116 we have:

$$\begin{aligned} C5 &= C23(S1Z_x - C1Z_y) - C23Z_z \\ S5 &= \pm\sqrt{1 - C5^2} && (3.119) \end{aligned}$$

Two symmetrical solutions are then possible for angle $\theta 5$ within $[-\pi, +\pi]$: Angle $\theta 4$ can be determined from the system (3-4) only when $S \neq 0$ (see next pages), we have:

$$\begin{aligned} S4 &= \frac{Z_xC1 + Z_yS1}{S5} \\ C4 &= \frac{C23(S1Z_x - C1Z_y) - S23Z_z}{S5} && (3.120) \end{aligned}$$

As $C4$ and $S4$ depend on the sign of $S5$, then two solutions are expected within $[-\pi, +\pi]$:

$$\{C4^+, S4^+\} \text{ and } \{C4^- = -C4^+ \ \ and \ \ S4^- = -S4^+\} \qquad (3.121)$$

The solutions $\theta 4^+$ and $\theta 4^-$ differ by π . In the domain $[-\pi, +\pi]$ we have: $\theta 4^+(C4^+, S4^+)$

$$\theta 4^- = \theta 4^+ - sign(\theta 4^+).\pi \qquad (3.122)$$

To determine $\theta 6$ when $\theta 5 \neq 0$ we assume the value of $C4, S4, C5$ and $S5$ have been computed according to equations 2.117 and 2.118. Let us express the rotation matrix M_5^6, we have:

$$M_5^6 = M_5^3.M_3^0.M_0^6 \qquad (3.123)$$

Matrices $M_5^3(\theta 4, \theta 5)$, $M_3^0(\theta_1, \theta_2, \theta_3)$, and M_0^6 are given. The matrix M_5^6 is a $ROTZ(\theta 6)$:

$$M_5^6 = \begin{bmatrix} C6 & -S6 & 0 \\ S6 & C6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.124)$$

Recall the matrices $M_3^0.M_0^6(3-3)$ and $M_3^5(3-1)$ that have been previously evaluated, we have:

$$M_5^3 = \begin{bmatrix} C4 & S4 & 0 \\ -S4C5 & C4C5 & S5 \\ S4S5 & -C4S5 & C5 \end{bmatrix} \qquad (3.125)$$

Expression of $C6$ and $S6$ are then obtained from the product $M_5^3.(M_3^0.M_0^6)$ as follows:

$$\begin{aligned} C6 &= C4(C1X_x + S1X_y) + S4(-S1C23X_x + C1C23X_y + S23X_z) \\ S6 &= -C4(C1Y_x + S1Y_y) + S4(S1C23Y_x - C1C23Y_y - S23Y_z) \end{aligned} \tag{3.126}$$

Depending on the sign of $S5$, i.e. $C4$ and $S4$, we determine two solution for $\theta 6$:

$$\begin{aligned} S5^+ &= +\sqrt{1 - C5^2} \rightarrow (C4^+, S4^+) \rightarrow (C6^+, S6^+) \\ \text{And } S5^- &= -S5^+ \rightarrow (C4^-, S4^-) \rightarrow (C6^-, S6^-) \end{aligned} \tag{3.127}$$

Since, two solutions are also expected for $\theta 6$:

$$\theta 6^+(C6^+, S6^+)$$

$$\theta 6^- = \theta 6^+ - Sign(\theta 6^+).\pi \tag{3.128}$$

Figure 2.13 shows how these solution can be obtained: consider one initial solution (Figure A) and consider the operations $\theta_5 \leftarrow -\theta_5$ (Figure B), $\theta_4 \leftarrow -\theta_4 + \Pi$ (Figure C), and $\theta_6 \leftarrow -\theta_6 + \Pi$ (Figure D). Obviously, this lead to obtain another possible solution for the position and the orientation of the arm.

