

# King Fahd University of Petroleum & Minerals Computer Engineering Dept

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COE 540 – Computer Networks  
Term 101  
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## Primer on Probability Theory

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- **Source: Chapter 2 and 3 of:  
Alberto Leon-Garcia, Probability and Random  
Processes for Electrical Engineering,  
Addison Wisely**

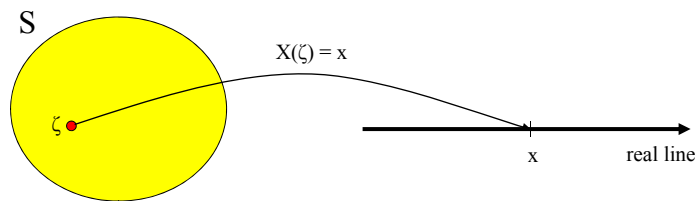
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## What is a Random Variable?

- **Random Experiment**
- **Sample Space**
- **Def: A random variable  $X$  is a function that assigns a number of  $X(\zeta)$  to each outcome  $\zeta$  in the sample space of  $S$  of the random experiment**



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## Set Functions

- Define  $\Omega$  as the set of all possible outcomes
- Define  $\mathbf{A}$  as set of events
- Define  $A$  as an event – subset of the set of all experiments outcomes
- Set operations:
  - Complementation  $A^c$ : is the event that event  $A$  does not occur
  - Intersection  $A \cap B$ : is the event that event  $A$  and  $B$  occur
  - Union  $A \cup B$ : is the event that event  $A$  or  $B$  occur
  - Inclusion  $A \subseteq B$ : An event  $A$  occurring implying events  $B$  occurs

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## Set Functions

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- Note:
  - Set of events  $\mathbf{A}$  is closed under set operations
  - $\Phi$  – empty set
  - $A \cap B = \Phi \rightarrow$  are mutually exclusive or disjoint

## Axioms of Probability

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- Let  $P(A)$  denote probability of event  $A$ :
  1. For any event  $A$  belongs  $\mathbf{A}$ ,  $P(A) \geq 0$ ;
  2. For set of all possible outcomes  $\mathbf{\Omega}$ ,  $P(\mathbf{\Omega}) = 1$ ;
  3. If  $A$  and  $B$  are disjoint events,  $P(A \cup B) = P(A) + P(B)$
  4. For countably infinite sets,  $A_1, A_2, \dots$  such that  $A_i$  ins  $A_j = \Phi$  for  $i \neq j$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

## Additional Properties

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- For any event,  $P(A) \leq 1$
- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A) \leq P(B)$  for  $A \subseteq B$

## Conditional Probability

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- Conditional probability is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

- $P(A/B)$  probability of event A conditioned on the occurrence of event B
- Note:
  - A and B are *independent* if  $P(A \cap B) = P(A)P(B) \rightarrow P(A/B) = P(A)$
  - Independent IS NOT EQUAL TO mutually exclusive

## The Law of Total Probability

- A set of events  $A_i, i = 1, 2, \dots, n$  partitions the set of experimental outcomes if

$$\bigcup_{i=1}^n A_i = \Omega$$

and

$$A_i \cap A_j = \Phi$$

Then we can write any event  $B$  in terms of  $A_i, i = 1, 2, \dots, n$  as

$$B = \bigcup_{i=1}^n A_i \cap B$$

Furthermore,

$$P(B) = \sum_{i=1}^n P(A_i \cap B)$$

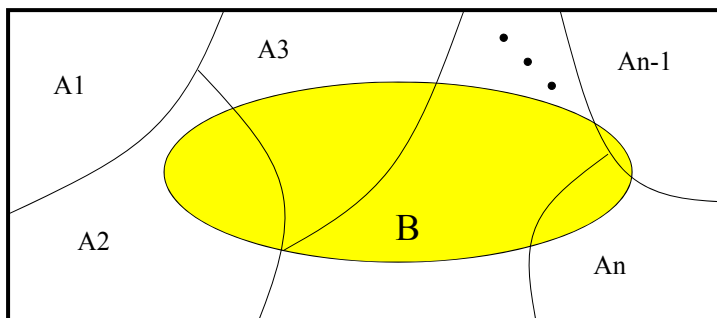
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## Bayes Rule

- Let  $A_1, A_2, \dots, A_n$  be a partition of a sample space  $S$ . Suppose the event  $A$  occurs**



A Partition of S into n disjoint sets

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## Bayes' Rule

- Using the total law of probability and applying it to the definition of the conditional probability, yields

$$\begin{aligned} P(A_i/B) &= \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i \cap B)}{\sum_{i=1}^n P(A_i \cap B)} \\ &= \frac{P(A_i)P(B/A_i)}{\sum_{i=1}^n P(A_i)P(B/A_i)} \end{aligned}$$

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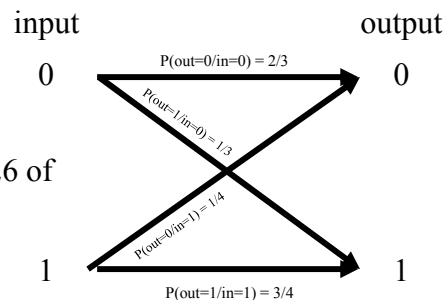
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## Example 1: Binary (Symmetric) Channel

- Given the binary symmetric channel depicted in figure, find  $P(\text{input} = j / \text{output} = i)$ ;  $i, j = 0, 1$ . Given that  $P(\text{input} = 0) = 0.4$ ,  $P(\text{input} = 1) = 0.6$ .

