A Single-Server Queue with Feedback

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Let us suppose that customers arrive at a counter in accordance with a Poisson process of density λ. The customers are served by a single server in order of arrival. The service times are identically distributed, mutually independent, positive random variables with distribution function $H(x)$. Suppose that after being served each customer either immediately joins the queue again with probability p or departs permanently with probability q ($p + q = 1$). In this paper we shall determine for a stationary process the distribution of the queue size as well as the Laplace-Stieltjes transform and the first two moments of the distribution function of the total time spent in the system by a customer.

I. INTRODUCTION

Although the problems discussed in this paper arose in the theory of telephone traffic, we use the terminology of queues. Thus instead of calls and holding times we shall speak about customers and service times respectively.

Let us suppose that in the time interval $(0, \infty)$ customers arrive at a counter in accordance with a Poisson process of density λ. Denote by τ_n ($n = 1, 2, \cdots$) the arrival time of the nth customer. Then the interarrival times $\tau_{n+1} - \tau_n$ ($n = 0, 1, \cdots ; \tau_0 = 0$) are identically distributed, mutually independent random variables with distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{1}$$

The customers are served by a single server in order of arrival. The server is idle if and only if there is no customer in the system. The service times are supposed to be identically distributed, mutually independent, positive random variables with distribution function $H(x)$, and independent of the input process. Suppose that after being served each customer either immediately joins the queue again with probability p or goes away permanently with probability q where $p + q = 1$. The event that a customer returns is independent of any other event involved and, in particular, independent of the number of his previous returns.

The process defined above is said to be of type $[F(x),H(x),p]$. If $p = 0$, then there is no feedback.

Let us denote by $\xi(t)$ the queue size at time t, that is, $\xi(t)$ is the number of customers (either waiting or being served) in the system at time t. Let ξ_n denote the queue size immediately before the arrival of the nth customer, that is, $\xi_n = \xi(\tau_n - 0)$. Denote by θ_n the total time spent in the system by the nth customer.

We are interested in finding the distribution of $\xi(t)$ for a stationary process, i.e., when $\xi(t)$ has the same distribution for all $t \geqq 0$ and the distribution of θ_n for a stationary process, i.e., when θ_n has the same distribution for every $n = 1, 2, \cdots$.

It is easy to prove that the limiting distribution $\lim_{t\to\infty} \mathbf{P}\{\xi(t) = k\}$ $(k = 0, 1, \cdots)$ exists and is independent of the initial state if and only if $\xi(t)$ has a stationary distribution and the limiting distribution is identical with the stationary distribution. Similarly the limiting distribution $\lim_{n\to\infty} \mathbf{P}\{\theta_n \leqq x\}$ exists and is independent of the initial state if and only if θ_n has a stationary distribution and the limiting distribution is identical with the stationary distribution.

Throughout this paper we use the following notation:

$$\Psi(s) = \int_0^\infty e^{-sx}\, dH(x) \qquad (\Re(s) \geqq 0) \tag{2}$$

for the Laplace-Stieltjes transform of $H(x)$,

$$\alpha_r = \int_0^\infty x^r\, dH(x) \qquad (r = 0, 1, \cdots) \tag{3}$$

for the rth moment of $H(x)$, and

$$\alpha = \int_0^\infty x\, dH(x) \tag{4}$$

for the average service time, i.e., $\alpha = \alpha_1$.

Further denote by $H^*(x)$ the distribution function of the total service time of a customer. We have

$$H^*(x) = q \sum_{k=1}^\infty p^{k-1} H_k(x) \tag{5}$$

where $H_k(x)$ denotes the kth iterated convolution of $H(x)$ with itself. For qp^{k-1} $(k = 1, 2, \cdots)$ is the probability that a customer joins the queue k times, and if he joins k times, his total service time is equal to the sum of k mutually independent random variables each of which has the distribution function $H(x)$. If we introduce the notation

$$\Psi^*(s) = \int_0^\infty e^{-sx}\, dH^*(x) \qquad (\Re(s) \geqq 0), \tag{6}$$

then by (5) we obtain that

$$\Psi^*(s) = q \sum_{k=1}^{\infty} p^{k-1}[\Psi(s)]^k = \frac{q\Psi(s)}{1 - p\Psi(s)}. \tag{7}$$

Let

$$\alpha_r^* = \int_0^\infty x^r\, dH^*(x) \qquad (r = 0, 1, \cdots;) \tag{8}$$

and $\alpha^* = \alpha_1^*$. By (7), $\alpha^* = \alpha/q$,

$$\alpha_2^* = \frac{\alpha_2}{q} + \frac{2p\alpha_1^2}{q^2}, \tag{9}$$

and in general α_r^* can be obtained by the following recurrence formula

$$\alpha_r^* = \alpha_r + \frac{p}{q} \sum_{j=1}^{r} \binom{r}{j} \alpha_j \alpha_{r-j}^*. \tag{10}$$

II. THE STATIONARY DISTRIBUTION OF THE QUEUE SIZE

If we know the stationary distribution of the queue size for a process of type $[F(x),H(x)0]$, then that for a process of type $[F(x),H(x),p]$ can be obtained immediately.