Conclusion on the case $\theta 5 \neq 0$

Two set of solutions are expected:

$$(\theta 5^+, \theta 4^+, \theta 6^+) \text{ and } (\theta 5^+, \theta 4^+, \theta 6^+)$$

where

$$\begin{aligned} \theta 5^+ &\quad \text{and} \quad \theta 5^- \text{are symmetrical} \\ \theta 4^+ &\quad \text{and} \quad \theta 4^- \text{Differ by} \pi \\ \theta 6^+ &\quad \text{and} \quad \theta 6^- \text{Differ by} \pi \end{aligned} \tag{3.129}$$

By trajectory continuity, we may identify a solution. For example, by comparing $(\theta 4^+, \theta 4^-)$ to the previous value of $\theta 4$ (Initial):
Example:

$$\text{If } |\theta 4_I - \theta 4^+| \quad > \quad |\theta 4_I - \theta 4^-| \ THEN$$

$$\theta 4 = \theta 4^+; \theta 5 = \theta^+; \theta 6 = \theta 6^+ \tag{3.130}$$

$$ELSE \tag{3.131}$$

$$\theta 4 = \theta 4^-; \theta 5^+; \theta 6 = -\theta 6^+ \tag{3.132}$$

$$END \tag{3.133}$$

Given the hand center $O_0O_{6,0}$ and the orientation matrix M_0^6, it is possible to find at least two configurations for the effector part $(\theta 4, \theta 5, \theta 6)$:

$$(\theta 5, \theta 4^+, \theta 6^+) \text{ and } (-\theta 5, \theta 4^-, \theta 6^-) \tag{3.134}$$

Case where $\theta 5 = 0$
This case corresponds to a singular configuration as shown on Figure 2.14.In this situation, the rotation axes of $\theta 4$ and $\theta 6$ are co-linear and concurrent, we have:

$$M_3^6 = M_3^4.M_4^5.M_5^6 = M_3^4.M_5^6 = ROTZ(\theta 4^\theta 6) \tag{3.135}$$

And

$$M_3^6 = \begin{bmatrix} C46 & -S46 & 0 \\ C46 & C46 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.136}$$

In order to express C46 and S46, we may use the following equation:

$$M_3^6 = M_3^0.M_0^6 \tag{3.137}$$

The product $M_3^0.M_0^6$ has been previously expressed, we have:

$$\begin{aligned} C46 &= C1X_x + S1X_y \\ S46 &= -(C1Y_X + S1Y_Y) \end{aligned} \tag{3.138}$$

Since $\theta 4$ and $\theta 6$ cannot be expressed independently from each other. In fact the mechanical analysis confirms this mathematical result, and then heuristics can be applied to obtain a possible alternative. For example we may keep unchanged one angle, i.e. $\theta 4$ or $\theta 6$, and determine the other angle from their sum $(\theta 4 + \theta 6)$ as it is identified using $C46$ and $S46$. Another alternative consists of estimating one angle, i.e. $\theta 4$ or $\theta 6$, by extrapolating its time function according to the previous values. Assume the previous values of $\theta 4$ are:

$$\theta 4(t-k), ..., \theta 4(t-1), \theta 4(t) \tag{3.139}$$

And

$$\theta 6(t-k), ..., \theta 6(t-1), \theta 6(t) \tag{3.140}$$

A polynomial approximation with degree K is given by:

$$\theta(t^1) = \sum_{i=0}^{K} a_i \theta(t-i) \tag{3.141}$$

Higher accuracy is obtained for lower degree polynomial. Since we may select to extrapolate either $\theta 4$ or $\theta 6$, according to the lowest polynomial degree. Assume $\theta 4$ is obtained by extrapolation, then $\theta 6$ could now be simply obtained as $\theta 6 = \theta 46 - \theta 4$.

Exercises

1. The hand center of a robot arm is given by: $O_0O_{n,0} = \sum_{i=1}^{n} M_o^i.O_{i-1}O_{i,i}$ Determine the coordinate of vector O_0O_n in the frame R_n .

2. A 2-degrees of freedom robot arm is defined by:
   Link 1 (Revolute (Z), L_1 on $X1$)
   Link 2 (Revolute (Z), L_2 on $X2$)
   Where the first row means that the link L_1 is defined along $Z1$ axis and this link is revolute about Z axis. Link L_2 is similarly defined.
   (a) Determine the coordinate of point O_2 with respect to frame R_0 .
   (b) Obtain the inverse geometrical transform: $(X, Y, Z) \rightarrow (\theta_1, \theta_2)$
   where, θ_1 and θ_2 are the angles of links L_1 and L_2, respectively.
   (c) Discuss the solution of Question B with regards to multiple solution and possible singularity.