### Solution:

Refer to examples 2.23 and 2.26 of Garcia's textbook



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## The Cumulative Distribution Function

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- **The cumulative distribution function (cdf) of a random variable  $X$  is defined as the probability of the event  $\{X \leq x\}$ :**

$$F_X(x) = \text{Prob}\{X \leq x\} \quad \text{for } -\infty < x < \infty$$

**i.e. it is equal to the probability the variable  $X$  takes on a value in the set  $(-\infty, x]$**

- **A convenient way to specify the probability of all semi-infinite intervals**

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## Properties of the CDF

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- $0 \leq F_X(x) \leq 1$
- $\lim_{x \rightarrow \infty} F_X(x) = 1$
- $\lim_{x \rightarrow -\infty} F_X(x) = 0$
- $F_X(x)$  is a nondecreasing function  $\rightarrow$  if  $a < b \rightarrow F_X(a) \leq F_X(b)$
- $F_X(x)$  is continuous from the right  $\rightarrow$  for  $h > 0$ ,  
$$F_X(b) = \lim_{h \rightarrow 0} F_X(b+h) = F_X(b^+)$$
- $\text{Prob}[a < X \leq b] = F_X(b) - F_X(a)$
- $\text{Prob}[X = b] = F_X(b) - F_X(b^-)$

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## Example 2: Exponential Random Variable

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- **Problem:** The transmission time  $X$  of a message in a communication system obey the exponential probability law with parameter  $\lambda$ , that is

$$\text{Prob } [X > x] = e^{-\lambda x} \quad x > 0$$

**Find the CDF of  $X$ . Find Prob  $[T < X \leq 2T]$  where  $T = 1/\lambda$**

## Example 2: Exponential Random Variable - cont'd

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- **Answer:**

**The CDF of  $X$  is**

$$\begin{aligned} F_X(x) &= \text{Prob } \{X \leq x\} = 1 - \text{Prob } \{X > x\} \\ &= 1 - e^{-\lambda x} \quad x \geq 0 \\ &= 0 \quad x < 0 \end{aligned}$$

$$\begin{aligned} \text{Prob } \{T < X \leq 2T\} &= F_X(2T) - F_X(T) \\ &= 1 - e^{-2} - (1 - e^{-1}) \\ &= \mathbf{0.233} \end{aligned}$$



## Example 3: Use of Bayes Rule

- **Problem:** The waiting time  $W$  of a customer in a queueing system is zero if he finds the system idle, and an exponentially distributed random length of time if he finds the system busy. The probabilities that he finds the system idle or busy are  $p$  and  $1-p$ , respectively. Find the CDF of  $W$

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## Example 3: cont'd

- **Answer:**  
The CDF of  $W$  is found as follows:

$$\begin{aligned}F_X(x) &= \text{Prob}\{W \leq x\} \\ &= \text{Prob}\{W \leq x/\text{idle}\}p + \text{Prob}\{W \leq x/\text{busy}\}(1-p)\end{aligned}$$

**Note**  $\text{Prob}\{W \leq x/\text{idle}\} = 1$  for any  $x > 0$

→

$$\begin{aligned}F_X(x) &= 0 & x < 0 \\ &= p + (1-p)(1 - e^{-\lambda x}) & x \geq 0\end{aligned}$$

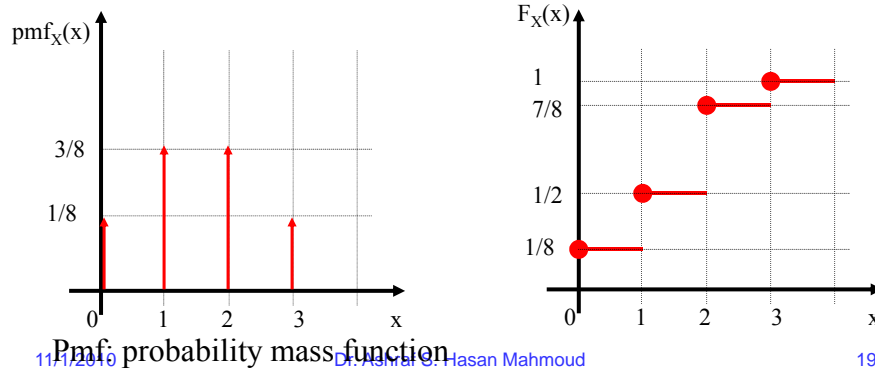
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## Types of Random Variables

- **(1) Discrete Random Variables**
  - **CDF is right continuous, staircase function of  $x$ , with jumps at countable set  $x_0, x_1, x_2, \dots$**



## Types of Random Variables

- **(2) Continuous Random Variables**
  - **CDF is continuous for all values of  $x \rightarrow \text{Prob} \{ X = x \} = 0$  (recall the CDF properties)**
  - **Can be written as the integral of some non negative function**

$$F_X(x) = \int_{-\infty}^{\infty} f(t) dt$$

Or

$$f(t) = \frac{dF_X(x)}{dx}$$

11/1 f(t) is referred to as the probability density function or PDF 20

## Types of Random Variables

- **(3) Random Variables of Mixed Types**

$$F_X(x) = p F_1(x) + (1-p) F_2(x)$$

## Probability Density Function

- **The PDF of X, if it exists, is define as the derivative of CDF  $F_X(x)$ :**

$$f_x(x) = \frac{dF_x(x)}{dx}$$

## Properties of the PDF

- $f_x(x) \geq 0$

- $P\{a \leq x \leq b\} = \int_a^b f_x(x) dx$

- $F_X(x) = \int_{-\infty}^x f_x(t) dt$

- $1 = \int_{-\infty}^{\infty} f_x(t) dt$

A valid pdf can be formed from any nonnegative, piecewise continuous function  $g(x)$  that has a finite integral:

$$\int_{-\infty}^{\infty} g(x) dx = c < \infty$$

By letting  $f_x(x) = g(x)/c$ , we obtain a function that satisfies the normalization condition.