Theorem 1: If $\lambda\alpha < q$, then the process $\{\xi(t), 0 \leqq t < \infty\}$ has a unique stationary distribution $\mathbf{P}\{\xi(t) = j\} = P_j^*(j = 0, 1, \cdots)$ and for $|z| \leqq 1$

$$U^*(z) = \sum_{j=0}^{\infty} P_j^* z^j = \left(1 - \frac{\lambda\alpha}{q}\right) \frac{q(1 - z)\Psi(\lambda(1 - z))}{(q + pz)\Psi(\lambda(1 - z)) - z}. \tag{11}$$

If $\lambda\alpha \geqq q$, then a stationary distribution does not exist.

Proof: To find the distribution of the queue size we may assume without loss of generality that the customers join the queue only once and are served in one stretch; however, their service time is equal to the total service time that they would have if they were served in the original manner. Accordingly the distribution of the queue size for the process of type $[F(x),H(x),p]$ is the same as for the process of type

$$[F(x),H^*(x),0].$$

For the latter a stationary distribution $\{P_j^*\}$ exists if and only if $\lambda\alpha^* < 1$, that is, $\lambda\alpha < q$, and is given by A. Y. Khintchine's formula:

$$U^*(z) = \sum_{j=0}^{\infty} P_j^* z^j = \frac{(1 - \lambda\alpha^*)(1 - z)\Psi^*(\lambda(1 - z))}{\Psi^*(\lambda(1 - z)) - z}. \tag{12}$$

(Cf., e.g., Ref. 1 or 2.) This proves (11).

Remark 1. Denote by $B_r^*(r = 0, 1, \cdots)$ the rth binomial moment of $\{P_j^*\}$, that is,

$$B_r^* = \sum_{j=r}^{\infty} \binom{j}{r} P_j^*. \tag{13}$$

If α_{r+1} is finite, then B_r^* is also finite. We have $B_0^* = 1$,

$$B_1^* = \frac{\lambda^2\alpha_2^*}{2(1 - \lambda\alpha_1^*)} + \lambda\alpha_1^* = \frac{\lambda^2\alpha_2 + 2\lambda\alpha_1(1 - \lambda\alpha_1)}{2(q - \lambda\alpha_1)} \tag{14}$$

and for $r = 2, 3, \cdots$

$$B_r^* = \frac{\lambda^r}{r!}\left[\sum_{\nu=1}^{r} \frac{\lambda^\nu \nu!}{(1 - \lambda\alpha^*)^\nu} Y_{r,\nu} + \frac{r}{\lambda}\sum_{\nu=1}^{r-1} \frac{\lambda^\nu \nu!}{(1 - \lambda\alpha^*)^\nu} Y_{r-1,\nu}\right] \tag{15}$$

where

$$Y_{n,\nu} = \sum_{\substack{j_1+j_2+\cdots+j_n=\nu \\ j_1+2j_2+\cdots+nj_n=n}} \frac{n!\alpha_2^{*j_1}\alpha_3^{*j_2}\cdots\alpha_{n+1}^{*j_n}}{j_1!j_2!\cdots j_n!(2!)^{j_1}(3!)^{j_2}\cdots((n+1)!)^{j_n}} \tag{16}$$

The proof of (14) and (15) can be found in Ref. 3.

III. THE STATIONARY PROCESS

Let us denote by χ_n the time needed to complete the current service (if any) at the instant $t = \tau_n - 0$, i.e., immediately before the arrival of the nth customer. If $\xi_n = 0$, then $\chi_n = 0$. It is easy to see that the vector sequence $\{\xi_n, \chi_n; n = 1, 2, \cdots\}$ is a Markov sequence. We shall prove that if $\lambda\alpha < q$, then $\{\xi_n, \chi_n\}$ has a unique stationary distribution, whereas if $\lambda\alpha \geqq q$, then a stationary distribution does not exist. For a stationary sequence $\{\xi_n, \chi_n\}$ introduce the following notation:

$$P_j = \mathbf{P}\{\xi_n = j\} \qquad (j = 0, 1, \cdots), \tag{17}$$

$$P_j(x) = \mathbf{P}\{\chi_n \leqq x, \xi_n = j\} \qquad (j = 1, 2, \cdots) \tag{18}$$

and

$$\Pi_j(s) = \int_0^\infty e^{-sx} dP_j(x) \qquad (j = 1, 2, \cdots). \tag{19}$$

Theorem 2: If $\lambda\alpha < q$, then the Markov sequence $\{\xi_n, \chi_n; n = 1, 2, \cdots\}$ has a unique stationary distribution, for which $P_j = P_j^*$ defined by (11) and

$$U(s,z) = \sum_{j=1}^{\infty} \Pi_j(s)z^j$$
$$= \left(1 - \frac{\lambda\alpha}{q}\right) \frac{\lambda z(1 - z)[\Psi(s) - \Psi(\lambda(1 - z))]}{[z - (q + pz)\Psi(\lambda(1 - z))][s - \lambda(1 - z)]}. \tag{20}$$

If $\lambda\alpha \geqq q$, then $\{\xi_n, \chi_n\}$ has no stationary distribution.

