

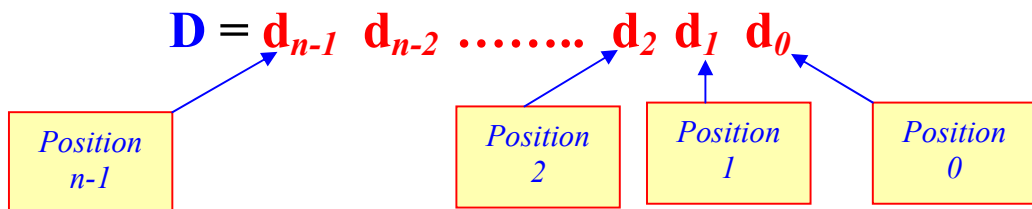
Number Systems

Introduction & Objectives:

- Before the inception of *digital* computers, the only number system that was in common use is the *decimal* number system (النظام العشري) which has a total of 10 digits (0 to 9).
- As discussed in the previous lesson, signals in *digital* computers may represent a digit in some number system. It was also found that the binary number system is more reliable to use compared to the more familiar decimal system
- In this lesson, you will learn:
 - What is meant by a weighted number system.
 - Basic features of weighted number systems.
 - Commonly used number systems, e.g. decimal, binary, octal and hexadecimal.
 - Important properties of these systems.

Weighted Number Systems:

- A number **D** consists of n digits with each digit has a particular *position*.



- Every digit *position* is associated with a *fixed weight*.
- If the weight associated with the i^{th} position is w_i , then the value of **D** is given by:

$$D = d_{n-1} w_{n-1} + d_{n-2} w_{n-2} + \dots + d_2 w_2 + d_1 w_1 + d_0 w_0$$

Example of Weighted Number Systems:

- The Decimal number system (النظام العشري) is a weighted system.
- For Integer decimal numbers, the weight of the rightmost digit (*at position 0*) is **1**, the weight of *position 1* digit is **10**, that of *position 2* digit is **100**, *position 3* is **1000**, etc.

Thus,

$w_0 = 1, w_1 = 10, w_2 = 100, w_3 = 1000, \text{ etc.}$

Example Show how the value of the decimal number **9375** is estimated

Position	3	2	1	0
Number	9	3	7	5
Weight	1000	100	10	1
Value	9 x 1000	3x100	7x10	5x1
Value	9000 + 300 + 70 + 5			

Diagram annotations: A box labeled "First Position Index" with an arrow pointing left above the table. Another box labeled "First Position Index (0)" with an arrow pointing to the right above the '0' position in the table.

The Radix (Base)

1. For *digit position* i , most weighted number systems use weights (w_i) that are *powers of some constant value* called the **radix (r)** or the **base** such that $w_i = r^i$.
2. A number system of radix r , typically has a set of r allowed digits $\in \{0,1, \dots,(r-1)\}$ → *See the next example*
3. The leftmost digit has the highest weight → **Most Significant Digit (MSD)** → *See the next example*
4. The rightmost digit has the lowest weight → **Least Significant Digit (LSD)** → *See the next example*

Example Decimal Number System

1. Radix (Base) = Ten
2. Since $w_i = r^i$, then
 - $w_0 = 10^0 = 1$,
 - $w_1 = 10^1 = 10$,
 - $w_2 = 10^2 = 100$,
 - $w_3 = 10^3 = 1000$, etc.
3. Number of Allowed Digits is Ten $\rightarrow \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Thus:

MSD

LSD

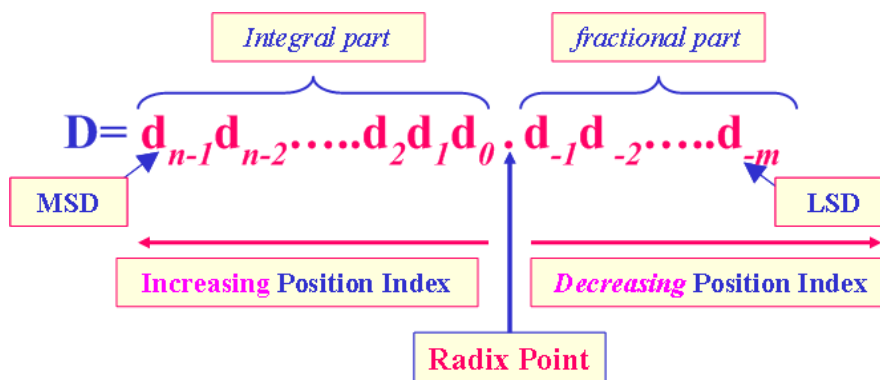
$$\begin{aligned}
 9375 &= 5 \times 10^0 + 7 \times 10^1 + 3 \times 10^2 + 9 \times 10^3 \\
 &= 5 \times 1 + 7 \times 10 + 3 \times 100 + 9 \times 1000
 \end{aligned}$$