This is the scheme we use to generate pdfs from simulation results!

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## Conditional PDFs and CDFs

- **If some event A concerning X is given, then conditional CDF of X given A is defined by**

$$P\{[X \leq x] \cap A\}$$

$$F_X(x/A) = \frac{P\{[X \leq x] \cap A\}}{P\{A\}} \quad \text{if } P\{A\} > 0$$

**The conditional pdf of X given A is then defined by**

$$f_X(x/A) = \frac{d}{dx} F_X(x/A)$$

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## Expectation of a Random Variable

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- **Expectation of the random variable X can be computed by**

$$E[X] = \sum_{\forall i} x_i P[X = x_i]$$

**for discrete variables, or**

$$E[X] = \int_{-\infty}^{\infty} t f_x(t) dt$$

**for continuous variables.**

## $n^{\text{th}}$ Expectation of a Random Variable

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- **The  $n^{\text{th}}$  expectation of the random variable X can be computed by**

$$E[X^n] = \sum_{\forall i} x_i^n P[X = x_i]$$

**for discrete variables, or**

$$E[X^n] = \int_{-\infty}^{\infty} t^n f_x(t) dt$$

**for continuous variables.**

## The Characteristic Function

- **The characteristic function of a random variable X is defined by**

$$\begin{aligned}\Phi_x(\omega) &= E[e^{j\omega X}] \\ &= \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx\end{aligned}$$

- **Note that  $\Phi_x(\omega)$  is simply the Fourier Transform of the PDF  $f_X(x)$  (with a reversal in the sign of the exponent)**
- **The above is valid for continuous random variables only**

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## The Characteristic Function (2)

- **Properties:**

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_x(\omega) \Big|_{\omega=0}$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) e^{-j\omega x} d\omega$$

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## The Characteristic Function (3)

- **For discrete random variables,**

$$\begin{aligned}\Phi_x(\omega) &= E[e^{j\omega X}] \\ &= \sum_{\forall k} p_X(x_k) e^{j\omega x_k}\end{aligned}$$

- **For integer valued random variables,**

$$\Phi_x(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}$$

## The Characteristic Function (4)

- **Properties**

$$p_X(k) = \frac{1}{2\pi} \int_0^{2\pi} \Phi_x(\omega) e^{-j\omega k} d\omega$$

**for  $k=0, \pm 1, \pm 2, \dots$**

## Expectation of a Function of the Random Variable

- **Let  $g(x)$  be a function of the random variable  $x$ , the expectation of  $g(x)$  is given by**

$$E[g(x)] = \sum_{\forall i} g(x_i) P[X = x_i]$$

**for discrete variables, or**

$$E[g(x)] = \int_{-\infty}^{\infty} g(t) f_x(t) dt$$

**for continuous variables.**

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## Probability Generating Function

- **A matter of convenience – compact representation**
- **The same as the z-transform**
- **If  $N$  is a non-negative integer-valued random variable, the probability generating function is defined as**

$$\begin{aligned} G_N(z) &= E[z^N] \\ &= \sum_{k=0}^{\infty} p_N(k) z^k \\ &= p_N(0) + p_N(1)z + p_N(2)z^2 + \dots \end{aligned}$$

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## Probability Generating Function (2)

- **Properties:**

- **1** 
$$p_N(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_N(z) \Big|_{z=0}$$

- **2** 
$$E[N] = G'_N(1)$$

- **3** 
$$Var[N] = G''_N(1) + G'_N(1) - [G'_N(1)]^2$$

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## Probability Generating Function (3)

- **For non-negative continuous random variables, let us define the Laplace transform of the PDF**

$$X^*(s) = \int_0^{\infty} f_X(x) e^{-sx} dx$$

**Properties:** 
$$= E[e^{-sx}]$$

$$E[X^n] = (-1)^n \frac{d^n}{ds^n} X^*(s) \Big|_{s=0}$$

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## Some Important Random Variables – Discrete Random Variables

- **Bernoulli**
- **Binomial**
- **Geometric**
- **Poisson**

$$\sum_{n=1}^M n = \frac{1}{2}M(M+1)$$

$$\sum_{n=0}^M \binom{M}{n} r^n = (1+r)^M; |r| < 1$$

$$\sum_{n=0}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2}; |r| < 1$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}; |r| < 1$$

$$\sum_{n=0}^M r^n = \frac{1-r^{M+1}}{1-r}; |r| < 1, M = 1, 2, \dots$$

$$\sum_{n=0}^M nr^{n-1} = \frac{1+(Mr-M-1)r^M}{(1-r)^2}; |r| < 1$$

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## Bernoulli Random Variable

- Let  $A$  be an event related to the outcomes of some random experiment. The indicator function for  $A$  is defined as

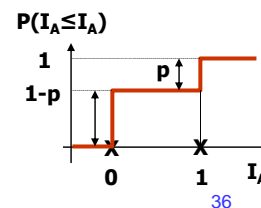
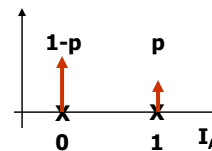
$$I_A(\zeta) = \begin{cases} 0 & \text{if } \zeta \text{ not in } A \\ 1 & \text{if } \zeta \text{ is in } A \end{cases}$$