Proof. First consider the process of type $[F(x),H^*(x),0]$. It is proved in Ref. 3 that in this case $\{\xi_n, \chi_n\}$ has a unique stationary distribution if and only if $\lambda\alpha < q$. Namely $\mathbf{P}\{\xi_n = j\} = P_j^*$ $(j = 0, 1, \cdots)$ given by (11) and the generating function corresponding to (20) is given by

$$U^*(s,z) = \frac{(1 - \lambda\alpha)\lambda z(1 - z)[\Psi^*(s) - \Psi^*(\lambda(1 - z))]}{[z - \Psi^*(\lambda(1 - z))][s - \lambda(1 - z)]}. \tag{21}$$

The distribution of the queue size is the same for both the types

$$[F(x),H(x),p] \quad \text{and} \quad [F(x),H^*(x),0].$$

The only difference between the process of type $[F(x),H(x),p]$ and $[F(x),H^*(x),0]$ is that in the latter χ_n the remaining length of the current service at the arrival of the nth customer is replaced by the remaining part of the total service time of the customer just being served at the arrival of the nth customer. The time added to χ_n is independent of the queue size and has the distribution function

$$\hat{H}(x) = q \sum_{k=0}^{\infty} p^k H_k(x), \tag{22}$$

because the probability that a departing customer will join the queue k more times is qp^k and in this case the additional total service time has the distribution function $H_k(x)$; $H_0(x) = 1$ if $x \geqq 0$ and $H_0(x) = 0$ if $x < 0$. The Laplace-Stieltjes transform of (22) is

$$\hat{\Psi}(s) = \int_0^{\infty} e^{-sx}\, d\hat{H}(x) = q \sum_{k=0}^{\infty} p^k[\Psi(s)]^k = \frac{q}{1 - p\Psi(s)}. \tag{23}$$

Accordingly we have

$$U^*(s,z) = U(s,z)\hat{\Psi}(s), \tag{24}$$

whence (20) follows.

Remark 2. If $\chi(t)$ denotes the time needed to complete the current service (if any) at time t ($\chi(t) = 0$ if $\xi(t) = 0$), then the vector process $\{\xi(t),\chi(t); 0 \leqq t < \infty\}$ is a Markov process. $\{\xi(t),\chi(t)\}$ has a stationary distribution if and only if $\lambda\alpha < q$ and it agrees with the stationary distribution of $\{\xi_n, \chi_n\}$.

IV. THE STATIONARY DISTRIBUTION OF θ_n

If the joint distribution of ξ_n and χ_n is known, then the distribution of θ_n is determined uniquely. If $\{\xi_n, \chi_n\}$ has a stationary distribution, then every $\theta_n (n = 1, 2, \cdots)$ has the same distribution. In case of a stationary process let

$$\mathbf{E}\{e^{-s\theta_n}\} = \Phi(s) \qquad (\Re(s) \geqq 0). \tag{25}$$

Theorem 3: If $\lambda\alpha < q$, then θ_n has a unique stationary distribution $\mathbf{P}\{\theta_n \leqq x\}$, *which is given by the following Laplace-Stieltjes transform*

$$\Phi(s) = q \sum_{k=1}^{\infty} p^{k-1} U_k(s,1) \qquad (\Re(s) \geqq 0), \tag{26}$$

where

$$U_1(s,z) = P_0\Psi(s + \lambda(1 - z)) + U(s + \lambda(1 - z), (q + pz)\Psi(s + \lambda(1 - z))) \tag{27}$$

for $\Re(s) \geqq 0$ *and* $|z| \leqq 1$, $P_0 = 1 - \lambda\alpha/q$, $U(s,z)$ *is defined by* (20), *and*

$$U_{k+1}(s,z) = \Psi(s + \lambda(1 - z))U_k(s,(q + pz)\Psi(s + \lambda(1 - z))) \tag{28}$$

for $k = 1, 2, \cdots$.

