

# King Fahd University of Petroleum & Minerals Computer Engineering Dept

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COE 241 – Data and Computer  
Communications

Term 141

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## Lecture Contents

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1. Fourier Analysis
  - a. Fourier Series Expansion
  - b. Fourier Transform
  - c. Ideal Low/band/high pass filters
2. Introduction to Z-Transform

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## Signals

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- **A signal is a function representing information**
  - **Voice signal** – microphone output
  - **Video signal** – camera output
  - **Etc.**
- **Types of Signals**
  - **Analog** – continuous-value continuous-time
  - **Discrete** – discrete-value continuous-time
    - **Digital** – predetermined discrete levels – much easier to reproduce at receiver with no errors
    - **Binary** – only two predetermined levels: e.g. 0 and 1

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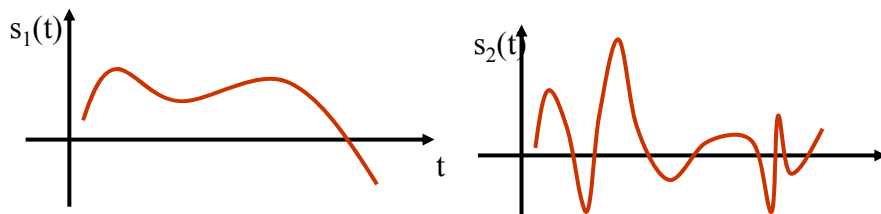
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## Example of Continuous-Value Continuous-time signal

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- **$s_1(t)$  and  $s_2(t)$  are two example of analog signals**



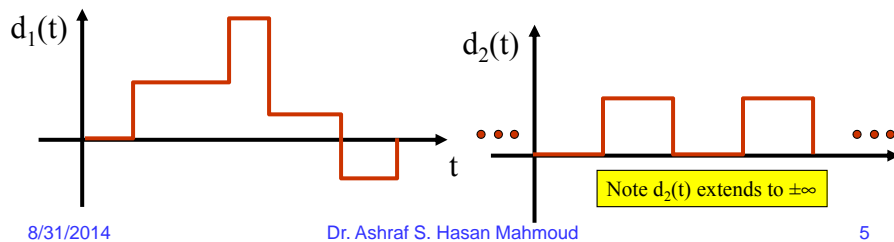
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## Example of Discrete-Value Continuous-time signal

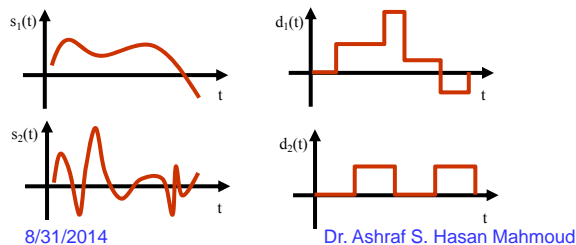
- $d_1(t)$  and  $d_2(t)$  are two example of discrete signals
  - $d_1(t)$  – takes more than two levels
  - $d_2(t)$  – takes only two levels - binary



Applies to BOTH analog and digital signals

## Time Domain Representation

- **Time domain representation** – we plot value (voltage, current, electric field intensity, etc.) versus time
  - Can infer rate of change (speed or frequency) information – e.g.  $s_2(t)$  seems faster than  $s_1(t)$ 
    - Using calculus terms: *rate of change* for  $s_2(t) >$  rate of change for  $s_1(t)$



Applies to BOTH analog and digital signals

## Frequency - Bandwidth

- **$s_2(t)$  faster than  $s_1(t)$  →**
  - **$s_2(t)$  contains higher frequencies than those contained in  $s_1(t)$**
- **$s_1(t)$  and  $s_2(t)$  contain more than one frequency**
  - **Minimum frequency =  $f_{\min}$**
  - **Maximum frequency =  $f_{\max}$**
- **Bandwidth = Range of frequencies contained in signal**  
**=  $f_{\max} - f_{\min}$**

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Applies to BOTH analog and digital signals

## Frequency - Bandwidth (2)

- **For our example signals, assume:**
  - **$S_1(t)$ :  $f_{\min} = 10$  Hz,  $f_{\max} = 500$  Hz**
  - **$S_2(t)$ :  $f_{\min} = 5$  Hz,  $f_{\max} = 1000$  Hz**
- **This means:**
  - **BW for  $s_1(t) = 500 - 10 = 490$  Hz**
  - **BW for  $s_2(t) = 1000 - 5 = 995$  Hz**
- **Note that: because  $s_2(t)$  is "faster than"  $s_1(t)$  it should contain frequencies higher than those in  $s_1(t)$** 
  - **E.g.  $s_2(t)$  contains frequencies (500,1000] which do not exist in  $s_1(t)$**

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Applies to BOTH analog  
and digital signals

## Frequency – Bandwidth (3)

- Consider the discrete signals  $d_1(t)$  and  $d_2(t)$
- The function plots have points of infinite slope
  - rate of change =  $\infty \rightarrow$  frequency =  $\infty$
- Therefore for signals that look like  $d_1(t)$  and  $d_2(t)$ ,  $f_{\max} = \infty$
- Furthermore,  $BW = \infty$
- Example:
  - $d_2(t)$  contains frequencies from some minimum  $f_{\min}$  Hz to  $f_{\max} = \infty$  Hz

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## Example of Signal BW

- Consider the human speech
- Typically  $f_{\min} \sim 100\text{Hz}$
- $f_{\max} \sim 3500\text{ Hz}$
- BW of the human speech signal = 3100 Hz

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## Bandwidth for Systems

- **For a system to respond (amplify, process, Tx, Rx, etc.) for a particular signal with all its details, the system should have an equal or greater bandwidth compared to that of the signal**
- **Example:**
  - **The system required to process  $s_2(t)$  should have a greater bandwidth than the system required to process  $s_1(t)$**

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## Bandwidth for Systems (2)

- **Example 2: consider the human ear system**
  - Responds to a range of frequencies only
  - $f_{\min} = 20 \text{ Hz}$   $f_{\max} = 20,000 \text{ Hz} \rightarrow \text{BW} = 19,980 \text{ Hz}$
  - It does not respond to sounds with frequencies outside this range
- **Example 3: consider the copper wire**
  - It passes (electric) signals only between a certain  $f_{\min}$  and a certain  $f_{\max}$
  - The higher the quality of the wire – the wider the BW
- **More on Systems BW later!**

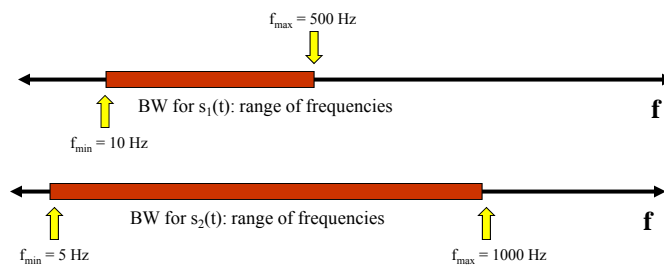
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## Frequency Representation

- **How to represent signals and indicate their frequency content?**
- **The X-axis: frequency (in Hertz or Hz)**
- **What is the Y-axis then? – the answer will be postponed!**



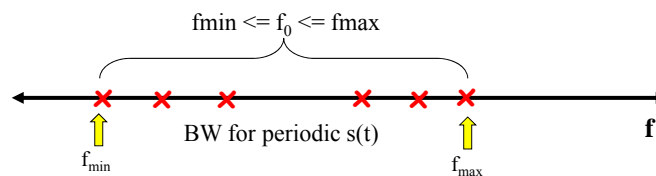
## Periodic Signals

- **A periodic signal repeats itself every T seconds**
  - **Period  $\rightarrow$  T seconds**
- **In calculus terms:**
  - **$s(t)$  is periodic if  $s(t) = s(t+T)$  for any  $-\infty < t < \infty$**
- **For previous examples:  $s_1(t)$ ,  $s_2(t)$ , and  $d_1(t)$  are not periodic – however,  $d_2(t)$  is periodic**

Applies to BOTH analog and digital signals

## Periodic Signals (2)

- A periodic signal has a **FUNDAMENTAL FREQUENCY** –  $f_0$ 
  - $f_0 = 1 / T$  – where  $T$  is the period
- A periodic signal may also have frequencies other than the fundamental frequency  $f_0$



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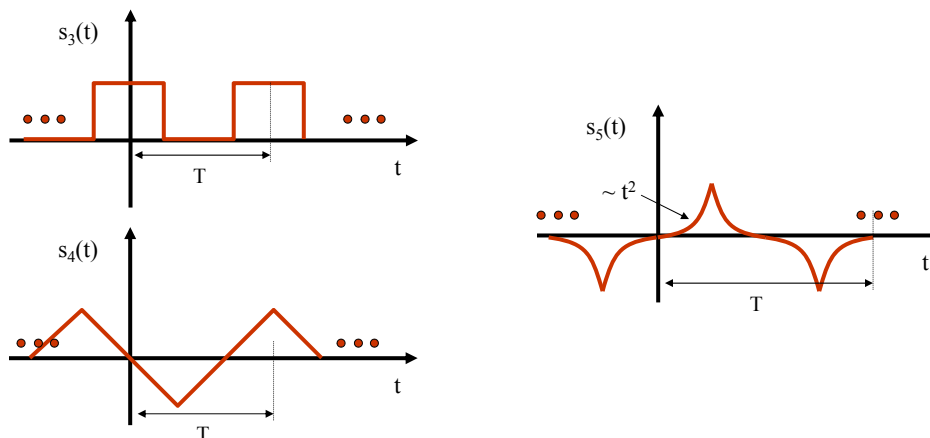
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Applies to BOTH analog and digital signals