- $I_A$  is random variable since it assigns a number to each outcome in  $S$
- It is discrete r.v. that takes on values from the set  $\{0,1\}$
- PMF is given by

$$p_1(0) = 1-p, p_1(1) = p$$

where  $P\{A\} = p$

- Describes the outcome of a Bernoulli trial
- $E[X] = p, \text{VAR}[X] = p(1-p)$
- $G_x(z) = (1-p+pz)$



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## Binomial Random Variable

- Suppose a random experiment is repeated  $n$  independent times; let  $X$  be the number of times a certain event  $A$  occurs in these  $n$  trials

$$X = I_1 + I_2 + \dots + I_n$$

i.e.  $X$  is the sum of Bernoulli trials ( $X$ 's range =  $\{0, 1, 2, \dots, n\}$ )

- $X$  has the following pmf

$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

for  $k=0, 1, 2, \dots, n$

- $E[X] = np$ ,  $\text{Var}[X] = np(1-p)$
- $G_X(z) = (1-p + pz)^n$

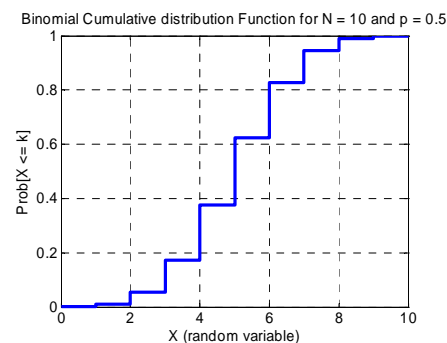
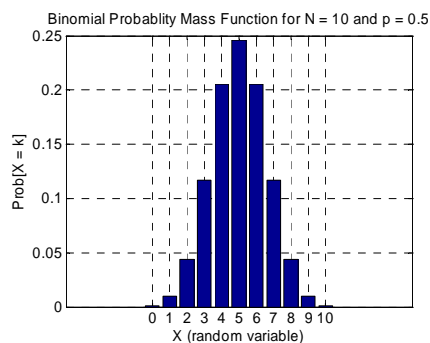
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## Binomial Random Variable – cont'd

- Example



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## Geometric Random Variable

- **Suppose a random experiment is repeated - We count the number of  $M$  of independent Bernoulli trials UNTIL the first occurrence of a success**
- **$M$  is called geometric random variable**
  - Range of  $M = 1, 2, 3, \dots$
- **$X$  has the following pmf**

$$\Pr[X = k] = (1 - p)^{k-1} p$$

for  $k=1, 2, 3, \dots$

- **$E[X] = 1/p, \quad \text{Var}[X] = (1-p)/p^2$**
- **$G_X(z) = pz/(1-(1-p)z)$**

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## Geometric Random Variable - 2

- **Suppose a random experiment is repeated - We count the number of  $M$  of independent Bernoulli trials BEFORE the first occurrence of a success**
- **$M$  is called geometric random variable**
  - Range of  $M = 0, 1, 2, 3, \dots$
- **$X$  has the following pmf**

$$\Pr[X = k] = (1 - p)^k p$$

for  $k=0, 1, 2, 3, \dots$

- **$E[X] = (1-p)/p, \quad \text{Var}[X] = (1-p)/p^2$**
- **$G_X(z) = pz/(1-(1-p)z)$**

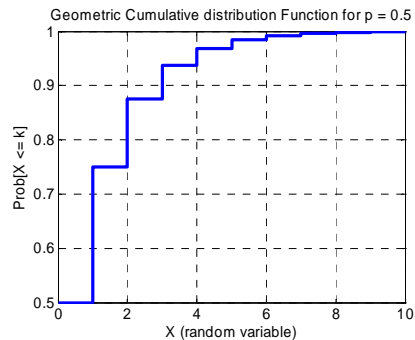
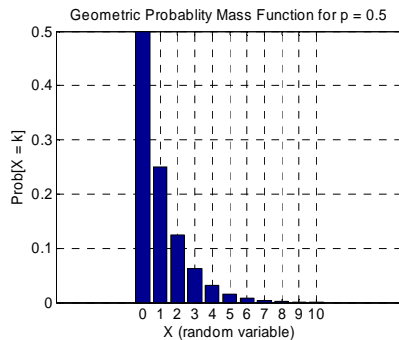
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## Geometric Random Variable – cont'd

- **Example:  $p = 0.5$ ;  $X$  is number of failures BEFORE a success (2<sup>nd</sup> type)**
- **Note Matlab's version of geometric distribution is the 2<sup>nd</sup> type**



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## Poisson Random Variable

- **In many applications we are interested in counting the number of occurrences of an event in a certain time period**

- **The pmf is given by**

$$\Pr[X = k] = \frac{\alpha^k}{k!} e^{-\alpha}$$

**For  $k=0, 1, 2, \dots$ ;  $\alpha$  is the average number of event occurrences in the specified interval**

- **$E[X] = \alpha$ ,  $\text{Var}[X] = \alpha$**
- **$G_X(z) = e^{\alpha(z-1)}$**
- **Remember: time between events is exponentially distributed!**
- **Poisson is the limiting case for Binomial as  $n \rightarrow \infty$ ,  $p \rightarrow 0$ , such that  $np = \alpha$**