3. The same as in problem 2.2 but with the following robot arm:
   Link 1 (Prismatic (Z), L_1 on $Z1$)
   Link 2 (Revolute (X), L_2 on $Z2$)

4. A three degrees of freedom arm is defined by:
   Link 1 (Prismatic (Z), L_1 on $Z1$)
   Link 2 (Revolute (Z), L_2 on $Y2$) Link 31 (Prismatic (Z), L_1 on $Z1$)
   Link 2 (Revolute (X), L_3 on $Y3$)
   (a) Find the geometrical model. For this express the vector O_0O_3 as function of the degrees of freedom θ_1, θ_2, and θ_3 .
   (b) Find the inverse geometrical transform. For this, obtain the expression θ_1, θ_2, and θ_3 as function of X, Y, and Z which are the coordinate of O_0O_3 in frame R_0 .
   (c) Determine the largest domains for θ_1, θ_2 and θ_3 for which only one solution $(\theta_1, \theta_2, \theta_3)$ exists.
   (d) Study the case of singularities for this arm and examine the solution the point O_3 is on the Z_0 axis .
   (e) A singularity in the geometrical model occurs when infinite number of solution for $\theta = G^{-1}(X, Y, Z)$ is observed. Propose method for getting continuous trajectory when passing by a singular point. For this, use information on previous values of $\theta(t-1), \theta(t-2), ..., \theta(t-k)$ in order to find solution for $\theta(t)$.

5. Give the block diagram of a motion coordination system by using the geometric method. The desired points will be the coordinate of the robot hand center and orientation. In which space correction of the motion is achieved in this method.

6. Explain the principle of motion coordination by refering to the discrete nature of the input points and the transient of the robot controller.

7. A six degrees of freedom robot arm is defined by:
   (a) The first 3 degrees of freedom are those defined in problem 2.4.
   (b) The last 3 degrees of freedom are:
   Link 4 (Revolute (Z), $L4$ on $Z4$)
   Link 5 (Revolute (X), $L5$ on $Z5$) Link 6 (Revolute (Z), $L6$ on $Z6$)
   i. The last 3 degrees of freedom have concurrent axes. Prove that the system can be uncoupled as follows: $O_0O_3 = OO_6 - (L5^L6)Z6$ where $Z6$ is the Z axis of frame $R6$.

ii. Find the geometrical model. For this find the expression of $O_0O_6, 0$ and M_0^6 as function of $\theta_1, ..., \theta 6$.

iii. Find the inverse geometrical transform. For this, find the expression of $\theta_1, ..., \theta 6$ as function of the vector $O_0O_6 = (XYZ)$ and $M_0^6 = \{X_6Y_6Z_6\}$.

Bibliography

[1] Journal IEEE Trans. on Robotics and Automation.

[2] Journal IEEE Trans. on Sys. Man and Cybernetics.

[3] International Journal of Robotics Research.

[4] Proceedings of the Inter. Symp. on Robotics and Automation.

[5] Proceedings of the Inter. Symp. on Robotics and Manufacturing.

[6] IEEE Proceedings on Sys. Man and Cybernetics.

[7] IEEE Conf. on Robotics and Automation.

[8] Craig, J., "Introduction to robotics: Mechnics and control", Addition-Wesley: Reading, Mass., 1986.

[9] Yoshikawa, T., "Analysis and control of robot manipulators with redundency", First Int. Symp. on Robotics and Research, Eds M. Brady and R. Paul, Cambridge, Mass., MIT press, 1984, pp. 735-748.

[10] Paul, R.C., "Robot manipulators: mathematics, programming and control", Cambridge, Mass., MIT press, 1981.

[11] Lee, C.S.G., "A geometric approach in solving kinematics of PUMA robots", IEEE Trans. on Aerospace and Electronic Systems, Vol. AES-20, No. 6, 1984, pp.696-706.

[12] Tommy Wang, L.-C. and CHih Cheng Chen, "On the numerical kinematic analysis of general parallel robotic manipulators", IEEE Trans. on Robotics and Automation, Vol 9, No 3, June 1993, pp. 272–285.

[13] Wang, L.T. and Ravani, B., "Recursive computations of kinematic and dynamic equations mechnical manipulators, IEEE Trans. on Robotics and Automation, Vol RA-1, 1985, pp. 124–131.

[14] Baillieul, J., "Kinematic programming alternatives for redundent manipulators", Proc. IEEE Int. Conf. Robotics and Automation, Vol RA-1, 1985, pp. 722–728.

[15] M. Al-Mouhamed and H. Al-Mohammad. A multi-microprocessor system for the control of robots. *Euromicro Journal on Microprocessing and Microprogramming*, 23:123–128, March 1988.

[16] P. K. Allen, A. Timcenko, B. Yoshimi, and P. Michelman. Automated tracking and grasping of a moving object with a robotic hand-eye system. *IEEE Trans. on Robotics and Automation*, Vol 9, No 2:152–165, Apr 1993.