

King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 540 – Computer Networks
Term 112
Dr. Ashraf S. Hasan Mahmoud
Rm 22-420
Ext. 1724
Email: ashraf@kfupm.edu.sa

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

1

Primer on Probability Theory

- **Source: Chapter 2 and 3 of:
Alberto Leon-Garcia, Probability and Random
Processes for Electrical Engineering,
Addison Wisely**

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

2

Expectation of a Random Variable

- **Expectation of the random variable X can be computed by**

$$E[X] = \sum_{\forall i} x_i P[X = x_i]$$

for discrete variables, or

$$E[X] = \int_{-\infty}^{\infty} t f_x(t) dt$$

for continuous variables.

n^{th} Expectation of a Random Variable

- **The n^{th} expectation of the random variable X can be computed by**

$$E[X^n] = \sum_{\forall i} x_i^n P[X = x_i]$$

for discrete variables, or

$$E[X^n] = \int_{-\infty}^{\infty} t^n f_x(t) dt$$

for continuous variables.

Expectation of a Function of the Random Variable

- **Let $g(x)$ be a function of the random variable x , the expectation of $g(x)$ is given by**

$$E[g(x)] = \sum_{\forall i} g(x_i)P[X = x_i]$$

for discrete variables, or

$$E[g(x)] = \int_{-\infty}^{\infty} g(t)f_x(t)dt$$

for continuous variables.

Some Important Random Variables – Discrete Random Variables

- **Bernoulli**
- **Binomial**
- **Geometric**
- **Poisson**

$$\sum_{n=1}^M n = \frac{1}{2}M(M+1)$$

$$\sum_{n=0}^M \binom{M}{n} r^n = (1+r)^M; |r| < 1$$

$$\sum_{n=0}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2}; |r| < 1$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}; |r| < 1$$

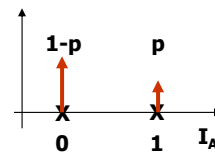
$$\sum_{n=0}^M r^n = \frac{1-r^{M+1}}{1-r}; |r| < 1, M = 1, 2, \dots$$

$$\sum_{n=0}^M nr^{n-1} = \frac{1+(Mr-M-1)r^M}{(1-r)^2}; |r| < 1$$

Bernoulli Random Variable

- Let A be an event related to the outcomes of some random experiment. The indicator function for A is defined as

$$I_A(\zeta) = \begin{cases} 0 & \text{if } \zeta \text{ not in } A \\ 1 & \text{if } \zeta \text{ is in } A \end{cases}$$

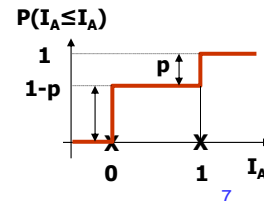


- I_A is random variable since it assigns a number to each outcome in S
- It is discrete r.v. that takes on values from the set $\{0,1\}$
- PMF is given by

$$p_I(0) = 1-p, p_I(1) = p$$

where $P\{A\} = p$

- Describes the outcome of a Bernoulli trial
- $E[X] = p, \text{VAR}[X] = p(1-p)$
- $G_X(z) = (1-p+pz)$



3/18/2012

Dr. Ashraf S. Hasan Mahmoud

7

Binomial Random Variable

- Suppose a random experiment is repeated n independent times; let X be the number of times a certain event A occurs in these n trials

$$X = I_1 + I_2 + \dots + I_n$$

i.e. X is the sum of Bernoulli trials (X 's range = $\{0, 1, 2, \dots, n\}$)