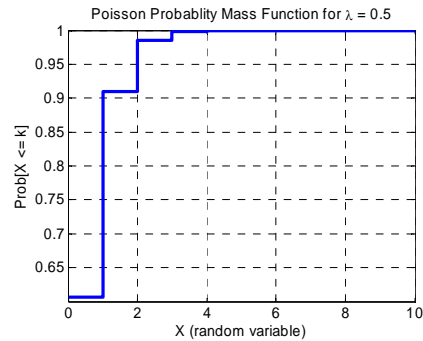
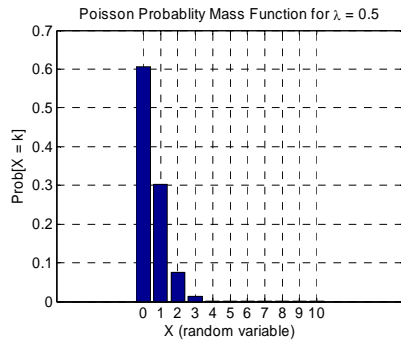
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## Poisson Random Variable – cont'd

- **Example:**



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## Matlab Code to Plot Distributions

```

0001 % plot distributions
0002 % see "help stats"
0003 clear all
0004 FontSize = 14;
0005 LineWidth = 3;
0006 % Binomial
0007 N = 10; X = [0:1:N]; P = 0.5;
0008 ybp = binopdf(X, N, P); % get PMF
0009 ybc = binocdf(X, N, P); % get CDF
0010 figure(1); set(gca,'FontSize', FontSize);
0011 bar(X, ybp);
0012 title(['Binomial Probability Mass Function for
N = ' ...
0013 num2str(N) ' and p = ' num2str(P)]);
0014 xlabel('X (random variable)');
0015 ylabel('Prob[X = k]'); grid
0016 figure(2); set(gca,'FontSize', FontSize);
0017 stairs(X, ybc,'LineWidth', LineWidth);
0018 title(['Binomial Cumulative distribution
Function for N = ' ...
0019 num2str(N) ' and p = ' num2str(P)]);
0020 xlabel('X (random variable)');
0021 ylabel('Prob[X <= k]'); grid
0022 % Geometric
0023 P = 0.5; X = [0:10];
0024 ygp = geopdf(X, P); % get pdf
0025 ygc = geocdf(X, P); % get cdf
0026 figure(3); set(gca,'FontSize', FontSize);
0027 bar(X, ygp);
0028 title(['Geometric Probability Mass Function for
p = ' num2str(P)]);
0029 xlabel('X (random variable)');
0030 ylabel('Prob[X = k]'); grid
0031 figure(4); set(gca,'FontSize', FontSize);
0032 stairs(X, ygc,'LineWidth', LineWidth);
0033 title(['Geometric Cumulative distribution
Function for p = ' num2str(P)]);
0034 xlabel('X (random variable)');
0035 ylabel('Prob[X <= k]'); grid
0036 % Poisson
0037 Lambda = 0.5; X = [0:10];
0038 ypp = poisspdf(X, Lambda);
0039 ypc = poisscdf(X, Lambda);
0040 figure(5); set(gca,'FontSize', FontSize);
0041 bar(X, ypp);
0042 title(['Poisson Probability Mass Function for
lambda = ' num2str(Lambda)]);
0043 xlabel('X (random variable)');
0044 ylabel('Prob[X = k]'); grid
0045 figure(6); set(gca,'FontSize', FontSize);
0046 stairs(X, ypc,'LineWidth', LineWidth);
0047 title(['Poisson Probability Mass Function for
lambda = ' num2str(Lambda)]);
0048 xlabel('X (random variable)');
0049 ylabel('Prob[X <= k]'); grid

```

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## Some Important Random Variables – Continuous Random Variables

- **Uniform**
- **Exponential**
- **Gaussian (Normal)**
- **Rayleigh**
- **Gamma**
- **M-Erlang**
- ....

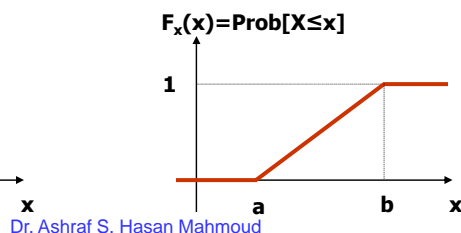
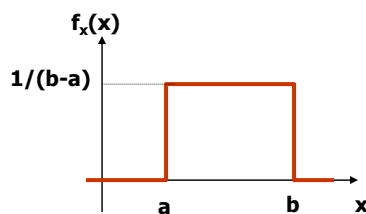
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## Uniform Random Variables

- Realizations of the r.v. can take values from the interval  $[a, b]$
- PDF  $f_x(x) = 1/(b-a) \quad a \leq x \leq b$
- $E[X] = (a+b)/2, \quad \text{Var}[X] = (b-a)^2/12$
- $\Phi_x(\omega) = [e^{j\omega b} - e^{j\omega a}]/(j\omega(b-a))$



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## Exponential Random Variables

- The exponential r.v.  $X$  with parameter  $\lambda$  has pdf

- And CDF given by
 
$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

- Range of  $X$ :  $[0, \infty)$

- $E[X] = 1/\lambda$ ,  $\text{Var}[X] = 1/\lambda^2$

- $\Phi_X(\omega) = \lambda/(\lambda - j\omega)$



This means:  
 $\text{Prob}[X \leq x] = 1 - e^{-\lambda x}$ , or  
 $\text{Prob}[X > x] = e^{-\lambda x}$