## Periodic Signals (3)

- **Examples of other periodic signals:**



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Applies to BOTH analog and digital signals

## Energy/Power of Signals

- **Energy for any signal is defined as**

$$E_s = \int |s(t)|^2 dt$$

where the integral is carried over ALL range of t

- **In other words,  $E_s$  is the area under the absolute squared of the signal**
- **The unit of energy is Joules**

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Applies to BOTH analog and digital signals

## Energy/Power of Signals (2)

- **Note that for periodic signal  $E_s$  is equal to infinity since it is defined on  $(-\infty, \infty)$** 
  - **However power is FINITE for these type of signals**

- **Power is defined as the average of the absolute squared of the signal, i.e.**

$$P_s = \frac{1}{T} \int_0^T |s(t)|^2 dt$$

Note the integral can be performed on  $[0, T]$ ,  $[-T/2, T/2]$ , or any continuous interval of length T

- **The unit of power is Joules/sec or Watt**

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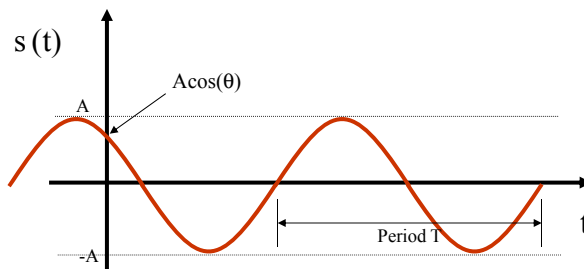
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## A VERY SPECIAL Analog Signal

- **A function of the form**

$$s(t) = A \cos(2\pi ft + \theta)$$



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## Characteristics of COSINE

- **Completely specified by:**
  - **Amplitude – A**
  - **Phase -  $\theta$**
  - **Frequency – f**
- **$s(t = 0) = A \cos(\theta)$**
- **Periodic signal – repeats itself every T seconds**
  - **$T = 1 / f$**
- **Time to review your trigonometry !!**
  - **E.g.  $\sin(x) = \cos(x - \pi/2)$**

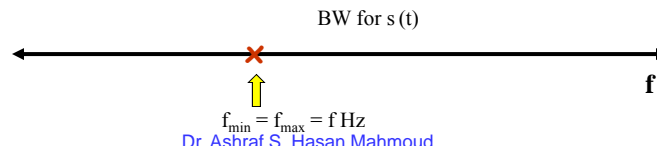
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## Characteristics of COSINE (2)

- **Energy for this signal,  $E_s = \text{infinity}$**
- **Power for this signal,  $P_g = A^2/2$** 
  - Note  $P_g$  is dependent only on the amplitude  $A$
  - **Exercise: Verify the above results using the power formula**
- **It contains ONLY ONE frequency  $f$** 
  - The "purest" form of analog signals
- **Frequency representation:**



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## Characteristics of COSINE (3)

- **Very Useful Properties ( $f = 1/T$ )**

$$\int_0^T \cos(2\pi ft + \theta) dt = 0$$

$$\frac{1}{T} \int_0^T \cos^2(2\pi ft + \theta) dt = 1/2$$

$$\int_0^T \cos(2\pi mft + \theta) dt = 0$$

$$\frac{1}{T} \int_0^T \cos^2(2\pi mft + \theta) dt = 1/2$$

$$\frac{1}{T} \int_0^T \cos(2\pi nft) \cos(2\pi mft) dt = \begin{cases} 0 & n \neq m \\ 1/2 & n = m \end{cases}$$

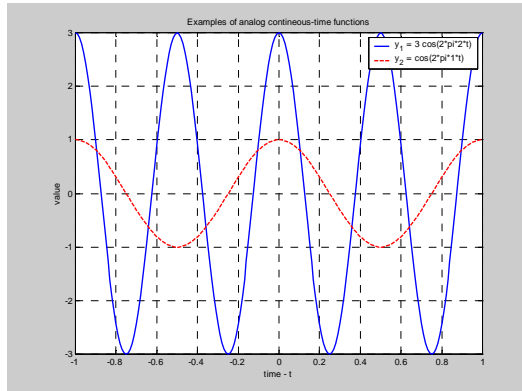
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## Example of Cosine Functions

- $Y_1(t)$  – has
  - a frequency  $f$  of 2 Hz ( $T = 1/2$  sec)
  - An amplitude of 3
  - $P_{Y1} = 3^2/2 = 4.5$  Watts
- $Y_2(t)$  - has
  - a frequency  $f$  of 1 Hz ( $T = 1/1 = 1$  sec)
  - An amplitude of 1
  - $P_{Y2} = 1^2/2 = 0.5$  Watts



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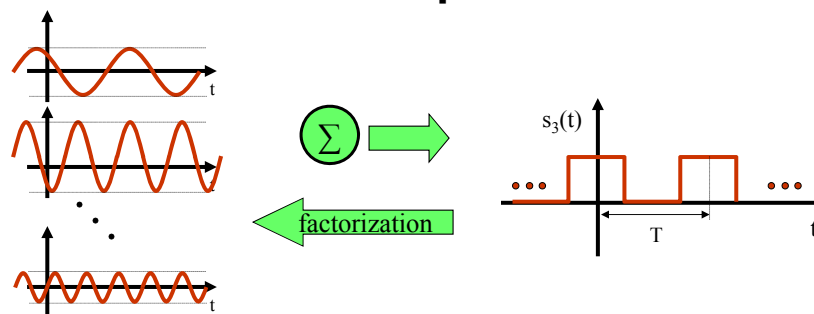
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ONLY FOR PERIODIC SIGNALS

## Fourier Series Expansion

- Can we use the basic cosine functions to represent periodic signals?
- YES – Fourier Series Expansion



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## Fourier Series Expansion (2)

- For a periodic signal  $s(t)$  can be represented as a sum of sinusoidal signals as in

$$s(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)]$$

where the coefficients are computed using:

$$A_0 = \frac{2}{T} \int_0^T s(t) dt$$

$f_0$  is the fundamental frequency of  $s(t)$  and is equal to  $1/T$

$$A_n = \frac{2}{T} \int_0^T s(t) \cos(2\pi n f_0 t) dt \quad B_n = \frac{2}{T} \int_0^T s(t) \sin(2\pi n f_0 t) dt$$

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## Fourier Series Expansion (3)

- Another form for the series:

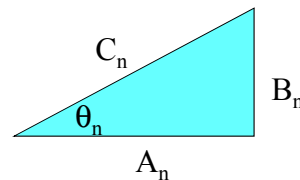
$$s(t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(2\pi n f_0 t + \theta_n)$$

where the coefficients are computed using:

$$C_0 = A_0$$

$$C_n = \sqrt{A_n^2 + B_n^2}$$

$$\theta_n = \tan^{-1} \left( \frac{-B_n}{A_n} \right)$$



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## Notes on Fourier Series Expansion

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- **The representation (the sum of sinusoids) is completely identical and equivalent to the original specification of  $s(t)$**
- **It applies to any periodic signal – analog or digital!**

**Very powerful tool - it reveals all frequencies contained in the original periodic signal  $s(t)$**

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## Notes on Fourier Series Expansion (2)

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- **In general,  $s(t)$  contains**
  - **DC term – the zero frequency term =  $A_0/2$**
  - **A (possibly infinite) number of harmonics (or sinusoids) at multiples of the fundamental frequency,  $f_0$**
- **The contribution of a harmonic with frequency  $nf_0$  is proportional to  $|A_n^2 + B_n^2|$  or  $C_n^2$** 
  - **E.g. if  $C_n^2 \sim 0$ , then we say the harmonic at  $nf_0$  (or higher) does not contribute significantly towards building  $s(t)$  – more on this when we discuss total power!**

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## Notes on Fourier Series Expansion (3)

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- **A harmonic with frequency equal to  $nf_0$  ( $n > 0$ ), has a period of  $1/(nT)$**
- **In general the series expansion of  $s(t)$  contains INFINITE number of terms (harmonics)**
- **However for less than 100% accurate representation one can ignore higher terms – terms with frequencies greater than certain  $n \cdot f_0$**

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## Notes on Fourier Series Expansion (4)

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- **Lets define the following function:**

$$s\_e(n=k)$$

**To be the series expansion of  $s(t)$  up to and including the  $n = k$  term**

**It should be noted that  $s\_e(n=k)$  is periodic with period  $T$**

- **Examples:**

$$s\_e(n=0) = A_0 / 2$$

$$\begin{aligned} s\_e(n=1) &= A_0 / 2 + A_1 \cos(2\pi f_0 t) + B_1 \sin(2\pi f_0 t) \\ &= A_0 / 2 + C_1 \cos(2\pi f_0 t + \theta_1) \end{aligned}$$

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## Notes on Fourier Series Expansion (5)

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- **Examples – cont'd:**

$$\begin{aligned} s_{-e}(n=2) &= A_0/2 + A_1 \cos(2\pi f_0 t) + B_1 \sin(2\pi f_0 t) \\ &\quad + A_2 \cos(2\pi \times 2 f_0 t) + B_2 \sin(2\pi \times 2 f_0 t) \\ &= A_0/2 + C_1 \cos(2\pi f_0 t + \theta_1) + C_2 \cos(2\pi \times 2 f_0 t + \theta_2) \end{aligned}$$