Proof: The probability that a customer joins the queue exactly k times (including the original arrival) is qp^{k-1} ($k = 1, 2, \cdots$). Denote by $\theta_n^{(k)}$ the total time spent in the system by the nth customer until his kth departure (if he joins the queue at least k times). Denote by $\zeta_n^{(k)}$ the queue size immediately after the kth departure of the nth customer. Let

$$U_k(s,z) = \mathbf{E}\{\exp[-s\theta_n^{(k)}]z^{\zeta_n^{(k)}}\} \qquad (k = 1, 2, \cdots). \tag{29}$$

We can easily see that for a stationary sequence $\{\xi_n, \chi_n\}$

$$\begin{aligned} U_1(s,z) &= P_0\Psi(s + \lambda(1 - z)) \\ &\quad + \sum_{j=1}^{\infty} \Pi_j(s + \lambda(1 - z))(q + pz)^j[\Psi(s + \lambda(1 - z))]^j \\ &= P_0\Psi(s + \lambda(1 - z)) \\ &\quad + U(s + \lambda(1 - z), (q + pz)\Psi(s + \lambda(1 - z))) \end{aligned} \tag{30}$$

where $P_0 = 1 - \lambda\alpha/q$ and $U(s,z)$ is defined by (20). Now we shall prove that (28) holds for $k = 1, 2, \cdots$. Under the conditions $\zeta_n^{(k)} = j$, $\theta_n^{(k)} = x$ and that after the kth service the nth customer joins the queue again, the difference $\theta_n^{(k+1)} - \theta_n^{(k)}$ is equal to the length of $j + 1$ services, the distribution function of which is $H_{j+1}(x)$. If $\theta_n^{(k+1)} - \theta_n^{(k)} = y$, then $\zeta_n^{(k+1)}$ is equal to the sum of two independent random random variables: the first is the number of customers arriving at the counter during the time interval of length y, which has a Poisson distribution with parameter λy, and the second is the number of returning customers, which has a Bernoulli distribution with parameters j and p. Thus

$$\mathbf{E}\{\exp[-s\theta_n^{(k+1)}]z^{\zeta_n^{(k+1)}} \mid \zeta_n^{(k)} = j, \theta_n^{(k)} = x\} = e^{-sx}(q + pz)^j \cdot \tag{31}$$

$$\int_0^\infty \exp[-sy - \lambda(1 - z)y]\, dH_{j+1}(y)$$

$$= e^{-sx}(q + pz)^j[\Psi(s + \lambda(1 - z))]^{j+1},$$

or

$$\mathbf{E}\{\exp[-s\theta_n^{(k+1)}]z^{\zeta_n^{(k+1)}} \mid \zeta_n^{(k)}, \theta_n^{(k)}\}$$
$$= \Psi(s + \lambda(1 - z)) \exp[-s\theta_n^{(k)}].[(q + pz)\Psi(s + \lambda(1 - z))]^{\zeta_n^{(k)}} \tag{32}$$

and unconditionally

$$U_{k+1}(s,z) = \Psi(s + \lambda(1 - z))U_k(s,(q + pz)\Psi(s + \lambda(1 - z))) \tag{33}$$

which proves (28). It is to be noted that stationarity has been used only in the determination of $U_1(s,z)$. The recurrence relation (28) is valid for any process. Finally,

$$\mathbf{E}\{\exp[-s\theta_n]\} = q \sum_{k=1}^{\infty} p^{k-1}\mathbf{E}\{\exp[-s\theta_n^{(k)}]\} = q \sum_{k=1}^{\infty} p^{k-1}U_k(s,1) \tag{34}$$

which was to be proved.

V. THE MOMENTS OF θ_n

Although it seems very complicated to find a closed formula for $\Phi(s)$, the moments of θ_n can be determined explicitly. We shall prove that $\Phi(s) = \Phi(s,1)$ where $\Phi(s,z)$ satisfies a functional equation. This observation makes it possible to find explicit formulas for the moments of θ_n.

Theorem 4: If $\lambda\alpha < q$ *and* θ_n *has a stationary distribution, then*

$$\mathbf{E}\{\theta_n\} = \frac{\lambda\alpha_2 + 2\alpha_1(1 - \lambda\alpha_1)}{2(q - \lambda\alpha_1)} \tag{35}$$

provided that α_2 *is finite, and*

$$\mathbf{E}\{\theta_n^2\} = \frac{q^2 - 2q}{6(q - \lambda\alpha_1)^2[q^2 - q(2 + \lambda\alpha_1) + \lambda\alpha_1]} \cdot \{2q[6\lambda\alpha_1^3 - 6\alpha_1^2 - 6\lambda\alpha_1\alpha_2 + 3\alpha_2 + \lambda\alpha_3] - [12\lambda\alpha_1^3 - 12\alpha_1^2 - 6\lambda\alpha_1\alpha_2 + 2\lambda^2\alpha_1\alpha_3 - 3\lambda^2\alpha_2^2]\}, \tag{36}$$

provided that α_3 is finite.