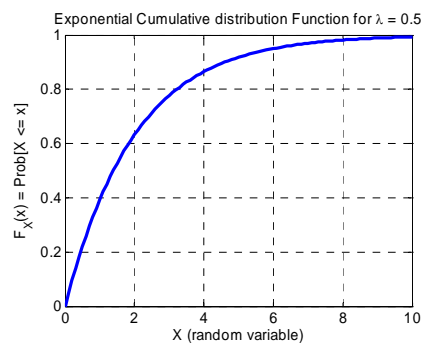
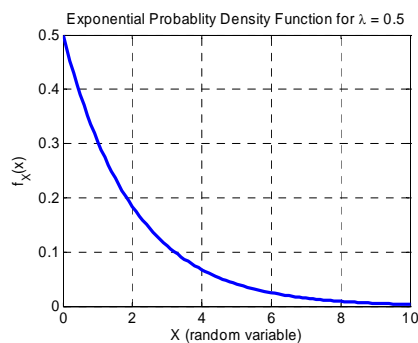
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## Exponential Random Variables – cont'd

- Example:
  - Note the mean is  $1/\lambda = 2$



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## Exponential Random Variables – Memoryless Property

- The exponential r.v. is the only continuous r.v. with the memoryless property!!
- Memoryless Property:  
 $P[X > t+h / X > t] = P[X > h]$

i.e. the probability of having to wait  $h$  additional seconds given that one has already been waiting  $t$  second IS EXACTLY equal to the probability of waiting  $h$  seconds when one first begins to wait

**Proof:**

$$\begin{aligned}
 P[X > t+h / X > t] &= \frac{P[(X > t+h) \cap (X > t)]}{P[X > t]} \\
 &= \frac{P[X > t+h]}{P[X > t]} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} \\
 &= e^{-\lambda h} \\
 &= P[X > h]
 \end{aligned}$$

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## Gaussian (Normal) Random Variable

- Rises in situations where a random variable  $X$  is the sum of a large number of "small" random variables – central limit theorem

- PDF  $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$

For  $-\infty < x < \infty$ ;  $\mu$  and  $\sigma > 0$  are real numbers

- $E[X] = \mu$ ,  $\text{Var}[X] = \sigma^2$
- $\Phi_X(\omega) = e^{j\mu\omega - \sigma^2\omega^2/2}$
- Under wide range of conditions  $X$  can be used to approximate the sum of a large number of independent random variables

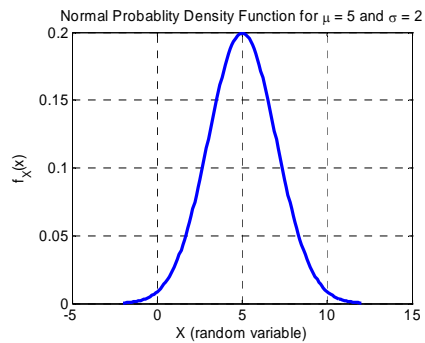
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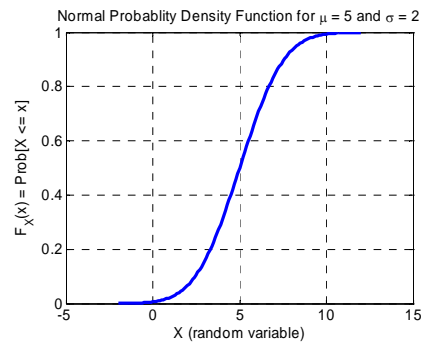
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## Gaussian (Normal) Random Variable – cont'd

- **Example:**



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## Rayleigh Random Variable

- **Arises in modeling of mobile channels**
- **Range:  $[0, \infty)$**

- **PDF:** 
$$f_X(x) = \frac{x}{\alpha^2} e^{-x^2/(2\alpha^2)}$$

- **For  $x \geq 0, \alpha > 0$**

- **$E[X] = \alpha\sqrt{\pi/2}, \text{ Var}[X] = (2-\pi/2)\alpha^2$**

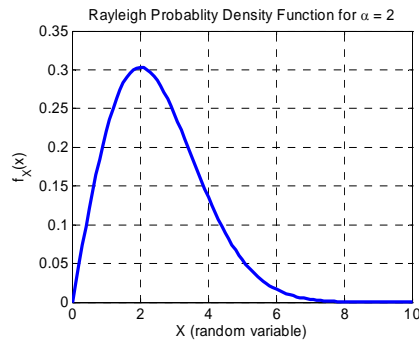
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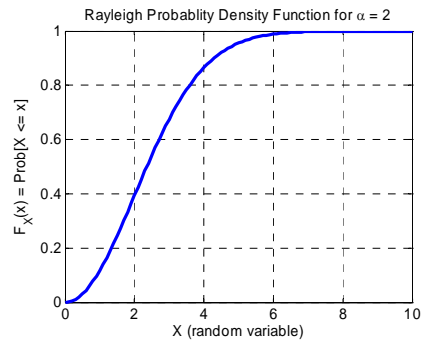
## Rayleigh Random Variable – cont'd

- **Example:**
  - Note that for  $\text{Alpha} = 2$ , the mean is  $2\sqrt{(\pi/2)}$