•••

$$\begin{aligned} s_{-e}(n=\infty) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)] \\ &= A_0/2 + \sum_{n=1}^{\infty} C_n \cos(2\pi n f_0 t + \theta_n) \end{aligned}$$

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## Notes on Fourier Series Expansion (6)

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- **It is obvious that  $s(t)$  is 100% represented by  $s_{-e}(n=\infty)$**
- **$s_{-e}(n = n^* < \infty)$  produces a less than 100% accurate representation of the original  $s(t)$**
- **For most practical periodic signals  $s_{-e}(n=10)$  provides a more than enough accuracy in representing  $s(t)$** 
  - **No need for infinite number of terms**

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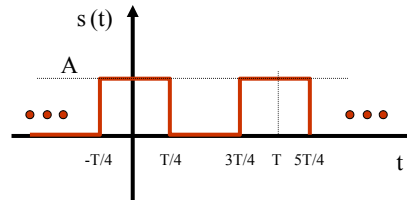


## Example 1:

- Consider the following  $s(t)$

- Over one period, the signal is defined as

$$s(t) = \begin{cases} A & -T/4 < t \leq T/4 \\ 0 & T/4 < t \leq 3T/4 \end{cases}$$



- Finding the Series Expansion:

- The DC term  $A_0$

$$A_0 = \frac{2}{T} \int_{-T/4}^{T/4} s(t) dt = \frac{2}{T} \times \frac{T}{2} \times A = A$$

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## Example 1: cont'd

- The term  $A_n$ :

$$A_n = \frac{2}{T} \int_{-T/4}^{T/4} s(t) \cos(2\pi f_0 t) dt = \frac{2A}{T} \int_{-T/4}^{T/4} \cos(2\pi f_0 t) dt$$

$$= \frac{2A}{2\pi f_0 T} \sin(2\pi f_0 t) \Big|_{t=-T/4}^{t=T/4} = \frac{A}{\pi} \times 2 \times \sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0 & n = 2, 4, 6, \dots \\ \frac{2A}{\pi} & n = 1, 5, 9, \dots \\ -\frac{2A}{\pi} & n = 3, 7, 11, \dots \end{cases}$$

### Remember

- $f_0 = 1/T$
- $\int \cos(ax) dx = -1/a \sin(ax)$
- $\sin(n\pi) = 0$  for integer  $n$
- $\sin(n\pi/2) = 1$  for  $n=1, 5, 9, \dots$
- $\sin(n\pi/2) = -1$  for  $n=3, 7, 11, \dots$

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## Example 1: cont'd

- Therefore  $A_n$  is given by:

$$= \begin{cases} 0 & n = 2, 4, 6, \dots \\ (-1)^{(n-1)/2} \times \frac{2A}{\pi n} & n = 1, 3, 5, 7, \dots \end{cases}$$

### Remember

$$\begin{aligned} (-1)^{(n-1)/2} &= 1 \text{ for } n = 1, 5, 9, \dots \\ &= -1 \text{ for } n = 3, 7, 11, \dots \end{aligned}$$

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## Example 1: cont'd

- The term  $B_n$ :

$$\begin{aligned} B_n &= \frac{2}{T} \int_{-T/4}^{T/4} s(t) \sin(2\pi n f_0 t) dt = \frac{2A}{T} \int_{-T/4}^{T/4} \sin(2\pi n f_0 t) dt \\ &= \frac{-2A}{2\pi n f_0 T} \cos(2\pi n f_0 t) \Big|_{t=-T/4}^{t=T/4} = \frac{-2A}{\pi n} \times \left\{ \cos\left(\frac{n\pi}{2}\right) - \cos\left(-\frac{n\pi}{2}\right) \right\} \\ &= 0 \end{aligned}$$

### Remember

1.  $\int \cos(ax) dx = -1/a \sin(ax)$
2.  $\cos(x) = \cos(-x)$

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## Example 1: cont'd

- Therefore, the overall series expansion is given by

$$s(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \times \cos(2\pi n f_0 t)$$

$$s(t) = \frac{A}{2} + \frac{2A}{\pi} \times \cos(2\pi f_0 t) - \frac{2A}{3\pi} \cos(2\pi \times 3 f_0 t) + \frac{2A}{5\pi} \times \cos(2\pi \times 5 f_0 t) - \frac{2A}{7\pi} \cos(2\pi \times 7 f_0 t) + \dots$$

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## Example 1: cont'd

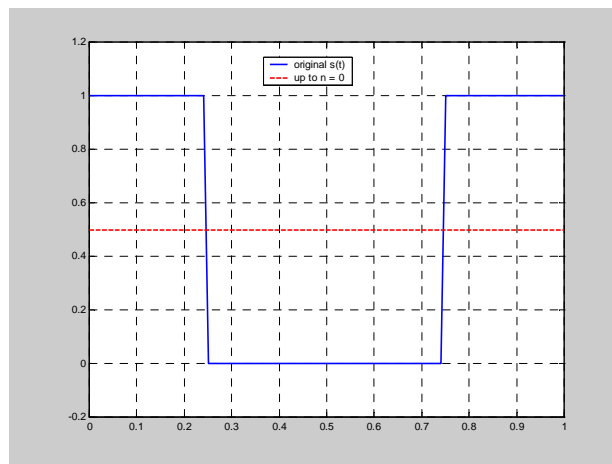
- Original  $s(t)$  and the series up to and including  $n = 0$

- i.e. Comparing:

$s(t)$

vs.

$s_e(n=0) = A/2$



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## Example 1: cont'd

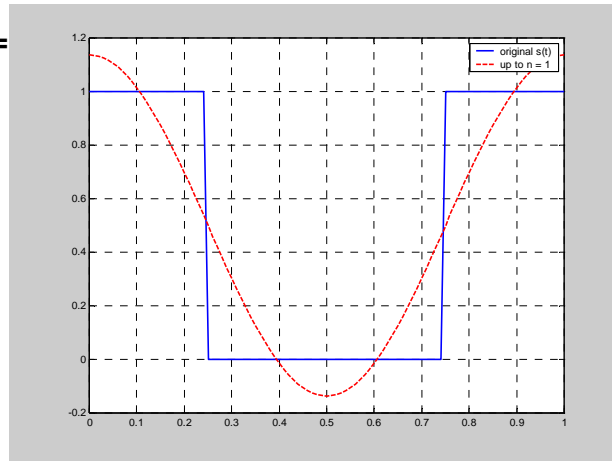
- Original  $s(t)$  and the series up to and including  $n = 1$

- i.e. Comparing:

$s(t)$

vs.

$$s_e(n=1) = \frac{A}{2} + \frac{2A}{\pi} \cos(2\pi f_0 t)$$



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## Example 1: cont'd

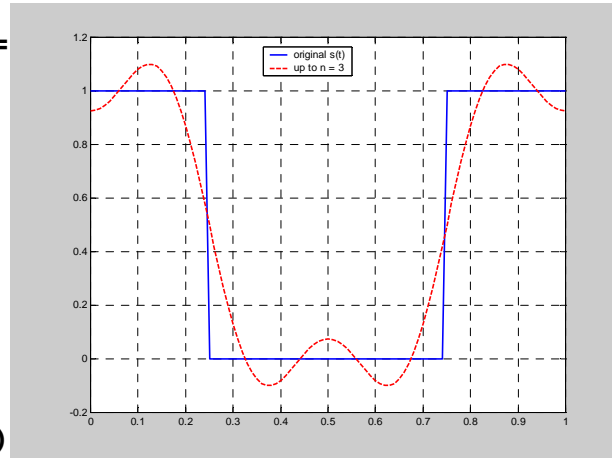
- Original  $s(t)$  and the series up to and including  $n = 3$

- i.e. Comparing:

$s(t)$

vs.

$$s_e(n=3) = \frac{A}{2} + \frac{2A}{\pi} \cos(2\pi f_0 t) - \frac{2A}{(3\pi)} \cos(2\pi 3f_0 t)$$



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## Example 1: cont'd

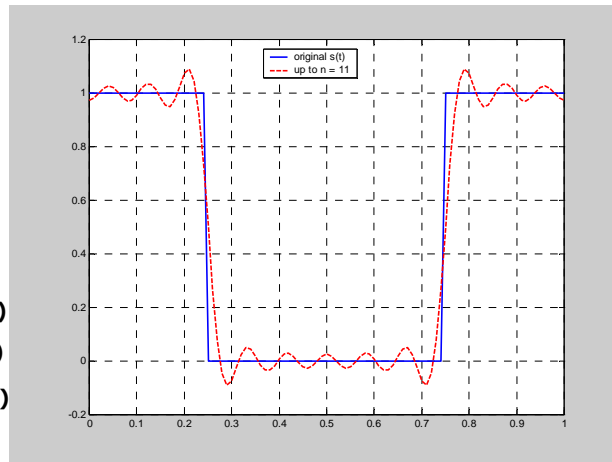
- Original  $s(t)$  and the series up to and including  $n = 11$

- i.e. Comparing:

$s(t)$

vs.

$$s_e(n=11) = \frac{A}{2} + \frac{2A}{\pi} \cos(2\pi f_0 t) - \frac{2A}{3\pi} \cos(2\pi 3f_0 t) + \frac{2A}{5\pi} \cos(2\pi 5f_0 t) - \frac{2A}{7\pi} \cos(2\pi 7f_0 t) + \frac{2A}{11\pi} \cos(2\pi 11f_0 t)$$



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## Example: cont'd

```
clear all

T = 1;
A = 1;
t = -1:0.01:1;
n_max = 11;

s = (A*square(2*pi/T*(t+T/4))+A)/2;

figure(1)
plot(t, s);
grid
axis([0 1 -0.2 1.2]);

s_e = A/2*ones(size(t));

for n=1:2:n_max
    s_e = s_e + (-1)^((n-1)/2) * 2*A/(n*pi) * cos(2*pi*n/T*t);
end

figure(2)
plot(t, s, 'b-', t, s_e, 'r--');
axis([0 1 -0.2 1.2]);
legend('original s(t)', 'up to n = 11');
grid
```

- The matlab code for plotting and evaluating the Fourier Series Expansion
- This code builds the series incrementally using the “for” loop
- Make sure you study this code!!