- X has the following pmf

$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k=0, 1, 2, \dots, n$

- $E[X] = np, \text{Var}[X] = np(1-p)$
- $G_X(z) = (1-p + pz)^n$

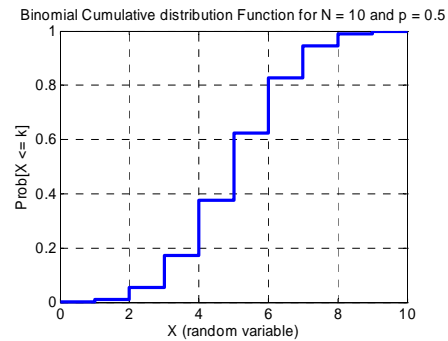
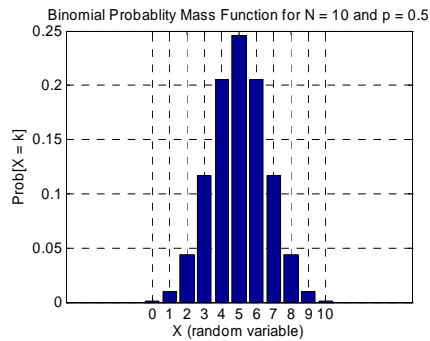
3/18/2012

Dr. Ashraf S. Hasan Mahmoud

8

Binomial Random Variable – cont'd

- **Example**



3/18/2012

Dr. Ashraf S. Hasan Mahmoud

9

Geometric Random Variable

- **Suppose a random experiment is repeated - We count the number of M of independent Bernoulli trials UNTIL the first occurrence of a success**
- **M is called geometric random variable**
 - Range of M = 1, 2, 3, ...
- **X has the following pmf**

$$\Pr[X = k] = (1 - p)^{k-1} p$$

for k=1, 2, 3, ...

- **$E[X] = 1/p$, $\text{Var}[X] = (1-p)/p^2$**
- **$G_X(z) = pz/(1-(1-p)z)$**

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

10

Geometric Random Variable - 2

- **Suppose a random experiment is repeated - We count the number of M of independent Bernoulli trials BEFORE the first occurrence of a success**
- **M is called geometric random variable**
 - Range of $M = 0, 1, 2, 3, \dots$
- **X has the following pmf**

$$\Pr[X = k] = (1 - p)^k p$$

for $k=0, 1, 2, 3, \dots$

- **$E[X] = (1-p)/p, \quad \text{Var}[X] = (1-p)/p^2$**
- **$G_X(z) = pz/(1-(1-p)z)$**

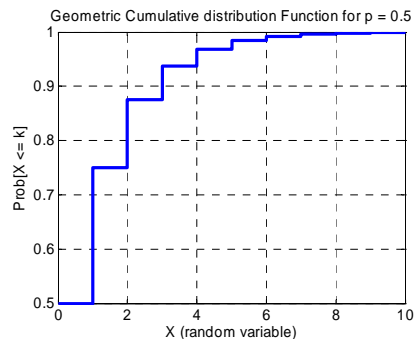
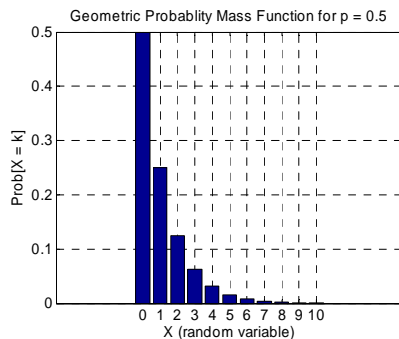
3/18/2012

Dr. Ashraf S. Hasan Mahmoud

11

Geometric Random Variable – cont'd

- **Example: $p = 0.5$; X is number of failures BEFORE a success (2nd type)**
- **Note Matlab's version of geometric distribution is the 2nd type**



3/18/2012

Dr. Ashraf S. Hasan Mahmoud

12

Poisson Random Variable

- In many applications we are interested in counting the number of occurrences of an event in a certain time period

- The pmf is given by

$$\Pr[X = k] = \frac{\alpha^k}{k!} e^{-\alpha}$$

For $k=0, 1, 2, \dots$; α is the average number of event occurrences in the specified interval

- $E[X] = \alpha$, $\text{Var}[X] = \alpha$
- $G_X(z) = e^{\alpha(z-1)}$
- Remember: time between events is exponentially distributed!
- Poisson is the limiting case for Binomial as $n \rightarrow \infty$, $p \rightarrow 0$, such that $np = \alpha$