Position	3	2	1	0
	1000	100	10	1
Weight	$= 10^3$	$= 10^2$	$= 10^1$	$= 10^0$

The Radix Point

Consider a number system of radix r,

- A number D of n integral digits and m fractional digits is represented as shown



- Digits to the left of the radix point (*integral digits*) have positive position indices, while digits to the right of the radix point (*fractional digits*) have negative position indices
- Position *indices* of digits to the left of the *radix point* (the *integral part of D*) start with a **0** and are incremented as we move lefts ($d_{n-1}d_{n-2}\dots d_2d_1d_0$.)
- Position *indices* of digits to the right of the *radix point* (the *fractional part of D*) are *negative* starting with **-1** and are decremented as we move rights ($d_{-1}d_{-2}\dots d_{-m}$).
- The *weight* associated with digit position i is given by $w_i = r^i$, where i is the position index
 - $\forall i = -m, -m+1, \dots, -2, -1, 0, 1, \dots, n-1$

- The Value of **D** is Computed as :

$$D = \sum_{i=-m}^{n-1} d_i r^i$$

Example Show how the value of the following decimal number is estimated

$$D = 52.946$$

Number	5	2	.	9	4	6
Position	1	0	.	-1	-2	-3
Weight	10^1 = 10	10^0 = 1	.	10^{-1} = 0.1	10^{-2} = 0.01	10^{-3} = 0.001
Value	5 x 10	2 x 1	.	9 x 0.1	4 x 0.01	6 x 0.001
Value	50 + 2 + 0.9 + 0.02 + 0.006					

$$D = 5 \times 10^1 + 2 \times 10^0 + 9 \times 10^{-1} + 4 \times 10^{-2} + 6 \times 10^{-3}$$

Notation

- Let $(D)_r$ denotes a number D expressed in a number system of radix r .

Note: In this notation, r will be expressed in decimal

Example:

- $(29)_{10}$ Represents a decimal value of 29. The radix “10” here means ten.
- $(100)_{16}$ is a Hexadecimal number since $r = “16”$ here means sixteen. This number is equivalent to a decimal value of 16^2 .
- $(100)_2$ is a Binary number (radix =2, i.e. two) which is equivalent to a decimal value of $2^2 = 4$.

Important Number Systems

The Decimal System

- $r = 10$ (*ten* → Radix is not a Power of 2)
 - Ten Possible Digits {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

The Binary System

- $r = 2$
- Two Allowed Digits {0, 1}
- A Binary Digit is referred to as **Bit**
- The leftmost bit has the highest weight → **Most Significant Bit (MSB)**
- The rightmost bit has the lowest weight → **Least Significant Bit (LSB)**

Examples

Find the decimal value of the two Binary numbers $(101)_2$ and $(1.101)_2$

MSB LSB

↓ ↓

- $(101)_2 = 1x2^0 + 0x2^1 + 1x2^2$
- $= 1x1 + 0x2 + 1x4$
- $= (5)_{10}$

MSB LSB

↓ ↓

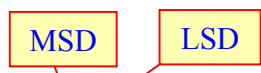
- ❖ $(1.101)_2 = 1x2^0 + 1x2^{-1} + 0x2^{-2} + 1x2^{-3}$
- ❖ $= 1 + 0.5 + 0.25 + 0.125$
- ❖ $= (1.875)_{10}$


Octal System:

- $r = 8$ (*Eight* = 2^3)
 - **Eight** Allowed Digits {0, 1, 2, 3, 4, 5, 6, 7}

Examples

Find the decimal value of the two Octal numbers $(375)_8$ and $(2.746)_8$


$$\begin{aligned}(375)_8 &= 5 \times 8^0 + 7 \times 8^1 + 3 \times 8^2 \\ &= 5 \times 1 + 7 \times 8 + 3 \times 64 \\ &= (253)_{10}\end{aligned}$$

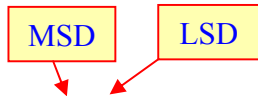

$$\begin{aligned}(2.746)_8 &= 2 \times 8^0 + 7 \times 8^{-1} + 4 \times 8^{-2} + 6 \times 8^{-3} \\ &= (2.94921875)_{10}\end{aligned}$$


Hexadecimal System:

- $r = 16$ (*Sixteen* = 2^4)
- **Sixteen** Allowed Digits {0-to-9 and A, B, C, D, E, F}
 - Where: $A = \text{ten}$, $B = \text{Eleven}$, $C = \text{Twelve}$,
 $D = \text{Thirteen}$, $E = \text{Fourteen}$ & $F = \text{Fifteen}$.
- **Q:** Why is the digit following 9 assigned the character **A** and not “**10**”?
- **A:** What we need is a *single* digit whose value is ten, but “**10**” is actually two digits not *one*.
 - Thus, in Hexadecimal system the 2-digit number $(10)_{16}$ actually represents a value of sixteen not ten $\{(10)_{16} = 0 \times 16^0 + 1 \times 16^1 = (16)_{10}\}$.