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## Notes Previous Example

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- **The more terms included in the series expansion → the closer the representation to the original  $s(t)$**
- i.e. comparing  $s(t)$  with  $s_e(n=n^*)$ , the greater the  $n^*$  the closer the representation is
- **How to measure “closeness”?**
- **Answer: Let’s use power!!**

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## Power Calculation Using Fourier Series Expansion

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- **Rule: if  $s(t)$  is represented using Fourier Series expansion, then its power can be calculated using:**

$$\begin{aligned} P_s &= \frac{1}{T} \int_0^T |s(t)|^2 dt = \frac{A_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} [A_n^2 + B_n^2] \\ &= \frac{A_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2 \end{aligned}$$

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## Power Calculation Using Fourier Series Expansion (2)

- **The previous result is based on the following two facts:**
  - (1) For  $f(t) = \text{constant}$   
→ power of  $f(t) = \text{constant}^2$

**Proof:**

$$\begin{aligned}\text{power} &= 1/T \times \int_0^T \text{constant}^2 dt \\ &= 1/T \times \text{constant}^2 \times T \\ &= \text{constant}^2 \text{ Watts}\end{aligned}$$

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## Power Calculation Using Fourier Series Expansion (3)

- **The previous result is based on the following facts (continued):**
  - (2) For  $f(t) = A \cos(2\pi n f_0 t + \theta)$   
→ power of  $f(t) = A^2/2$

**Proof:**

$$\begin{aligned}P_f &= \frac{1}{T} \int_0^T |f(t)|^2 dt = \frac{A^2}{T} \int_0^T \cos^2(2\pi n f_0 t + \theta) dt \\ &= \frac{A^2}{T} \int_0^T \left[ \frac{1}{2} + \frac{1}{2} \cos(4\pi n f_0 t + 2\theta) \right] dt \\ &= \frac{A^2}{T} \left[ \frac{T}{2} + 0 \right] = \frac{A^2}{2}\end{aligned}$$

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## Example 2:

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- **Problem: What is the power of the signal  $s(t)$  used in previous example? And find  $n^*$  such that the power contained in  $s_e(n=n^*)$  is 95% of that existing in  $s(t)$ ?**

- **Solution:**

Let the power of  $s(t)$  be given by  $P_s$

$$P_s = \frac{1}{T} \int_0^T |s(t)|^2 dt = \frac{1}{T} \times A^2 \times \frac{T}{2} = \frac{A^2}{2} = 0.5A^2$$

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## Example 2: cont'd

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- **Now it is desired to compute the power using the Fourier Series Expansion**
- **What is the power in  $s_e(n=0) = A/2$ ?**
- **Ans: we apply the power formula:**

$$\begin{aligned} P_{s_e(n=0)} &= \frac{1}{T} \int_0^T |s_e(n=0)|^2 dt \\ &= \frac{1}{T} \times \frac{A^2}{4} \times T = \frac{A^2}{4} = 0.25A^2 \end{aligned}$$

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## Example 2: cont'd

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- **What is the power in**  
 $s_{-e}(n=1) = A/2 + 2A/\pi \cos(2\pi f_0 t)$
- **Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:**

$$P_{s_{-e}(n=1)} = \frac{1}{T} \int_0^T |s_{-e}(n=1)|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2}$$
$$= \left( \frac{1}{4} + \frac{2}{\pi^2} \right) A^2 = 0.4526A^2$$

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## Example 2: cont'd

---

- **What is the power in**  
 $s_{-e}(n=3) = A/2 + 2A/\pi \cos(2\pi f_0 t) - 2A/(3\pi) \cos(2\pi 3f_0 t)$
- **Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:**

$$P_{s_{-e}(n=3)} = \frac{1}{T} \int_0^T |s_{-e}(n=3)|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2} + \frac{2A^2}{9\pi^2}$$
$$= \left( \frac{1}{4} + \frac{2}{\pi^2} + \frac{2}{9\pi^2} \right) A^2 = 0.4752A^2$$

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## Example 2: cont'd

- **What is the power in**

$$s_{-e}(n=5) = A/2 + 2A/\pi \cos(2\pi f_0 t) - \\ 2A/(3\pi) \cos(2\pi 3f_0 t) + \\ 2A/(5\pi) \cos(2\pi 5f_0 t)$$

- **Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:**

$$P_{s_{-e}(n=5)} = \frac{1}{T} \int_0^T |s_{-e}(n=5)|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2} + \frac{2A^2}{9\pi^2} + \frac{2A^2}{25\pi^2} \\ = \left( \frac{1}{4} + \frac{2}{\pi^2} + \frac{2}{9\pi^2} + \frac{2}{25\pi^2} \right) A^2 = 0.4833A^2$$

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## Example 2: cont'd

- **What is the power in**

$$s_{-e}(n=\infty) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \times \cos(2\pi n f_0 t)$$

- **Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:**

$$P_{s_{-e}(n=\infty)} = \frac{1}{T} \int_0^T |s_{-e}(n=\infty)|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \\ = \left( \frac{1}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right) A^2 = 0.5A^2$$

This the EXACT SAME power contained in  $s(t)$  -  
This is expected since  $s(t)$  is 100% represented by  $s_{-e}(n=\infty)$

## Example 2: cont'd

$s_e(n=k)$	Expression	Power	% Power <sup>+</sup>
$k = 0$	$A/2$	$0.25 A^2$	$(0.25A^2)/(0.5A^2) = 50\%$
$k = 1$	$A/2 + 2A/\pi\cos(2\pi f_0 t)$	$0.4526 A^2$	$(0.4526A^2)/(0.5A^2) = 90.5\%$
$k = 2^*$	$A/2 + 2A/\pi\cos(2\pi f_0 t)$	$0.4526 A^2$	90.5%
$k = 3$	$A/2 + 2A/\pi\cos(2\pi f_0 t) - 2A/(3\pi)\cos(2\pi 3f_0 t)$	$0.4752 A^2$	95.0%
$k = 5$	$A/2 + 2A/\pi\cos(2\pi f_0 t) - 2A/(3\pi)\cos(2\pi 3f_0 t) + 2A/(5\pi)\cos(2\pi 5f_0 t)$	$0.4833 A^2$	96.7%

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<sup>+</sup> % power = power of  $s_e(n=k)$  relative to original power in  $s(t)$  which is equal to  $0.5A^2$   
<sup>\*</sup> For  $k = 2$ , the expression  $s_e(n=k)$  is the same as that for  $s_e(k=1)$ . Why?

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## Example 2: cont'd

- Therefore,  $s_e(n=n^*)$  such that 95% of power is contained  $\rightarrow n^* = 3$

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## Power Spectral Density Function

- **Fourier Series Expansion:**
  - Specifies all the basic harmonics contained in the original function  $s(t)$
  - $C_n^2/2 = (A_n^2+B_n^2) / 2$  determines the power contribution of the  $n$ th harmonic with frequency  $nf_0$
- The power Spectral Density function is a function specifying: how much power is contributed by a given frequency

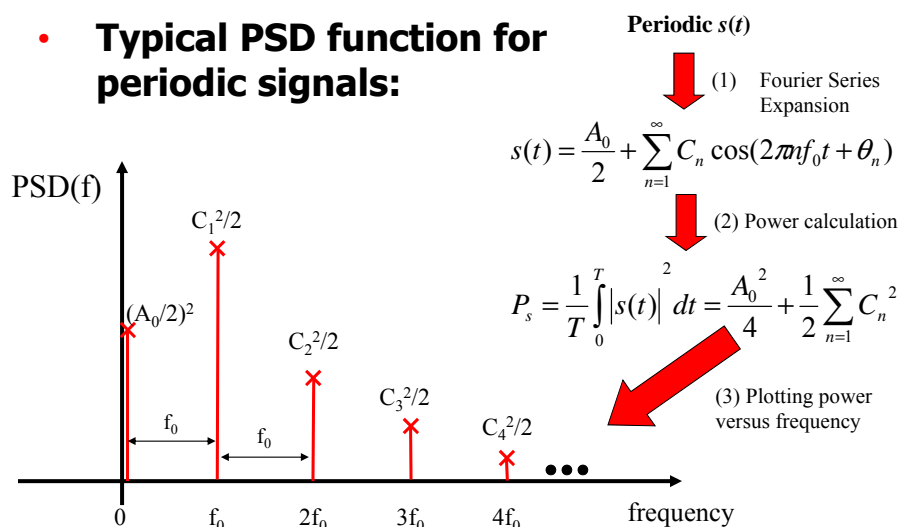
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## Power Spectral Density Function (2)

- **Typical PSD function for periodic signals:**



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## Power Spectral Density Function (3)

- **A mathematical expression for PSD(f) can be written as**

$$PSD(f) = \begin{cases} A_0^2/4 & f = 0 \\ C_n^2/2 & f = n \times f_0 \\ 0 & \text{otherwise} \end{cases}$$