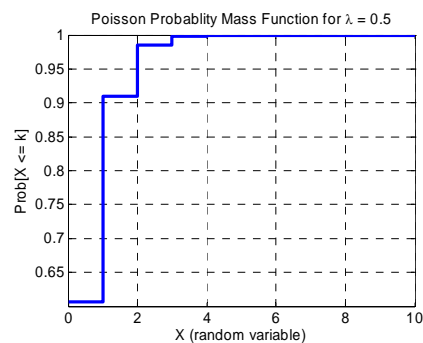
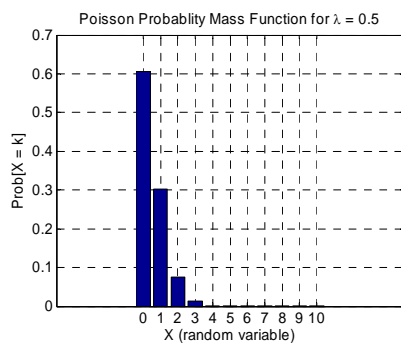
3/18/2012

Dr. Ashraf S. Hasan Mahmoud

13

Poisson Random Variable – cont'd

- Example:



3/18/2012

Dr. Ashraf S. Hasan Mahmoud

14

Matlab Code to Plot Distributions

```
0001 % plot distributions
0002 % see "help stats"
0003 clear all
0004 FontSize = 14;
0005 LineWidth = 3;
0006 % Binomial
0007 N = 10; X = [0:1:N]; P = 0.5;
0008 ybp = binopdf(X, N, P); % get PMF
0009 ybc = binocdf(X, N, P); % get CDF
0010 figure(1); set(gca,'FontSize', FontSize);
0011 bar(X, ybp);
0012 title(['Binomial Probability Mass Function for
N = ' ...
num2str(N) ' and p = ' num2str(P)]);
0013 xlabel('X (random variable)');
0014 ylabel('Prob[X = k]'); grid
0015 figure(2); set(gca,'FontSize', FontSize);
0016 stairs(X, ybc,'LineWidth', LineWidth);
0017 title(['Binomial Cumulative distribution
Function for N = ' ...
num2str(N) ' and p = ' num2str(P)]);
0018 xlabel('X (random variable)');
0019 ylabel('Prob[X <= k]'); grid
0020 % Geometric
0021 P = 0.5; X = [0:10];
0022 ygp = geopdf(X, P); % get pdf
0023 ygc = geocdf(X, P); % get cdf
0024 figure(3); set(gca,'FontSize', FontSize);
0025 bar(X, ygp);
0026 title(['Geometric Probability Mass Function for
p = ' num2str(P)]);
0027 xlabel('X (random variable)');
0028 ylabel('Prob[X = k]'); grid
0029 figure(4); set(gca,'FontSize', FontSize);
0030 stairs(X, ygc,'LineWidth', LineWidth);
0031 title(['Geometric Cumulative distribution
Function for p = ' num2str(P)]);
0032 xlabel('X (random variable)');
0033 ylabel('Prob[X <= k]'); grid
0034 % Poisson
0035 Lambda = 0.5; X = [0:10];
0036 ypp = poispdf(X, Lambda);
0037 ypc = poisscdf(X, Lambda);
0038 figure(5); set(gca,'FontSize', FontSize);
0039 bar(X, ypp);
0040 title(['Poisson Probability Mass Function for
\lambda = ' num2str(Lambda)]);
0041 xlabel('X (random variable)');
0042 ylabel('Prob[X = k]'); grid
0043 figure(6); set(gca,'FontSize', FontSize);
0044 stairs(X, ypc,'LineWidth', LineWidth);
0045 title(['Poisson Probability Mass Function for
\lambda = ' num2str(Lambda)]);
0046 xlabel('X (random variable)');
0047 ylabel('Prob[X <= k]'); grid
```

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

15

Some Important Random Variables – Continuous Random Variables

- **Uniform**
- **Exponential**
- **Gaussian (Normal)**
- **Rayleigh**
- **Gamma**
- **M-Erlang**
-