Proof: Let

$$\Phi(s,z) = q\sum_{k=1}^{\infty} p^{k-1} U_k(s,z) \tag{37}$$

where $U_k(s,z)$ is defined by (27) and (28). By using the recurrence relation (28) we obtain that

$$\Phi(s,z) = qU_1(s,z) + p\Psi(s + \lambda(1 - z))\Phi(s,(q + pz)\Psi(s + \lambda(1 - z))). \tag{38}$$

Let

$$\Phi_{ij} = \left(\frac{\partial^{i+j}\Phi(s,z)}{\partial s^i \partial z^j}\right)_{s=0,z=1}. \tag{39}$$

If we form Φ_{ij} by (38) for $i + j = r$ $(i = 0, 1, \cdots, r)$, then we obtain $r + 1$ linear equations for the determination of Φ_{ij}. These equations can be solved successively for $r = 1, 2, \cdots$. By (26)

$$\mathbf{E}\{\theta_n^r\} = (-1)^r \Phi_{r0} \qquad (r = 0, 1, \cdots) \tag{40}$$

for a stationary process.

VI. A PARTICULAR CASE

If we suppose, in particular, that the service times have an exponential distribution

$$H(x) = \begin{cases} 1 - e^{-\mu x} & \text{for } x \geqq 0, \\ 0 & \text{for } x < 0, \end{cases} \tag{41}$$

then $H^*(x) = 1 - e^{-\mu q x}$ for $x \geqq 0$, $\Psi(s) = \mu/(\mu + s)$ and $\Psi^*(s) = \mu q/(\mu q + s)$. In this case (27) and (28) reduce to

$$U_{k+1}(s,z) = \frac{\mu}{\mu + s + \lambda(1 - z)} U_k\left(s, \frac{\mu(q + pz)}{\mu + s + \lambda(1 - z)}\right) \tag{42}$$

for $k = 0, 1, \cdots$, where

$$U_0(s,z) = \left(1 - \frac{\lambda}{\mu q}\right) \Big/ \left(1 - \frac{\lambda z}{\mu q}\right). \tag{43}$$

By (42) we obtain that

$$U_k(s,1) = \frac{\left(1 - \frac{\lambda}{\mu q}\right)}{a_k(s) - b_k(s)} \tag{44}$$

where $a_k(s)$ and $b_k(s)$ $(k = 0, 1, \cdots)$ are given by the following matrix equation

$$\left\| \begin{matrix} a_k(s) \\ b_k(s) \end{matrix} \right\| = \left\| \begin{matrix} \frac{\mu + \lambda + s}{\mu}, & -q \\ \frac{\lambda}{\mu} & p \end{matrix} \right\|^k \left\| \begin{matrix} 1 \\ \frac{\lambda}{\mu q} \end{matrix} \right\|. \tag{45}$$

Now by (26) the Laplace-Stieltjes transform of $\mathbf{P}\{\theta_n \leq x\}$ is

$$\Phi(s) = \left(1 - \frac{\lambda}{\mu q}\right) \sum_{k=1}^{\infty} \frac{qp^{k-1}}{a_k(s) - b_k(s)}. \tag{46}$$

In this particular case

$$\mathbf{E}\{\theta_n\} = \frac{1}{\mu q - \lambda} \tag{47}$$

and

$$\mathbf{E}\{\theta_n^2\} = \frac{2\mu(2q - q^2)}{(\mu q - \lambda)^2[\mu(2q - q^2) - \lambda(1 - q)]}. \tag{48}$$

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APPENDIX—BY W. S. BROWN

Calculation of the Second Moment of θ_n

If $\lambda\alpha < q$ and if θ_n (the total time spent in the system by the nth customer) has a stationary distribution, then the moments of θ_n are determined by (38)–(40) as explained in Section V. The first moment can be calculated by hand without serious difficulty. The calculation of the second moment, with the aid of an IBM 7090 computer,[4] is described in this appendix.

It is convenient to replace z by $t = 1 - z$ and U_1 by $W = qU_1/(q - \lambda\alpha_1)$. Then the rth moment of θ_n is

$$\beta_r = (-1)^r \Phi_{r0} \tag{49}$$

where

$$\Phi_{ij} = \left[\left(\frac{\partial}{\partial s}\right)^i \left(\frac{\partial}{\partial t}\right)^j \Phi(s,t)\right]_{s=0,\ t=0}. \tag{50}$$

The function $\Phi(s,t)$ is implicitly defined by the equation

$$\Phi(s,t) = (q - \lambda\alpha_1)W(s,t) + p\psi(s + \lambda t)\Phi(s,\omega(s,t)) \tag{51}$$

where*

$$\begin{aligned} \psi(s) &= \sum_{r=0}^{\infty} \frac{(-1)^r \alpha_r s^r}{r!} \\ \omega(s,t) &= 1 - (1 - pt)\psi(s + \lambda t) \\ W(s,t) &= \psi(s + \lambda t) + S(s + \lambda t, \lambda\omega(s,t))T(\omega(s,t)) \end{aligned} \tag{52}$$

with

$$\begin{aligned} S(x,y) &= \frac{\psi(x) - \psi(y)}{x - y} \\ T(\omega) &= \frac{\lambda\omega(1 - \omega)}{1 - \omega - (1 - p\omega)\psi(\lambda\omega)}. \end{aligned} \tag{53}$$