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## Matlab Code to Plot Distributions

```

0001 % plot distributions
0002 % see "help stats"
0003 clear all
0004 FontSize = 14;
0005 LineWidth = 3;
0006 % exponential
0007 X = [0:.1:10]; Lambda = 0.5;
0008 yep = exppdf(X, 1/Lambda); % get PDF
0009 yec = expcdf(X, 1/Lambda); % get CDF
0010 figure(1); set(gca,'FontSize', FontSize);
0011 plot(X, yep, 'LineWidth', LineWidth);
0012 title(['Exponential Probability Density
Function for \lambda = ' ...
0013 num2str(Lambda)]);
0014 xlabel('X (random variable)');
0015 ylabel('f_X(x)'); grid
0016 figure(2); set(gca,'FontSize', FontSize);
0017 plot(X, yec, 'LineWidth', LineWidth);
0018 title(['Exponential Cumulative Distribution
Function for \lambda = ' ...
0019 num2str(Lambda)]);
0020 xlabel('X (random variable)');
0021 ylabel('F_X(x) = Prob[X <= x]'); grid
0022 % normal
0023 X = [-2:.1:12]; Mu = 5; Sigma = 2;
0024 ynp = normpdf(X, Mu, Sigma); % get PDF
0025 ync = normcdf(X, Mu, Sigma); % get CDF
0026 figure(3); set(gca,'FontSize', FontSize);
0027 plot(X, ynp, 'LineWidth', LineWidth);
0028 title(['Normal Probability Density Function
for \mu = ' ...
0029 num2str(Mu) ' and \sigma = '
num2str(Sigma)]);
0030 xlabel('X (random variable)');
0031 ylabel('f_X(x)'); grid
0032 figure(4); set(gca,'FontSize', FontSize);
0033 plot(X, ync, 'LineWidth', LineWidth);
0034 title(['Normal Probability Density Function
for \mu = ' ...
0035 num2str(Mu) ' and \sigma = '
num2str(Sigma)]);
0036 xlabel('X (random variable)');
0037 ylabel('F_X(x) = Prob[X <= x]'); grid
0038 % Rayleigh
0039 X = [0:.1:10]; Alpha = 2;
0040 yrp = raylpdf(X, Alpha); % get PDF
0041 yrc = raylcdf(X, Alpha); % get CDF
0042 figure(5); set(gca,'FontSize', FontSize);
0043 plot(X, yrp, 'LineWidth', LineWidth);
0044 title(['Rayleigh Probability Density Function
for \alpha = ' ...
0045 num2str(Alpha)]);
0046 xlabel('X (random variable)');
0047 ylabel('f_X(x)'); grid
0048 figure(6); set(gca,'FontSize', FontSize);
0049 plot(X, yrc, 'LineWidth', LineWidth);
0050 title(['Rayleigh Probability Density Function
for \alpha = ' ...
0051 num2str(Alpha)]);
0052 xlabel('X (random variable)');
0053 ylabel('F_X(x) = Prob[X <= x]'); grid

```

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## Gamma Random Variable

- **Versatile distribution** ~ appears in modeling of lifetime of devices and systems
- **Has two parameters:**  $\alpha > 0$  and  $\lambda > 0$

- **PDF:** 
$$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

- **For**  $0 < x < \infty$
- **The quantity**  $\Gamma(z)$  **is the gamma function and is specified by**

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$$

- **The gamma function has the following properties:**

- $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(z+1) = z\Gamma(z)$  for  $z > 0$
- $\Gamma(m+1) = m!$  For  $m$  nonnegative integer
- $E[X] = \alpha/\lambda$ ,  $\text{Var}[X] = \alpha/\lambda^2$
- $\Phi_X(\omega) = 1/(1-j\omega/\lambda)^\alpha$

If  $\alpha = 1 \rightarrow$  gamma r.v. becomes exponential

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## Joint Distributions of Random Variables

- **Def: The joint probability distribution of two r.v.s X and Y is given by**

$$F_{XY}(x,y) = P(X \leq x, Y \leq y)$$

where  $x$  and  $y$  are real numbers.

- **This refers to the JOINT occurrence of**  $\{X \leq x\}$  **AND**  $\{Y \leq y\}$
- **Can be generalized to any number of variables**

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## Joint Distributions of Random Variables - Properties

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- $F_{XY}(-\infty, -\infty) = 0$
- $F_{XY}(\infty, \infty) = 1$
- $F_{XY}(x_1, y) \leq F_{XY}(x_2, y)$  for  $x_1 \leq x_2$
- $F_{XY}(x, y_1) \leq F_{XY}(x, y_2)$  for  $y_1 \leq y_2$
- **The marginal distributions are given by**
  - $F_X(x) = F_{XY}(x, \infty)$
  - $F_Y(y) = F_{XY}(\infty, y)$

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## Joint Distributions of Random Variables - Properties - 2

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- **Density function:**  $f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$
- **or**  $F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(\alpha, \beta) d\alpha d\beta$
- **Marginal densities:**  $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$
- **and**  $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$

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## Joint Distributions of Random Variables – Independence

- **Two random variables are independent if the joint distribution functions are products of the marginal distributions:**

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

or

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

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## Joint Distributions of Random Variables – Discrete Nonnegative Variables

- **Def:**

$$F_{XY}(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P(X = x_i, Y = y_j)U(x - x_i)U(y - y_j)$$

where

**$P(X=x_i, Y=y_j)$  is the joint probability for the r.v.s X and Y**

**$U(x)$  is 1 for  $x \geq 0$  and 0 otherwise**

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## Example 4: Packet Segmentation

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- **Problem:** The number of bytes  $N$  in a message has a geometric distribution with parameter  $p$ . The message is broken into packets of maximum length  $M$  bytes. Let  $Q$  be the number of full packets in a message and let  $R$  be the number of bytes left over.  
A) Find the joint pmf for  $Q$  and  $R$ , and  
B) Find the marginal pmfs of  $Q$  and  $R$ .