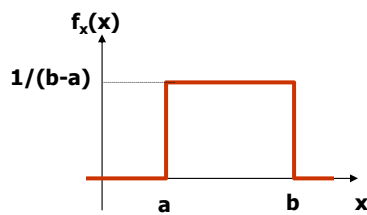
3/18/2012

Dr. Ashraf S. Hasan Mahmoud

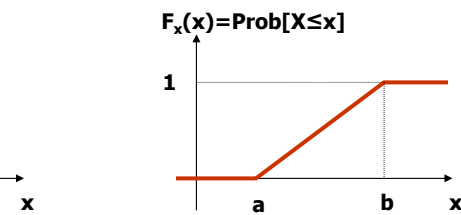
16

Uniform Random Variables

- Realizations of the r.v. can take values from the interval $[a, b]$
- PDF $f_X(x) = 1/(b-a)$ $a \leq x \leq b$
- $E[X] = (a+b)/2$, $\text{Var}[X] = (b-a)^2/12$
- $\Phi_X(\omega) = [e^{j\omega b} - e^{j\omega a}]/(j\omega(b-a))$



3/18/2012



Dr. Ashraf S. Hasan Mahmoud

17

Exponential Random Variables

- The exponential r.v. X with parameter λ has pdf

- And CDF given by

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

- Range of X : $[0, \infty)$
- $E[X] = 1/\lambda$, $\text{Var}[X] = 1/\lambda^2$
- $\Phi_X(\omega) = \lambda/(\lambda - j\omega)$



This means:
 $\text{Prob}[X \leq x] = 1 - e^{-\lambda x}$, or
 $\text{Prob}[X > x] = e^{-\lambda x}$

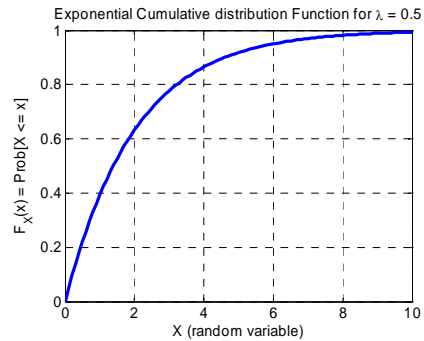
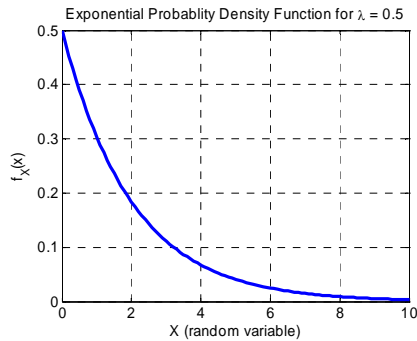
3/18/2012

Dr. Ashraf S. Hasan Mahmoud

18

Exponential Random Variables – cont'd

- **Example:**
 - Note the mean is $1/\lambda = 2$



3/18/2012

Dr. Ashraf S. Hasan Mahmoud

19

Exponential Random Variables – Memoryless Property

- The exponential r.v. is the only continuous r.v. with the memoryless property!!
- **Memoryless Property:**
 $P[X > t+h / X > t] = P[X > h]$

i.e. the probability of having to wait h additional seconds given that one has already been waiting t second IS EXACTLY equal to the probability of waiting h seconds when one first begins to wait

Proof:

$$\begin{aligned}
 P[X > t+h / X > t] &= \frac{P[(X > t+h) \cap (X > t)]}{P[(X > t)]} \\
 &= \frac{P[X > t+h]}{P[X > t]} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} \\
 &= e^{-\lambda h} \\
 &= P[X > h]
 \end{aligned}$$

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

20

Gaussian (Normal) Random Variable

- Rises in situations where a random variable X is the sum of a large number of "small" random variables – central limit theorem