This last pair of equations can be rewritten in the more useful form

$$\begin{aligned} S(x,y) &= \sum_{r=0}^{\infty} \frac{(-1)^{r+1}\alpha_{r+1}}{(r+1)!} C_r(x,y) \\ T(\omega) &= -\frac{\lambda(1 - \omega)}{q - \lambda(1 - p\omega)\varphi(\lambda\omega)} \end{aligned} \tag{54}$$

where

$$\begin{aligned} C_r(x,y) &= \frac{x^{r+1} - y^{r+1}}{x - y} = \sum_{k=0}^{r} x^k y^{r-k} \\ \varphi(x) &= \frac{1 - \psi(x)}{x} = \sum_{r=0}^{\infty} \frac{(-1)^r \alpha_{r+1} x^r}{(r+1)!}. \end{aligned} \tag{55}$$

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* For convenience we have assumed that all of the service moments α_r [see (3)] are finite. However for the calculation of β_r it is clearly sufficient to require only the finiteness of α_{r+1}.

It is now clear that

$$\begin{aligned} \psi(0) &= \alpha_o = 1 \\ S(0,0) &= -\alpha_1 \\ T(0) &= -\frac{\lambda}{q - \lambda\alpha_1}, \\ W(0,0) &= \frac{q}{q - \lambda\alpha_1}, \end{aligned} \tag{56}$$

so from (49)–(51)

$$\beta_0 = \Phi_{00} = \Phi(0,0) = 1 \tag{57}$$

as is required by the definition of the *zero*th moment.

Now suppose all of the quantities Φ_{ij} for $i + j < r$, where r is some positive integer, have been calculated and are expressed as rational functions of λ and p (or q) and the service moments α_k. Then by differentiation of (51) we can obtain a system of $r + 1$ linear equations in the $r + 1$ unknowns, Φ_{ij} with $i + j = r$. These equations will also contain the quantities Φ_{ij} with $i + j < r$, which can be replaced by their known values. The solutions of this linear system will again be rational functions of λ and p (or q) and the service moments α_k. Theoretically this procedure permits the calculation of arbitrarily many of the moments, but in practice the calculations are extremely lengthy. The reader may wish to try the first moment as an exercise.

We shall now outline the computation of the first two moments, β_1 and β_2. The first step was to fix the time scale by setting λ equal to one. The next step was to express $\Phi(s,t)$ as a Taylor series with coefficients Φ_{ij} and expand (51) to second order in s and t. Since rational function operations were not yet available, all denominators had to be eliminated by suitable premultiplications. The result obtained by the computer was a gigantic polynomial in s, t, Φ_{00}, Φ_{01}, Φ_{10}, Φ_{02}, Φ_{11}, Φ_{20}, α_1, α_2, α_3, and q. We chose to view it as a polynomial in s and t. Setting $\Phi_{00} = 1$, all the terms independent of both s and t vanished as expected.

The equations for Φ_{01} and Φ_{10}, obtained by setting the coefficients of t and s to zero, are

$$\begin{aligned} 2A\Phi_{01} + B &= 0 \\ 2C\Phi_{10} + 2D\Phi_{01} + E &= 0 \end{aligned} \tag{58}$$

where

$$\begin{aligned} A &= (q - \alpha_1)[q^2 - q(\alpha_1 + 2) + \alpha_1] \\ B &= -(2\alpha_1^2 - 2\alpha_1 - \alpha_2)[q^2 - q(\alpha_1 + 2) + \alpha_1] \\ C &= -q(q - \alpha_1) \\ D &= -\alpha_1(q - 1)(q - \alpha_1) \\ E &= (2\alpha_1^2 - 2\alpha_1 - \alpha_2)[(\alpha_1 + 1)q - \alpha_1]. \end{aligned} \tag{59}$$

These expressions were factored by hand. From (58) we have

$$\begin{aligned} \Phi_{01} &= -\frac{B}{2A} \\ \Phi_{10} &= \frac{BD - AE}{2AC}, \end{aligned} \tag{60}$$

so, using (59)

$$\Phi_{01} = \Phi_{10} = \frac{2\alpha_1^2 - 2\alpha_1 - \alpha_2}{2(q - \alpha_1)}. \tag{61}$$

Thus the mean time spent in the system (waiting time plus service time) is

$$\beta_1 = -\Phi_{10} = \alpha_1\left(\frac{1 - \alpha_1}{q - \alpha_1}\right) + \frac{\alpha_2}{2(q - \alpha_1)}. \tag{62}$$

In the absence of feedback ($p = 0$, $q = 1$) this reduces to

$$\beta_1^{(0)} = \alpha_1 + \frac{\alpha_2}{2(1 - \alpha_1)} \tag{63}$$

where the first term is the mean service time and the second term is the familiar expression[5] for the mean waiting time in a single server system.