Examples

Find the decimal value of the two Hexadecimal numbers $(9EI)_{16}$ and $(3B.C)_{16}$


$$\begin{aligned}(9EI)_{16} &= 1 \times 16^0 + E \times 16^1 + 9 \times 16^2 \\ &= 1 \times 1 + 14 \times 16 + 9 \times 256 \\ &= (2529)_{10}\end{aligned}$$


$$\begin{aligned}(3B.C)_{16} &= C \times 16^{-1} + B \times 16^0 + 3 \times 16^1 \\ &= 12 \times 16^{-1} + 11 \times 16^0 + 3 \times 16 \\ &= (59.75)_{10}\end{aligned}$$

Important Properties

1. The number of possible digits in any number system with radix r equals r . (Give examples in decimal, binary, octal and hexadecimal)
2. The smallest digit is 0 and the largest possible digit has a value = $(r-1)$
3. The Largest value that can be expressed in n integral digits is $(r^n - 1)$ → Prove (Hint add 1 to the LSD position of the largest number)
4. The Largest value that can be expressed in m fractional digits is $(1 - r^{-m})$ → Prove (Hint add 1 to the LSD position of the largest number)
5. The Largest value that can be expressed in n integral digits and m fractional digits is $(r^n - r^{-m})$ → Prove (Hint- add results of properties 3 & 4 above)
6. Total number of values (patterns) representable in n digits is r^n

Clarification (a)

Q. What is the result of adding 1 to the largest digit of some number system??

A.

- For the decimal number system, $(1)_{10} + (9)_{10} = (10)_{10}$
- For the octal number system, $(1)_8 + (7)_8 = (10)_8 = (8)_{10}$

OCTAL System

$$\begin{array}{r} 7 \\ + \\ 1 \\ \hline \cancel{8} \end{array} \quad \text{illegal octal digit}$$

⇓

$$10 = 0 \times 8^0 + 1 \times 8^1$$

- For the hex number system, $(1)_{16} + (F)_{16} = (10)_{16} = (16)_{10}$

HEX System

$$\begin{array}{r} F \\ + \\ 1 \\ \hline (16)_{10} \end{array}$$

⇓ *convert to HEX*

$$(10)_{16} = 0 \times 16^0 + 1 \times 16^1$$

- For the binary number system, $(1)_2 + (1)_2 = (10)_2 = (2)_{10}$

Conclusion. Adding 1 to the largest digit in any number system always has a result of (10) in that number system.

- This is easy to prove since the largest digit in a number system of radix r has a value of $(r-1)$. Adding 1 to this value the result is r which is always equal to $(10)_r = 0 \times r^0 + 1 \times r^1 = (r)_{10}$

Clarification (b)

Q. What is the largest value representable in 3-integral digits?

A. The largest value results when all 3 positions are filled with the largest digit in the number system.

-
- **For** the decimal system, it is $(999)_{10}$
 - **For** the octal system, it is $(777)_8$
 - **For** the hex system, it is $(FFF)_{16}$
 - **For** the binary system, it is $(111)_2$
-

Clarification (c)

Q. What is the result of adding 1 to the largest 3-digit number?

?

A.

- **For** the decimal system, $(1)_{10} + (999)_{10} = (1000)_{10} = (10^3)_{10}$
- **For** the octal system, $(1)_8 + (777)_8 = (1000)_8 = (8^3)_{10}$

OCTAL System

$$\begin{array}{r} 777 \\ + 1 \\ \hline \cancel{778} \\ 10 \end{array} \quad \begin{array}{r} 777 \\ + 1 \\ \hline 770 \end{array} \quad \begin{array}{r} 777 \\ + 1 \\ \hline 1000 \end{array}$$

➤ For the hex system, $(1)_{16} + (FFF)_{16} = (1000)_{16} = (16^3)_{16}$

HEX System

$$\begin{array}{r} FFF \\ + 1 \\ \hline 10 \end{array} \quad \begin{array}{r} FFF \\ + 1 \\ \hline FFF \end{array} \quad \begin{array}{r} FFF \\ + 1 \\ \hline 1000 \end{array}$$

➤ For the binary system, $(1)_2 + (111)_2 = (1000)_2 = (2^3)_{10}$

Binary System

$$\begin{array}{r} 111 \\ + 1 \\ \hline \cancel{112} \\ 10 \end{array} \quad \begin{array}{r} 111 \\ + 1 \\ \hline 110 \end{array} \quad \begin{array}{r} 111 \\ + 1 \\ \hline 1000 \end{array}$$

In general, for a number system of radix r , adding 1 to the largest n -digit number = r^n

Accordingly, the value of largest n -digit number = $r^n - 1$

Conclusions.