- PDF
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

For $-\infty < x < \infty$; μ and $\sigma > 0$ are real numbers

- $E[X] = \mu$, $\text{Var}[X] = \sigma^2$
- $\Phi_X(\omega) = e^{j\mu\omega - \sigma^2\omega^2/2}$
- Under wide range of conditions X can be used to approximate the sum of a large number of independent random variables

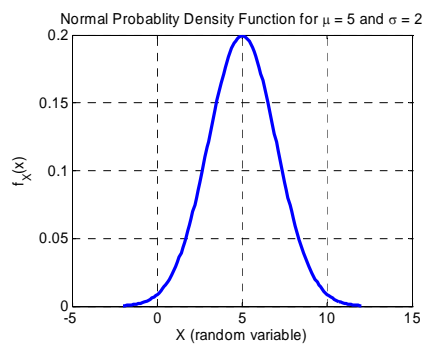
3/18/2012

Dr. Ashraf S. Hasan Mahmoud

21

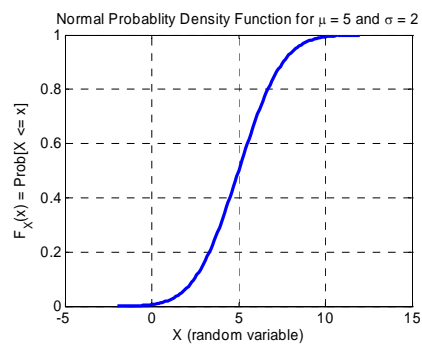
Gaussian (Normal) Random Variable – cont'd

- Example:



3/18/2012

Dr. Ashraf S. Hasan Mahmoud



22

Matlab Code to Plot Distributions

```

0001 % plot distributions
0002 % see "help stats"
0003 clear all
0004 FontSize = 14;
0005 LineWidth = 3;
0006 % exponential
0007 X = [0:.1:10]; Lambda = 0.5;
0008 yep = exppdf(X, 1/Lambda); % get PDF
0009 yec = expcdf(X, 1/Lambda); % get CDF
0010 figure(1); set(gca,'FontSize', FontSize);
0011 plot(X, yep, 'LineWidth', LineWidth);
0012 title(['Exponential Probability Density
Function for \lambda = ' ...
num2str(Lambda)]);
0013 xlabel('X (random variable)');
0014 ylabel('f_X(x)'); grid
0015 figure(2); set(gca,'FontSize', FontSize);
0016 plot(X, yec, 'LineWidth', LineWidth);
0017 title(['Exponential Cumulative Distribution
Function for \lambda = ' ...
num2str(Lambda)]);
0018 xlabel('X (random variable)');
0019 ylabel('F_X(x) = Prob[X <= x]'); grid
0020 % normal
0021 X = [-2:.1:12]; Mu = 5; Sigma = 2;
0022 ynp = normpdf(X, Mu, Sigma); % get PDF
0023 ync = normcdf(X, Mu, Sigma); % get CDF
0024 figure(3); set(gca,'FontSize', FontSize);
0025 plot(X, ynp, 'LineWidth', LineWidth);
0026 title(['Normal Probability Density Function
for \mu = ' ...
num2str(Mu) ' and \sigma = '
num2str(Sigma)]);
0027 xlabel('X (random variable)');
0028 ylabel('f_X(x)'); grid
0029 figure(4); set(gca,'FontSize', FontSize);
0030 plot(X, ync, 'LineWidth', LineWidth);
0031 title(['Normal Probability Density Function
for \mu = ' ...
num2str(Mu) ' and \sigma = '
num2str(Sigma)]);
0032 xlabel('X (random variable)');
0033 ylabel('F_X(x) = Prob[X <= x]'); grid
0034 % Rayleigh
0035 X = [0:.1:10]; Alpha = 2;
0036 yrp = raypdf(X, Alpha); % get PDF
0037 yrc = raycdf(X, Alpha); % get CDF
0038 figure(5); set(gca,'FontSize', FontSize);
0039 plot(X, yrp, 'LineWidth', LineWidth);
0040 title(['Rayleigh Probability Density Function
for \alpha = ' ...
num2str(Alpha)]);
0041 xlabel('X (random variable)');
0042 ylabel('f_X(x)'); grid
0043 figure(6); set(gca,'FontSize', FontSize);
0044 plot(X, yrc, 'LineWidth', LineWidth);
0045 title(['Rayleigh Probability Density Function
for \alpha = ' ...
num2str(Alpha)]);
0046 xlabel('X (random variable)');
0047 ylabel('F_X(x) = Prob[X <= x]'); grid