- **Another way (more compact) of writing PSD(f) is as follows:**

$$PSD(f) = \frac{A_0^2}{4} \times \delta(f) + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2 \times \delta(f - n f_0)$$

where  $\delta(f)$  is defined by

$$\delta(f) = \begin{cases} 1 & f = 0 \\ 0 & f \neq 0 \end{cases}$$

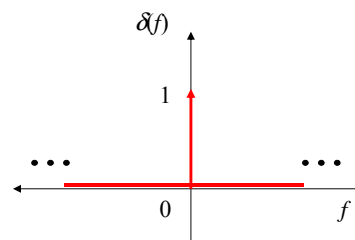
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## Power Spectral Density Function (4)

- **$\delta(f)$  is referred to as the dirac function or unit impulse function**



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## Note on the PSD Function

- **PSD function has units of Watts/Hz**
- **For periodic signals → PSD is a discrete function - defined for integer multiples of the fundamental frequency**
  - **Specifies the power contribution of every harmonic component  $C_n^2/2 \leftrightarrow nf_0$**
- **The separation between the discrete components is at least  $f_0$** 
  - **It is exactly  $f_0$  if all  $C_n$ 's are not zeros**
  - **E.g. for the previous  $s(t)$  example,  $C_n=0$  for even  $n \rightarrow$  separation =  $2f_0$**

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## Note on the PSD Function (2)

- **To calculate the total power of signal → Integrate PSD over all contained frequencies**
  - **For discrete PSD: integration = summation**
- **Therefore total power of  $s(t)$ ,**

$$P_s = (A_n/2)^2 + \sum C_n^2/2 \text{ in Watts}$$

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## Example 3:

- Find the PSD function of the periodic signal  $s(t)$  considered in Example 1.
- From Example 1,  $s(t)$  is given by

$$s(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \times \cos(2\pi n f_0 t)$$

- Using Example 2:
  - Power at the zero frequency =  $(A/2)^2 = A^2/4$
  - Power at the  $n$ th harmonic ( $n$  odd) is equal to  $2A^2/(n\pi)^2$
  - Power at the  $n$ th harmonic ( $n$  even) is zero
  - Therefore the PSD function is given by

$$PSD(f) = \frac{A^2}{4} \times \delta(f) + \frac{2A^2}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \times \delta(f - n f_0)$$

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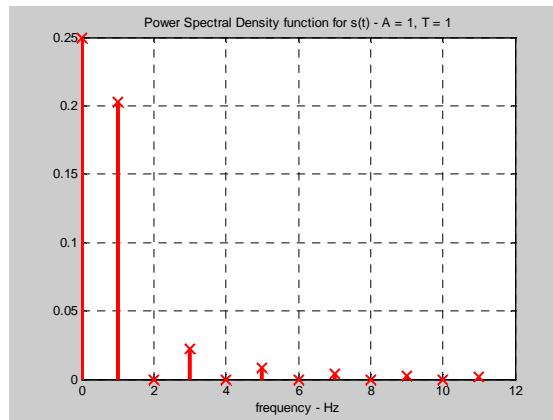
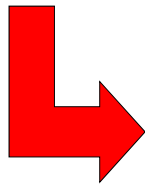
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## Example 3: cont'd

- The PSD is plotted as shown ( $A = 1, T = 1$ )

$$PSD(f) = \frac{A^2}{4} \times \delta(f) + \frac{2A^2}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \times \delta(f - n f_0)$$