The equations for Φ_{02}, Φ_{11}, and Φ_{20}, obtained by setting the coefficients of t^2, st, and s^2 to zero (and replacing Φ_{10} by Φ_{01}), are

$$\begin{aligned} A_{13}\Phi_{02} + B_1\Phi_{01} + C_1 &= 0 \\ A_{22}\Phi_{11} + A_{23}\Phi_{02} + B_2\Phi_{01} + C_2 &= 0 \\ A_{31}\Phi_{20} + A_{32}\Phi_{11} + A_{33}\Phi_{02} + B_3\Phi_{01} + C_3 &= 0 \end{aligned} \tag{64}$$

where

$$
\begin{aligned}
A_{13} &= 6(q-\alpha_1)^2[q^3 - q^2(2\alpha_1+3) + q(\alpha_1^2+4\alpha_1+3) - \alpha_1(\alpha_1+2)] \\
A_{22} &= -12(q-\alpha_1)^2[q^2 - q(\alpha_1+2) + \alpha_1] \\
A_{23} &= -12\alpha_1(q-1)(q-\alpha_1)^2(q-\alpha_1-1) \\
A_{31} &= 6q(q-\alpha_1)^2 \\
A_{32} &= 12\alpha_1(q-1)(q-\alpha_1)^2 \\
A_{33} &= 6\alpha_1^2(q-1)(q-\alpha_1)^2
\end{aligned} \tag{65}
$$

and

$$
\begin{aligned}
B_1 &= 6(q-1)(q-\alpha_1)^2[4q\alpha_1 - (2\alpha_1^2+4\alpha_1+\alpha_2)] \\
B_2 &= 12(q-1)(q-\alpha_1)^2[2q\alpha_1 - (2\alpha_1^2+3\alpha_1+\alpha_2)] \\
B_3 &= -6(q-1)(q-\alpha_1)^2(2\alpha_1^2+2\alpha_1+\alpha_2)
\end{aligned} \tag{66}
$$

and finally

$$
\begin{aligned}
C_1 = &-2q^4(6\alpha_1^3 - 6\alpha_1^2 - 6\alpha_1\alpha_2 + 3\alpha_2 + \alpha_3) + 3q^3(8\alpha_1^4 - 8\alpha_1^2\alpha_2 \\
&- 8\alpha_1^2 - 6\alpha_1\alpha_2 + 2\alpha_1\alpha_3 - \alpha_2^2 + 6\alpha_2 + 2\alpha_3) - 2q^2(6\alpha_1^5 \\
&+ 12\alpha_1^4 - 6\alpha_1^3\alpha_2 - 6\alpha_1^3 - 21\alpha_1^2\alpha_2 + 3\alpha_1^2\alpha_3 - 12\alpha_1^2 \\
&- 3\alpha_1\alpha_2^2 + 7\alpha_1\alpha_3 - 3\alpha_2^2 + 9\alpha_2 + 3\alpha_3) + q(12\alpha_1^5 \\
&- 18\alpha_1^3\alpha_2 + 2\alpha_1^3\alpha_3 - 3\alpha_1^2\alpha_2^2 - 12\alpha_1^2\alpha_2 + 10\alpha_1^2\alpha_3 \\
&- 12\alpha_1^2 - 9\alpha_1\alpha_2^2 + 12\alpha_1\alpha_2 + 10\alpha_1\alpha_3 - 6\alpha_2^2) \\
&+ \alpha_1(6\alpha_1^2\alpha_2 - 2\alpha_1^2\alpha_3 + 3\alpha_1\alpha_2^2 - 6\alpha_1\alpha_2 - 4\alpha_1\alpha_3 + 3\alpha_2^2)
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
C_2 = &\,2q^3(12\alpha_1^4 - 18\alpha_1^3 - 12\alpha_1^2\alpha_2 + 6\alpha_1^2 + 6\alpha_1\alpha_2 + 2\alpha_1\alpha_3 + 3\alpha_2 \\
&+ \alpha_3) - q^2(24\alpha_1^5 + 12\alpha_1^4 - 24\alpha_1^3\alpha_2 - 48\alpha_1^3 - 48\alpha_1^2\alpha_2 \\
&+ 8\alpha_1^2\alpha_3 + 12\alpha_1^2 - 6\alpha_1\alpha_2^2 + 30\alpha_1\alpha_2 + 12\alpha_1\alpha_3 + 3\alpha_2^2 \\
&+ 18\alpha_2 + 6\alpha_3) + q(24\alpha_1^5 - 12\alpha_1^4 - 36\alpha_1^3\alpha_2 + 4\alpha_1^3\alpha_3 \\
&- 12\alpha_1^3 - 6\alpha_1^2\alpha_2^2 - 12\alpha_1^2\alpha_2 + 14\alpha_1^2\alpha_3 - 9\alpha_1\alpha_2^2 + 30\alpha_1\alpha_2 \\
&+ 12\alpha_1\alpha_3 - 3\alpha_2^2) + \alpha_1(12\alpha_1^2\alpha_2 - 4\alpha_1^2\alpha_3 + 6\alpha_1\alpha_2^2 \\
&- 12\alpha_1\alpha_2 - 6\alpha_1\alpha_3 + 3\alpha_2^2)
\end{aligned} \tag{68}
$$