```

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

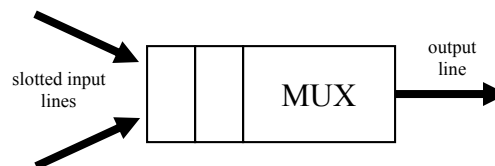
23

Discrete-Time Markov Chain

Example 10: Multiplexer

Problem: Data in the form of fixed-length packets arrive in slots on both of the input lines of a multiplexer. A slot contains a packet with probability p , independent of the arrivals during other slots or on the other line. The multiplexer transmits one packet per time slot and has the capacity to store two messages only. If no room for a packet is found, the packet is dropped.

- Draw the state diagram and define the matrix P
- Compute the throughput of the multiplexer for $p = 0.3$



3/18/2012

Dr. Ashraf S. Hasan Mahmoud

24

Example 10: Multiplexer – cont'd

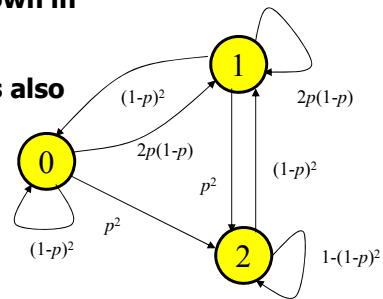
Solution: In any slot time, the arrivals pmf is given by

$$P(j \text{ cells arrive}) = \begin{matrix} (1-p)^2 & j=0 \\ 2p(1-p) & j=1 \\ p^2 & j=2 \end{matrix}$$

Let the state be the number of packets in the buffer, then the state diagram is shown in figure.

The corresponding transition matrix is also given below

$$P = \begin{bmatrix} (1-p)^2 & 2p(1-p) & p^2 \\ (1-p)^2 & 2p(1-p) & p^2 \\ 0 & (1-p)^2 & 1-(1-p)^2 \end{bmatrix}$$



3/18/2012

Dr. Ashraf S. Hasan Mahmoud

25

Example 10: Multiplexer – cont'd

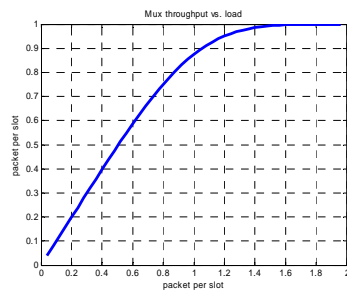
Solution-cont'd:

Load: average arrivals = $2p$ packets/slot

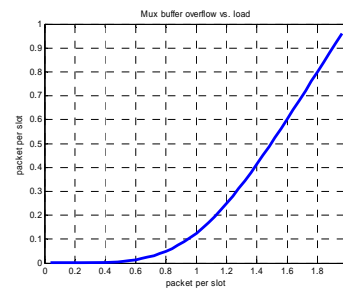
Throughput: $\pi_1 + \pi_2$

Buffer overflow = Prob(two packet arrivals while in state 2)
 = Prob(two arrivals) X π_2
 = $p^2 \pi_2$

The graphs below show the relation of load versus –throughput and buffer overflow for the MUX



S.



26

Example 10: Multiplexer – cont'd

Solution-cont'd:

The matlab code used for plotted previous results is shown below.