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## Example 3: cont'd

- **Matlab Code to plot PSD**

```
clear all

T = 1;
A = 1;
t = -1:0.01:1;
n_max = 11;

Frequency = [0:1:n_max];
PwrSepctralD = zeros(size(Frequency));

% Record the DC term power at f = 0
PwrSepctralD(1) = (A/2)^2;

% Record the nth harmonic power at f = nf0
for n=1:2:n_max
    PwrSepctralD(n+1) = (2*A/(n*pi))^2 / 2;
end

figure(1)
stem(Frequency, PwrSepctralD,'rx');
title('Power Spectral Density function for s(t) - A = 1, T = 1');
xlabel('frequency - Hz');
grid
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```

The “stem” function is typically used to plot discrete functions

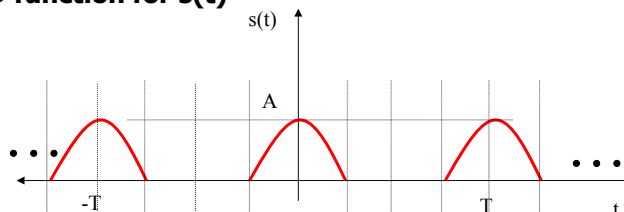
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## Example 4:

This is a typical exam question

- **Problem:** Consider the periodic half-wave rectified signal  $s(t)$  depicted in figure.
  - Write a mathematical expression for  $s(t)$
  - Calculate the Fourier Series Expansion for  $s(t)$
  - Calculate the total power for  $s(t)$
  - Find  $n^*$  such that  $s_e(n^*)$  has 95% of the total power
  - Determine the PSD function for  $s(t)$
  - Plot the PSD function for  $s(t)$



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## Example 4: cont'd

- **Answer:**

(a) To write a mathematical expression for  $s(t)$ , remember that the general form of a sinusoidal function is given by

$$A \cos(2\pi \times \text{Freq} \times t), \text{ or} \\ A \cos(2\pi / \text{Period} \times t)$$

Therefore  $s(t)$  is given by

$$s(t) = A \cos(2\pi/T t) \quad -T/4 < t \leq T/4 \\ = 0 \quad T/4 < t \leq 3T/4$$

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## Example 4: cont'd

- **Answer:**

(b) The F.S.E of  $s(t)$ :  $s(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)]$

The DC term is given by

$$A_0 = \frac{2}{T} \int_{-T/4}^{T/4} s(t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi t / T) dt \\ = \frac{A}{\pi} \times \sin(2\pi t / T) \Big|_{t=-T/4}^{t=T/4} = \frac{A}{\pi} [\sin(\pi/2) - \sin(-\pi/2)] \\ = \frac{2A}{\pi}$$

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## Example 4: cont'd

Remember:

$$\int [\cos(ax)\cos(bx)] dx = \frac{\sin(ax+bx)}{2(a+b)} + \frac{\sin(ax-bx)}{2(a-b)}$$

for  $a \neq b$   
 $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$

• **Answer:**

**The  $A_n$  term is given by (remember  $1/T = f_0$ )**

$$A_n = \frac{2}{T} \int_{-T/4}^{T/4} s(t) \cos(2\pi f_0 t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi t/T) \cos(2\pi f_0 t) dt$$

$$= \frac{2A}{T} \times \left[ \frac{\sin(2\pi(n+1)f_0 t)}{4\pi(n+1)f_0} + \frac{\sin(2\pi(n-1)f_0 t)}{4\pi(n-1)f_0} \right] \Bigg|_{t=-T/4}^{t=T/4} \quad \text{For } n \neq 1$$

$$= \frac{A}{\pi} \times \left[ \frac{\cos(n\pi/2)}{(n+1)} + \frac{-\cos(n\pi/2)}{(n-1)} \right] \quad \text{For } n \neq 1$$

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This means: the  $n=1$  should be special!

## Example 4: cont'd

**But**

$$\begin{aligned} \cos(n\pi/2) &= 0 & n &= \text{odd, } n \neq 1 \\ &= (-1)^{(1+n/2)} & n &= \text{even} \end{aligned}$$

**Therefore**

$$A_n = \frac{A}{\pi} \times \left[ \frac{(-1)^{(1+n/2)}}{(n+1)} + \frac{(-1)(-1)^{(1+n/2)}}{(n-1)} \right] \quad \text{For } n \text{ even}$$

$$= 0 \quad \text{For } n \text{ odd, } n \neq 1$$

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## Example 4: cont'd

The expression for  $A_n$  (for even  $n$ ) can be further simplified to

$$\begin{aligned}
 A_n &= \frac{A}{\pi} \times \left[ \frac{(-1)^{(1+n/2)}}{(n+1)} + \frac{(-1)(-1)^{(1+n/2)}}{(n-1)} \right] \\
 &= \frac{A}{\pi} \times \left[ \frac{(-1)^{(1+n/2)}(n-1) + (-1)(-1)^{(1+n/2)}(n+1)}{(n+1)(n-1)} \right] \\
 &= \frac{A}{\pi(n^2-1)} \times [(-1)^{(1+n/2)}(n-1) - (-1)^{(1+n/2)}(n+1)] \\
 &= \frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)}
 \end{aligned}$$

**For n even**

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## Example 4: cont'd

An is still not completely specified – we still need to calculate it for  $n=1$ ; in other words we need to calculate  $A_1$ :

$$A_{n=1} = \frac{2}{T} \int_{-T/4}^{T/4} s(t) \cos(2\pi \times 1 \times f_0 t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi / T) \cos(2\pi f_0 t) dt$$

Therefore:

$$\begin{aligned}
 A_1 &= \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos^2(2\pi f_0 t) dt \\
 &= \frac{2A}{T} \times \left[ \frac{t}{2} + \frac{1}{4 \times 2\pi f_0} \sin(4\pi f_0 t) \right] \Bigg|_{t=-T/4}^{t=T/4} = \frac{2A}{T} \times \left[ \frac{T}{4} + \frac{\sin(\pi) - \sin(-\pi)}{8\pi f_0} \right] \\
 &= \frac{A}{2}
 \end{aligned}$$

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## Example 4: cont'd

This mean  $A_n$  is equal to the following:

$$A_n = \begin{cases} 2A/\pi & n = 0 \\ 0 & n \text{ odd, } n \neq 1 \\ A/2 & n = 1 \\ \frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)} & n = 2, 4, 6, \dots \end{cases}$$

The above expression specifies  $A_n$  for ALL POSSIBLE values of  $n \rightarrow$  specification is complete

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## Example 4: cont

Remember:

$$\int [\sin(ax)\cos(bx)] dx = \frac{-\cos(ax+bx)}{2(a+b)} - \frac{\cos(ax-bx)}{2(a-b)} \text{ for } a \neq b$$

We still need to compute  $B_n$ :

$$\begin{aligned} B_n &= \frac{2}{T} \int_{-T/4}^{T/4} s(t) \sin(2\pi n f_0 t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi t / T) \sin(2\pi n f_0 t) dt \\ &= \frac{2A}{T} \times \left[ \frac{\cos(2\pi(n+1)f_0 t)}{4\pi(n+1)f_0} - \frac{\cos(2\pi(n-1)f_0 t)}{4\pi(n-1)f_0} \right] \Bigg|_{t=-T/4}^{t=T/4} \quad \text{For } n \neq 1 \\ &= \frac{A}{2\pi} \times \left[ \frac{-\cos(\pi/2(n+1)) + \cos(-\pi/2(n+1))}{(n+1)} - \frac{\cos(\pi/2(n-1)) - \cos(-\pi/2(n-1))}{(n-1)} \right] \\ &= 0 \quad \text{For } n \neq 1 \end{aligned}$$

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This means: the  $n=1$  should be special!

Remember:  
 $\sin(2ax) = 2 \cos(ax) \sin(ax)$

## Example 4: cont'd

**$B_n$  is still NOT completely specified – we still need to calculate it for  $n=1$ ; in other words we need to calculate  $B_1$ :**

$$B_{n=1} = \frac{2}{T} \int_{-T/4}^{T/4} s(t) \sin(2\pi \times 1 \times f_0 t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi / T) \sin(2\pi f_0 t) dt$$

**Therefore:**

$$\begin{aligned} B_1 &= \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi f_0 t) \sin(2\pi f_0 t) dt = \frac{A}{T} \times \int_{-T/4}^{T/4} \sin(4\pi f_0 t) dt \\ &= \frac{-A}{4\pi} \times \cos(4\pi f_0 t) \Big|_{t=-T/4}^{t=T/4} = \frac{-A}{4\pi} \times [\cos(\pi) - \cos(-\pi)] \\ &= 0 \end{aligned}$$

**→ This means  $B_n = 0$  for all  $n$**

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## Example 4: cont'd

- To summarize:

$$A_n = \begin{cases} 2A/\pi & n = 0 \\ 0 & n \text{ odd, } n \neq 1 \\ A/2 & n = 1 \\ \frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)} & n = 2, 4, 6, \dots \end{cases}$$

And

$$B_n = 0 \quad \text{for all } n$$

- Having computed  $A_n$  and  $B_n$  we are now in a position to write the Fourier Series Expansion for  $s(t)$

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## Example 4: cont'd

- The Fourier Series Expansion for  $s(t)$  is given by

$$s(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)]$$

$$= \frac{A}{\pi} + \frac{A}{2} \cos(2\pi f_0 t) + \frac{2A}{\pi} \sum_{n=2,4,6}^{\infty} \frac{(-1)^{(1+n/2)}}{n^2 - 1} \cos(2\pi n f_0 t)$$

The  $C_n$  terms (there is a typo in the textbook) are as follows:

$$C_0 = A/\pi$$

$$C_1 = A/2$$

$$C_n = \frac{2A(-1)^{(1+n/2)}}{\pi(n^2 - 1)}, \quad n = 2, 4, 6, \dots$$

$$0 \quad n \text{ odd, } n \neq 1$$