1. In any number system of radix r , the result of adding 1 to the *largest n -digit* number equals r^n .
2. Thus, the value of the *largest n -digit* number is equal to $(r^n - 1)$
3. Thus, *n digits* can represent r^n different values (digit combinations) starting from a 0 value up to the largest value of $r^n - 1$.

Appendix A. Summary of Number Systems Properties

The following table summarizes the basic features of the Decimal, Octal, Binary, and Hexadecimal number systems as well as a number system with a general radix r

	Decimal 10	Octal 8	Binary 2	Hexadecimal 16	General r
Allowed Digits	{0-9}	{0-7}	{0-1}	{0-9, A-F}	{0 - R} where $R = (r-1)$
Value of $a_{n-1} \dots a_2 a_1 a_0 \cdot a_{-1} a_{-2} \dots a_{-m}$	$a_{n-1} \times 10^{n-1} + a_{n-2} \times 10^{n-2} + \dots + a_2 \times 10^2 + a_1 \times 10^1 + a_0 \times 10^0 + a_{-1} \times 10^{-1} + a_{-2} \times 10^{-2} + \dots + a_{-m} \times 10^{-m}$ $a_i \in \{0-9\}$ $i = -m, \dots, 0, 1, \dots, n-1$	$a_{n-1} 8^{n-1} + \dots + a_2 8^2 + a_1 8^1 + a_0 8^0 + a_{-1} 8^{-1} + a_{-2} 8^{-2} + \dots + a_{-m} 8^{-m}$ $a_i \in \{0-7\}$	$a_{n-1} 2^{n-1} + \dots + a_2 2^2 + a_1 2^1 + a_0 2^0 + a_{-1} 2^{-1} + a_{-2} 2^{-2} + \dots + a_{-m} 2^{-m}$ $a_i \in \{0,1\}$		$a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r^1 + a_0 r^0 + a_{-1} r^{-1} + a_{-2} r^{-2} + \dots + a_{-m} r^{-m}$ $a_i \in \{0 - (r-1)\}$
Smallest n-digit number	000.....0	000.....0	000.....0	000.....0	000.....0
Largest n-digit number	$999 \dots 9 = 10^n - 1$	$77 \dots 7 = 8^n - 1$	$11 \dots 1 = 2^n - 1$	$FF \dots F = 16^n - 1$	$RR \dots R = r^n - 1$
Range of n-digit integers	$0 - (10^n - 1)$	$0 - (8^n - 1)$	$0 - (2^n - 1)$	$0 - (16^n - 1)$	$0 - (r^n - 1)$
# of Possible Combinations of n-digits	10^n	8^n	2^n	16^n	r^n
Max Value of m Fractional Digits	$1 - 10^{-m}$	$1 - 8^{-m}$	$1 - 2^{-m}$	$1 - 16^{-m}$	$1 - r^{-m}$

Appendix B. First 16 Binary Numbers & Their Decimal Equivalent
(All Possible Binary Combinations in 4-Bits)

Decimal	Bin. Equivelent	Decimal	Bin. Equivelent
0	0000	8	1000
1	0001	9	1001
2	0010	10	1010
3	0011	11	1011
4	0100	12	1100
5	0101	13	1101
6	0110	14	1110
7	0111	15	1111

Appendix C. Decimal Values of the First 10 Powers of 2

- One Kilo is defined as 1000.
- For example, one Kilogram is 1000 grams. A kilometer is 1000 meters.
- In the Binary system, the power of 2 value closest to 1000 is 2^{10} which equals 1024. This is referred to as one Kilo (or in short 1K) in binary systems.
- Thus, one Kilo (or 1K) in Binary systems is not exactly 1000 but rather equals 1024 or 2^{10}
- Thus, in binary systems $2K = 2 \times 1024 = 2048$, $4K = 4 \times 1024 = 4096$, and so on
- Similarly, a one Meg (one million) in binary systems is 2^{20} which equals 1,048,576.

Powers of 2	Decimal. Value
2^0	1
2^1	2
2^2	4
2^3	8
2^4	16
2^5	32
2^6	64
2^7	128
2^8	256
2^9	512
2^{10}	1024

1 Kilo = 1K
2K = 2048
4K = 4096