Make sure you understand the matrix formulation and the solution for the steady state probability vector π

```
clear all
Step = 0.02;
ArrivalProb = [Step:Step:1-Step];
A = zeros(4,3);
E = zeros(4,1);
E(4) = 1;
for i=1:length(ArrivalProb)
    p = ArrivalProb(i);
    P = [(1-p)^2 2*p*(1-p) p^2; ...
         (1-p)^2 2*p*(1-p) p^2; ...
         0 (1-p)^2 1-(1-p)^2];
    A(1:3,:) = (P - eye(3))';
    A(4,:) = ones(1,3);
    E(4) = 1;
    SteadyStateP = A\E;
    % Prob(packet is lost) = Prob(2 arrivals) X
    % Prob(being in state 2);
    DropProb(i) = p^2*SteadyStateP(3);
    Throughput(i) = sum(SteadyStateP(2:3));
end
```

```
% matlab code continued
figure(1),
h = plot(2*ArrivalProb, Throughput);
set(h, 'LineWidth', 3);
title('Mux throughput vs. load');
ylabel('packet per slot');
xlabel('packet per slot');
grid
figure(2),
h = plot(2*ArrivalProb, DropProb);
set(h, 'LineWidth', 3);
title('Mux buffer overflow vs. load');
ylabel('packet per slot');
xlabel('packet per slot');
grid
```

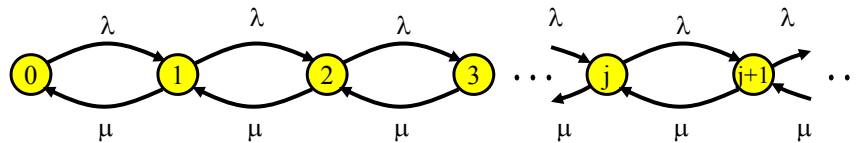
3/18/2012

Dr. Ashraf S. Hasan Mahmoud

27

Example 12:

- **Problem:** The M/M/1 single-server queueing system



The corresponding rate transition matrix is given by

$$\Gamma = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -(\lambda + \mu) & \lambda & 0 & \dots \\ 0 & \mu & -(\lambda + \mu) & \lambda & \dots \\ 0 & 0 & \mu & -(\lambda + \mu) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

28

Example 12: cont'd

- **Answer: The state transition rates:**

- Customers arrive with rate $\lambda \rightarrow \gamma_{i,i+1} = \lambda$ for $i = 0, 1, 2, \dots$
- When system is not empty, customers depart at rate $\mu \rightarrow \gamma_{i,i-1} = \mu$ for $i = 1, 2, 3, \dots$

- **The global balance equations:**

$$\begin{aligned} \lambda p_0 &= \mu p_1 && \text{for } j = 0 \\ (\lambda + \mu)p_j &= \lambda p_{j-1} + \mu p_{j+1} && \text{for } j=1, 2, \dots \end{aligned}$$

$$\begin{aligned} \rightarrow \lambda p_j - \mu p_{j+1} &= \lambda p_{j-1} - \mu p_j && \text{for } j=1, 2, \dots \\ &= \text{constant} \end{aligned}$$

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

29

Example 12: cont'd

- **Answer:**

For $j = 1$, we have

$$\lambda p_0 - \mu p_1 = \text{constant}$$

Therefore the constant is equal to zero.

Hence,

$$\begin{aligned} \lambda p_{j-1} &= \mu p_j \text{ or} \\ p_j &= (\lambda/\mu) p_{j-1} && \text{for } j=1, 2, \dots \end{aligned}$$

By simple induction:

$$p_j = \rho^j p_0$$

where $\rho = \lambda/\mu$

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

30

Example 12: cont'd

- **Answer:**

To obtain p_0 , we use the fact that

$$1 = \sum_j p_j = (1 + \rho + \rho^2 + \dots)p_0$$

note the above series converges only for $\rho < 1$ or equivalently $\lambda < \mu$

Therefore, $p_0 = 1 - \rho$

In general, the steady state pmf for the M/M/1 queue is given by

$$p_j = (1 - \rho)\rho^j$$

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

31

References

- **Alberto Leon-Garcia, Probability and Random Processes for Electrical Engineering, Addison Wesley, 1989**

3/18/2012

Dr. Ashraf S. Hasan Mahmoud

32