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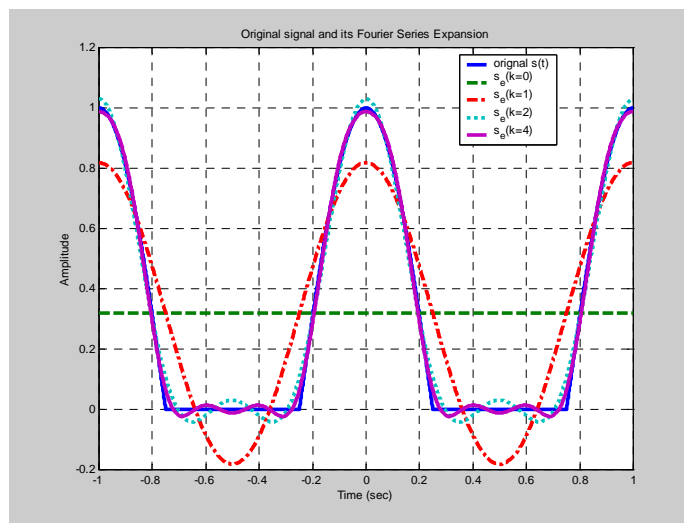
## Example 4: cont'd

Remember:

$$\sin(2ax) = \frac{1}{2} \cos(ax) \sin(ax)$$

Plot for  $s(t)$  and the Fourier Series Expansion for  $k=0, 1, 2,$  and  $4$

Note: As  $k$  increases  $s_e(n=k)$  approaches the original  $s(t)$



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## Example 4: cont'd

- The total power of  $s(t)$  is given by:

$$\begin{aligned}
 P_s &= \frac{1}{T} \int_{-T/4}^{3T/4} |s(t)|^2 dt = \frac{A^2}{T} \times \int_{-T/4}^{T/4} \cos^2(2\pi t / T) \\
 &= \frac{A^2}{T} \times \left[ \frac{t}{2} + \frac{\sin(4\pi t / T)}{8\pi / T} \right] \Bigg|_{t=-T/4}^{t=T/4} \\
 &= \frac{A^2}{4}
 \end{aligned}$$

Therefore total power of  $s(t) = 0.25 A^2$

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## Example 4: cont'd

- To find  $n^*$  such that power of  $s_e(n=n^*) = 95\%$  of total power:

$s_e(n=k)$	Expression	Power	% Power+
$k = 0$	$A/\pi$	$0.1013 A^2$	$(0.1013A^2)/(0.25 A^2) = 40.5\%$
$k = 1$	$A/\pi + A/2 \cos(2\pi f_0 t)$	$0.2263 A^2$	$(0.2262A^2)/(0.25A^2) = 90.5\%$
$k = 2$	$A/\pi + A/2 \cos(2\pi f_0 t) + 2A/(3\pi) \cos(2\pi 2f_0 t)$	$0.2488 A^2$	$(0.2488A^2)/(0.25A^2) = 99.5\%$

Therefore  $n^* = 2 \rightarrow$  power of  $s_e(n=2) = 0.2488 A^2$  which is 99.5% of total power of  $s(t)$

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## Example 4: cont'd

- The PSD function for  $s(t)$  is as follows:
  - Power for DC term =  $(A/\pi)^2$
  - Power for harmonic at  $f = f_0$ :  $(A/2)^2/2 = A^2/8$
  - Power for harmonic at  $f = nf_0$  ( $n=2,4,6, \dots$ ):  $[2A/(\pi(n^2-1))]^2/2 = 2A^2/(\pi(n^2-1))^2$

- Therefore PSD function equals to

$$PSD(f) = \left(\frac{A}{\pi}\right)^2 \delta(f) + \frac{A^2}{8} \delta(f - f_0) + \frac{2A^2}{\pi^2} \sum_{n=2,4,6}^{\infty} \frac{\delta(f - nf_0)}{(n^2 - 1)^2}$$

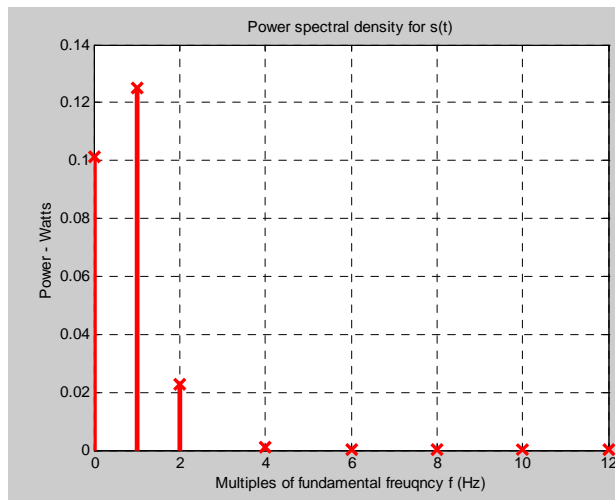
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## Example 4: cont'd

- Plot of The PSD function for  $s(t)$



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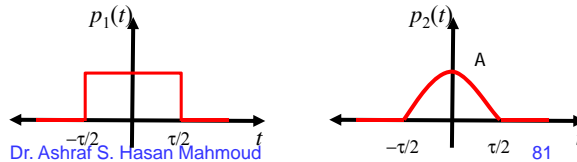
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## Fourier Transform

- **Fourier Series Expansion analysis is applicable for PERIODIC signals ONLY**
- **There are important signals that are not periodic such as**
  - **Your voice waveform**
  - **Pulse signal  $p(t)$  – used for modulation and transmission**
  - **Examples:  $p_1(t)$  and  $p_2(t)$**



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## Fourier Transform (2)

- **How to find the frequency content of such signals?**
- **Use FOURIER TRANSFORM**

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi jft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{2\pi jft} df$$

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## Notes on Fourier Transform

- **F.T describes a two-way transformation**

$$x(t) \quad \leftarrow \rightarrow \quad X(f)$$

where  $x(t)$  is the time representation of the signal, while  $X(f)$  is the frequency representation of the signal

- **$X(f)$  is defined on a continuous range of frequencies**
  - All frequencies within the range of  $X(f)$  where  $X(f)$  is not zero contribute towards building  $x(t)$

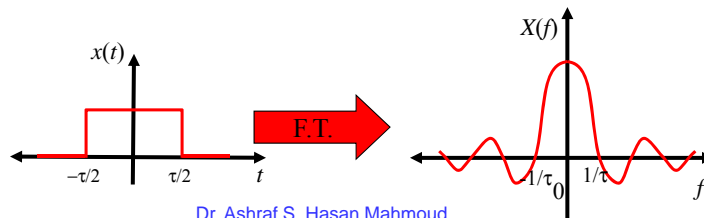
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## Notes on Fourier Transform (2)

- **The magnitude of the contribution of a particular frequency  $f^*$  in  $x(t)$  is proportional to  $|X(f^*)|^2$**
- **Example: Consider the F.T. pair shown below – clearly frequencies belonging to  $(-1/\tau, 1/\tau)$  contribute significantly more compared to frequencies belonging to  $(1/\tau, \infty)$  or  $(-\infty, -1/\tau)$**



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## Properties of Fourier Transform

- If  $x(t)$  is time-limited  $\rightarrow X(f)$  is not frequency-limited
  - i.e. the range of  $X(f) = (-\infty, \infty)$
- $X(f)$  is complex-valued (has magnitude and phase) in general, i.e.  $X(f) \in \mathbb{C}$
- If  $x(t)$  is a real-valued symmetric  $\rightarrow X(f)$  is real-valued, i.e.  $X(f) \in \mathbb{R}$

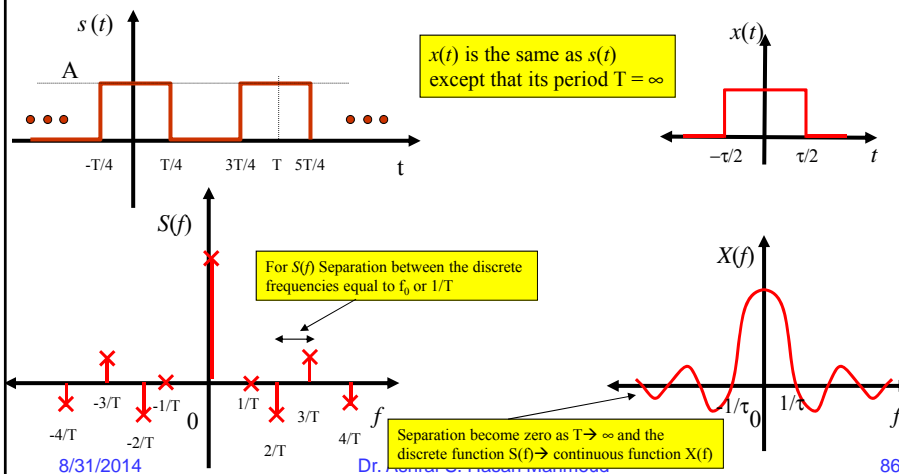
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## Relation between Fourier Series Expansion and Fourier Transform

- Consider the following two signals:



## Relation between Fourier Series Expansion and Fourier Transform (2)

- **The separation between spectral lines for a periodic signal is  $1/T$**
- **As  $T \rightarrow$  infinity and  $s(t)$  becomes non periodic  $\rightarrow$  the separation between spectral lines  $\rightarrow$  zero (i.e. it becomes continuous)**

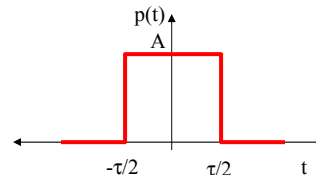
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## Example 5:

- **Problem: Consider the square pulse function shown in figure:**
  - **Write a mathematical expression for  $p(t)$**
  - **Find the Fourier transform for  $p(t)$**
  - **Plot  $P(f)$**



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## Example 5: cont'd

- **Answer:  $p(t)$  can be expressed as**

$$p(t) = A \quad |t| \leq \tau/2 \\ = 0 \quad \text{otherwise}$$

**The F.T. for  $p(t)$ ,  $P(f)$  is given by**

$$P(f) = \int_{-\infty}^{\infty} p(t)e^{-2\pi jft} dt$$

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**$P(f)$  is real-valued – for this specific example of  $p(t)$ ; why?**

## Example 5: cont'd

- **Which is equal to**

$$\begin{aligned} P(f) &= \int_{-\infty}^{\infty} p(t)e^{-2\pi jft} dt = \int_{-\tau/2}^{\tau/2} Ae^{-2\pi jft} dt \\ &= \frac{A}{-2\pi jf} \int_{-\tau/2}^{\tau/2} e^{-2\pi jft} dt = -\frac{A}{2\pi jf} \times (e^{-\pi jf\tau} - e^{\pi jf\tau}) \\ &= \frac{A}{\pi f} \times \frac{(e^{\pi jf\tau} - e^{-\pi jf\tau})}{2j} \\ &= A\tau \frac{\sin(\pi f\tau)}{\pi f\tau} \end{aligned}$$

Remember: Euler identity :  
 $e^{jx} = \cos(x) + j \sin(x)$ , OR  
 $\cos(x) = (e^{jx} + e^{-jx})/2$   
 $\sin(x) = (e^{jx} - e^{-jx})/(2j)$

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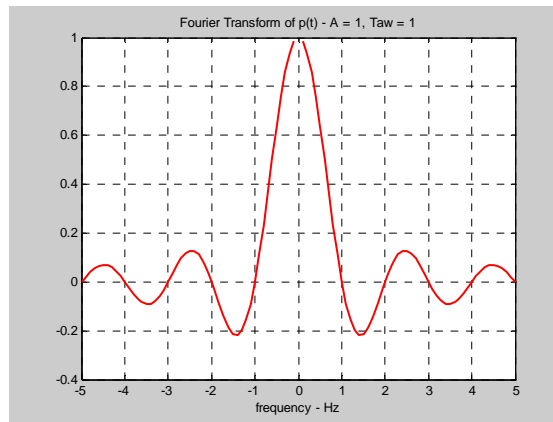
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## Example 5: cont'd

- **P(f) plot for A = 1 and  $\tau = 1$**

- **Note:**

- **P(f) is define on  $(-\infty, \infty)$**
- **P(f) is continuous (except at  $f = 0$  Hz)**
- **P(f) = ZERO for  $f = n/\tau$**
- **For practical pulses – P(f) approaches zero as  $f \rightarrow \pm\infty$**
- **Most of the energy of p(t) is contained in the period of  $(-1/\tau, 1/\tau)$**



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## Energy Spectral Density Function (ESDF)

- **ESDF is defined as**

$$ESD(f) = \frac{1}{2\pi} P(f)P^*(f)$$

where **P(f)** is the F.T. of the pulse **p(t)**. **P\*(f)** is the complex conjugate of **P(f)**.

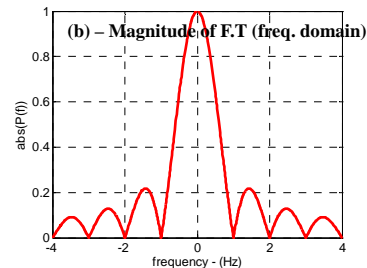
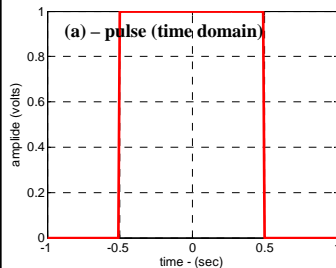
- **ESDF is a measure of how much energy is contained at a particular frequency f**
- **Units of ESDF is Joules per Hz**
- **How would you compare ESDF with PSDF?**

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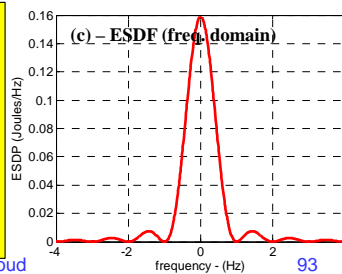
## Example 5 (For A = 1 Volts and Taw = 1 sec) - cont'd



$$p(t) = \begin{cases} A & |t| < \tau/2 \\ 0 & \text{otherwise} \end{cases} \quad \text{(a)}$$

$$P(f) = \int_{-\infty}^{\infty} p(t)e^{-2\pi jft} dt = A\tau \frac{\sin(\pi\tau f)}{\pi\tau f} \quad \text{(b)}$$

$$ESDF(f) = \frac{1}{2\pi} P(f)P^*(f) = \frac{1}{2\pi} \left| A\tau \frac{\sin(\pi\tau f)}{\pi\tau f} \right|^2 \quad \text{(c)}$$



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## Example 5 (For A = 1 Volts and Taw = 1 sec) - Matlab Code

- Code for producing plots on previous slide