$$\begin{aligned} C_3 = & -q^2(12\alpha_1^5 - 12\alpha_1^4 - 12\alpha_1^3\alpha_2 - 6\alpha_1^2\alpha_2 + 2\alpha_1^2\alpha_3 + 6\alpha_1\alpha_2 \\ & + 2\alpha_1\alpha_3 + 3\alpha_2^2 + 6\alpha_2 + 2\alpha_3) + q\alpha_1(12\alpha_1^4 - 12\alpha_1^3 \\ & - 18\alpha_1^2\alpha_2 + 2\alpha_1^2\alpha_3 - 3\alpha_1\alpha_2^2 + 4\alpha_1\alpha_3 + 12\alpha_2 + 4\alpha_3) \\ & + \alpha_1^2(6\alpha_1\alpha_2 - 2\alpha_1\alpha_3 + 3\alpha_2^2 - 6\alpha_2 - 2\alpha_3). \end{aligned} \tag{69}$$

These expressions were factored by the computer using the "divide if divisible" subroutine and a long list of trial divisors including all factors appearing in (59). The complete solution of (64) was obtained in three passes through the computer with some assistance from the author between passes. On the first pass the quantity

$$B_1\Phi_{01} + C_1$$

was calculated [using (61) for Φ_{01}] and an attempt was made to divide it by each of the factors of A_{13}. The division by the cubic factor was successful, and there followed the result

$$\Phi_{02} = \frac{2qF - G}{6(q - \alpha_1)^2} \tag{70}$$

where

$$\begin{aligned} F &= 6\alpha_1^3 - 6\alpha_1^2 - 6\alpha_1\alpha_2 + 3\alpha_2 + \alpha_3 \\ G &= 12\alpha_1^3 - 12\alpha_1^2 - 6\alpha_1\alpha_2 + 2\alpha_1\alpha_3 - 3\alpha_2^2. \end{aligned} \tag{71}$$

On the second pass the quantity

$$A_{23}\Phi_{02} + B_2\Phi_{01} + C_2$$

was calculated [using (70) for Φ_{02} and (61) for Φ_{01}] and an attempt was made to divide it by each of the factors of A_{22} and by the numerator of Φ_{02}. The latter division was successful, and the result

$$\Phi_{11} = \frac{[2qF - G][q^2 - q(\alpha_1 + 3) + \alpha_1]}{12(q - \alpha_1)^2[q^2 - q(\alpha_1 + 2) + \alpha_1]} \tag{72}$$

was thereby obtained. On the third pass the quantity

$$[q^2 - q(\alpha_1 + 2) + \alpha_1][A_{32}\Phi_{11} + A_{33}\Phi_{02} + B_3\Phi_{01} + C_3]$$

was calculated (the first factor being introduced in order to cancel the corresponding factor in the denominator of Φ_{11}) and an attempt was made to divide by each of the factors of A_{31}. Only the division by q was successful. This yielded the final result

$$\beta_2 = \Phi_{20} = \frac{(2qF - G)(q^2 - 2q)}{6(q - \alpha_1)^2[q^2 - q(\alpha_1 + 2) + \alpha_1]} \tag{73}$$

where F and G are the expressions defined in (71). Note that Φ_{02} appears as a factor in Φ_{11} and Φ_{20}. In the absence of feedback ($p = 0$, $q = 1$) (73) reduces to

$$\beta_2^{(0)} = \frac{\alpha_2}{1 - \alpha_1} + \frac{\alpha_2^2}{2(1 - \alpha_1)^2} + \frac{\alpha_3}{3(1 - \alpha_1)} \tag{74}$$

which is the correct result. To see this we use the law of composition to obtain

$$\beta_2^{(0)} = \alpha_2 + 2\alpha_1\omega_1 + \omega_2 \tag{75}$$

where α_1 and α_2 are the service moments while ω_1 and ω_2 are the waiting moments. The latter are given in Ref. 5 as

$$\begin{aligned} \omega_1 &= \frac{\alpha_2}{2(1 - \alpha_1)} \\ \omega_2 &= \frac{\alpha_2^2}{2(1 - \alpha_1)^2} + \frac{\alpha_3}{3(1 - \alpha_1)}. \end{aligned} \tag{76}$$

Substituting (76) into (75) we obtain (74) as expected. To obtain (36), substitute (71) into (73), replace each α_k by $\lambda^k\alpha_k$, and divide the resulting expression by λ^2.

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