```
clear all; LineWidth = 3; FontSize = 14;
%Example of rectangular pulse
A = 1; Taw = 1; % parameters for the rectangle pulse
t_step = 0.01; f_step = 0.01; Nmax = 4;
t = -Taw:t_step:Taw; % define the time axis
f = -Nmax/Taw:f_step:Nmax/Taw; % define the frequency axis

p_t = A*rectpuls(t/Taw); % The rectangle pulse of height A and width Taw
P_f = A*Taw*sin(pi*Taw*f)/(pi*Taw*f); % The corresponding F.T P(f)
ESDF_f = P_f.*conj(P_f)/(2*pi); % The ESDF

figure(1); clf; set(gca, 'FontSize', FontSize);
h = plot(t, p_t, '-r', 'LineWidth', LineWidth);
xlabel('time - (sec)'); ylabel('amplitude (volts)'); grid on;

figure(2); clf; set(gca, 'FontSize', FontSize);
h = plot(f, abs(P_f), '-r', 'LineWidth', LineWidth);
xlabel('frequency - (Hz)'); ylabel('abs(P(f))'); grid on;

figure(3); clf; set(gca, 'FontSize', FontSize);
h = plot(f, ESDF_f, '-r', 'LineWidth', LineWidth);
xlabel('frequency - (Hz)'); ylabel('ESDF (Joules/Hz)'); grid on;
```

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## Example 6:

- Repeat example 5 for triangular pulse shown in textbook Figure A.2 page 839. Let  $T$  be equal to 2 seconds and  $A$  be equal to 3 volts.

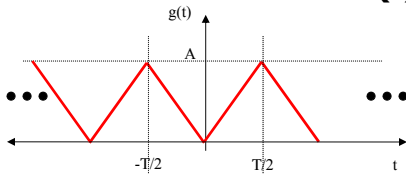
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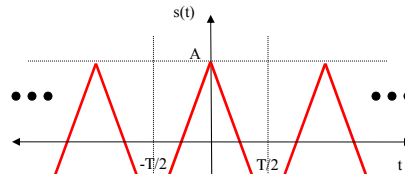
## Example 7:

- Problem:** If the FSE expansion for the function  $g(t)$  is as given below. Compute the FSE for the function  $s(t)$  without computing the FSE coefficients for  $s(t)$ .



$$g(t) = \begin{cases} \frac{-2A}{T}t & -\frac{T}{2} \leq t \leq 0 \\ \frac{2A}{T}t & 0 \leq t \leq \frac{T}{2} \end{cases}$$

$$g(t) = \frac{A}{2} + \sum_{n=1,3,5}^{\infty} \frac{(-4A)}{\pi^2 n^2} \cos\left(\frac{2\pi n}{T}t\right)$$



$$s(t) = \begin{cases} \frac{4A}{T}t + A & -\frac{T}{2} \leq t \leq 0 \\ \frac{-4A}{T}t + A & 0 \leq t \leq \frac{T}{2} \end{cases}$$

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## Z-Transform

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- **Digital signals**
  - Discrete-time signals – sampled continues-time signals
  - E.g. I/O signal used by micro-controllers
- **Systems described by difference equations**
  - E.g. CD player contains digital signal processing system (digital filter) to manipulate the digital audio signal
- **For such systems input and output are related by difference equations and the Z-transform is used to solve these systems**

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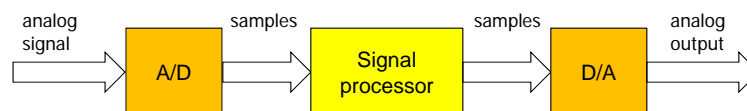
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## Z-Transform - Introduction

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- **For Modern systems**



**processing – filtering noise, equalizing music, adding effects to audio/video, etc.**

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## Z-Transform - Definition

- Let  $y(t)$  be a continuous-time signal (function)
- Define  $T$  – sampling period
- Then –  $y(k) = y_k = y(kT)$  for  $k = 0, 1, 2, 3, \dots$  defines uniformly spaced samples of the original signal  $y(t)$
- The Z-Transform for  $y_k$  is defined as

$$Z[y_k] = Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k}$$

if the summation converges

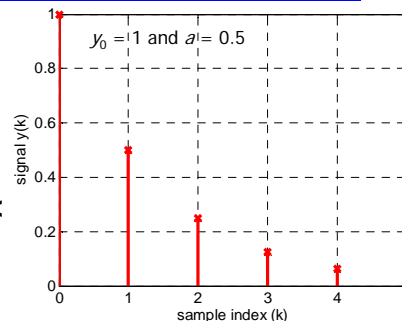
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## Z-Transform – Example 1

- Compute the Z-transform for the sequence  $y(k) = y_0 a^k$  for  $k = 0, 1, 2, \dots$  and  $0 < a < 1$
- Compute the Z-transform for the sequence
- A plot for  $y(k)$  is shown for  $0 < a < 1$
- Note the  $y(k)$  is defined only on specific indices (time samples)



- Applying the definition

$$Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} = \sum_{k=0}^{\infty} y_0 a^k z^{-k} = y_0 \sum_{k=0}^{\infty} (a/z)^k = \frac{y_0}{1 - a/z} = \frac{y_0 z}{z - a}$$

for  $|a/z| < 1$ .

$$y_0 a^k \leftrightarrow \frac{y_0 z}{z - a} \text{ pair (1)}$$

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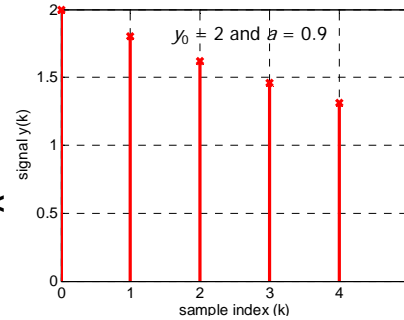
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for  $k=0, 1, 2, \dots$

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## Z-Transform – Example 2

- Compute the Z-transform for the sequence  
 $y(k) = 2(0.9)^k$   
 for  $k = 0, 1, 2, \dots$  and  $0 < a < 1$
- Compute the Z-transform for the sequence
- A plot for  $y(k)$  is shown for  $0 < a < 1$



- Using the previous result

$$Y(z) = \frac{2}{1 - 0.9/z} = \frac{2z}{z - 0.9}$$

for  $|0.9/z| < 1$ .

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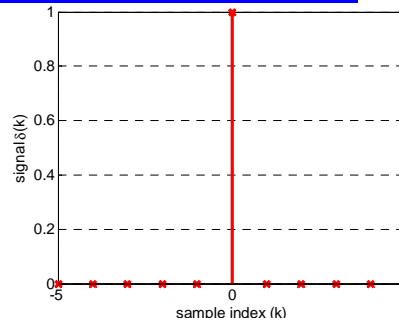
### Discrete-time Delta Function

## Z-Transform – Example 3

- Compute the Z-transform for the sequence

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

- Compute the Z-transform for the sequence



- Using the definition

$$\Delta(z) = \sum_{k=-\infty}^{\infty} \delta(k)z^{-k} = \delta(0)z^0 = 1$$

for all  $z$ .

$$y_0 \delta(k) \leftrightarrow y_0 \quad \text{pair (2)}$$

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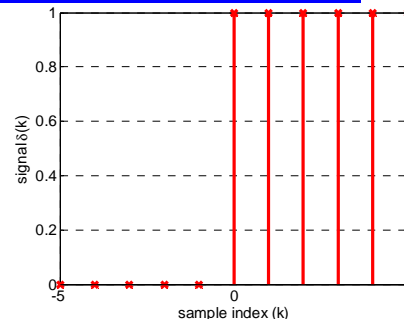
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## Z-Transform – Example 3

- Compute the Z-transform for the sequence

$$u(k) = \begin{cases} 1 & k = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

- Compute the Z-transform for the sequence



- Using the definition

$$U(z) = \sum_{k=-\infty}^{\infty} u(k)z^{-k} = \sum_{k=0}^{\infty} z^{-k} = z^0 + z^{-1} + z^{-2} + \dots = \frac{1}{1 - 1/z} = \frac{z}{z - 1}$$

for  $|1/z| < 1$ .

$$y_0 u(k) \leftrightarrow \frac{y_0 z}{z - 1} \text{ pair (3)}$$

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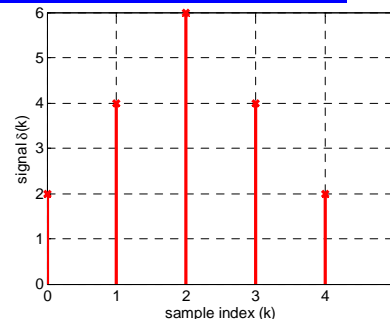
## Z-Transform – Example 4

- Compute the Z-transform for the sequence

$$y(k) = [2, 4, 6, 4, 2, 0]$$

for  $k = 0, 1, 2, 3, 4$ , and  $5$ , respectively

- Compute the Z-transform for the sequence



- Using the definition

$$Y(z) = \sum_{k=-\infty}^{\infty} y(k)z^{-k} = 2z^0 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4} + 0z^{-5}$$

for  $|1/z| < 1$ .

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## Inverse Z-Transform – Example 5

- **Compute the Inverse Z-Transform for**

$$Y(z) = \frac{2z}{z - 0.3}$$

- **We use pair (1) →  $y_0 = 2$  and  $a = 3$**   
**Or  $y(k)$  is given by**

$$y_k = 2(0.3)^k$$

**for  $k = 0, 1, 2, \dots$**

**$y(k)$  may be rewritten as**

$$y_k = 2(0.3)^k u(k)$$

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## Inverse Z-Transform – Example 6

- **Compute the Inverse Z-Transform for**

$$Y(z) = \frac{10}{(z - 0.3)(z - 0.9)}$$

- **Solution?**

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