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## Example 4: Packet Segmentation - cont'd

---

- **Solution:**

$$N \sim \text{geometric} \rightarrow P(N=k) = (1-p)p^k$$

Message of  $N$  bytes  $\rightarrow$   $Q$  full  $M$ -bytes packets +  
 $R$  remaining bytes

Therefore:  $Q \in \{0, 1, 2, \dots\}$ ,  $R \in \{0, 1, 2, \dots, M-1\}$

The joint pmf is given by:

$$P(Q=q, R=r) = P(N = qM+r) = (1-p)p^{(qM+r)}$$

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## Example 4: cont'd

- **Solution:**

**The marginal pmfs:**

$$\begin{aligned}P(Q = q) &= \sum_{r=0}^{M-1} P(Q = q, R = r) \\ &= \sum_{r=0}^{M-1} (1-p)p^{(qM+r)} \\ &= (1-p^M)(p^M)^q \quad q = 0, 1, 2, \dots\end{aligned}$$

**and**

$$\begin{aligned}P(R = r) &= \sum_{q=0}^{\infty} P(Q = q, R = r) \\ &= \sum_{q=0}^{\infty} (1-p)p^{(qM+r)} \\ &= \frac{(1-p)}{1-p^M} p^r \quad r = 0, 1, \dots, M-1\end{aligned}$$

Verify the marginal pmfs add to ONE!!  
P(R = r) is a truncated geometric r.v.

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## Independent Discrete R.V.s

- **For Discrete random variables:**

$$\mathbf{P(M=i, N=j) = P(M=i) P(N=j)}$$

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## Example 5:

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- **Problem: Are the Q and R random variables of Previous Example independent? Why?**

## Conditional Distributions

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- **Def: for continuous X and Y**

**Or**

$$F_{Y/X}(y/x) = P(Y \leq y / X \leq x) = \frac{F_{XY}(x, y)}{F_X(x)}$$
$$f_{Y/X}(y/x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

- **For discrete M and N**

$$P(M = i / N = j) = \frac{P(M = i, N = j)}{P(N = j)}$$

## Conditional Distributions - 2

- **For mixed types:**

$$\begin{aligned}F_X(x) &= \sum_{i=0}^{\infty} P(N = j, X \leq x) \\ &= \sum_{j=0}^{\infty} P(N = j)P(X \leq x / N = j)\end{aligned}$$

**or**

$$P(N = j) = \int_{-\infty}^{\infty} P(N = j / X = x) f_X(x) dx$$

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## Conditional Distributions - 3

- **For N random variables:**

$$\begin{aligned}F_{X_1, X_2, \dots, X_N}(x_1, x_2, \dots, x_N) &= \\ &= F_{X_1}(x_1) \times F_{X_2/X_1}(x_2/x_1) \times \dots \times F_{X_N/X_1, \dots, X_{N-1}}(x_N/x_1, \dots, x_{N-1})\end{aligned}$$

**or**

$$\begin{aligned}f_{X_1, X_2, \dots, X_N}(x_1, x_2, \dots, x_N) &= \\ &= f_{X_1}(x_1) \times f_{X_2/X_1}(x_2/x_1) \times \dots \times f_{X_N/X_1, \dots, X_{N-1}}(x_N/x_1, \dots, x_{N-1})\end{aligned}$$

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## Example 6:

- **Problem:** The number of customers that arrive at a service station during a time  $t$  is a Poisson random variable with parameter  $\beta t$ . The time required to service each customer is exponentially distributed with parameter  $\alpha$ . Find the pmf for the number of customers  $N$  that arrive during the service time  $T$  of a specific customer. Assume the customer arrivals are independent of the customer service time.

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## Example 6: cont'd

- **Solution:**  
The PDF for  $T$  is given by  $f_T(t) = \alpha e^{-\alpha t} \quad t \geq 0$   
Let  $N =$  number of arrivals during time  $t$   
→ the arrivals conditional pmf is given by

$$P(N = j | T = t) = \frac{(\beta t)^j e^{-\beta t}}{j!} \quad j = 0, 1, \dots \quad t \geq 0$$

To find the arrivals pmf during service time  $T$ , we use:

$$P(N = j) = \int_{-\infty}^{\infty} P(N = j | T = t) f_T(t) dt$$

this reduces to:

$$= \int_0^{\infty} \alpha \frac{(\beta t)^j}{j!} e^{-\alpha t} e^{-\beta t} dt$$

$$P(N = j) = \left( \frac{\alpha}{\alpha + \beta} \right) \left( \frac{\beta}{\alpha + \beta} \right)^j \quad j = 0, 1, \dots$$

**Note that:**

$$\Gamma(j+1) = \int_0^{\infty} t^j e^{-t} dt = j!$$

Thus  $N$  is geometrically distributed with probability of success equal to  $\alpha / (\alpha + \beta)$

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This is NOT a thorough treatment of the subject. For a fair Treatment of the subject please refer to the textbook or to Garcia's textbook

## Markov Processes

- **Brief Introduction into Stochastic/Random Processes**
- **A random process  $X(t)$  is a Markov Process if the future of the process given the present is independent of the past.**

- **For arbitrary times:  $t_1 < t_2 < \dots < t_k < t_{k+1}$**  Past or History
- $$\begin{aligned} \text{Prob}[X(t_{k+1}) = x_{k+1} / X(t_k) = x_k, X(t_{k-1}) = x_{k-1}, \dots, X(t_1) = x_1] \\ = \text{Prob}[X(t_{k+1}) = x_{k+1} / X(t_k) = x_k] \end{aligned}$$

for discrete-valued, or

$$\begin{aligned} \text{Prob}[a < X(t_{k+1}) \leq b / X(t_k) = x_k, \dots, X(t_1) = x_1] \\ = \text{Prob}[a < X(t_{k+1}) \leq b / X(t_k) = x_k] \end{aligned}$$

for continuous-valued

**Markov Property**

Markov  $\equiv$  Memoryless

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## Continuous-Time Markov Chain

- **An integer-valued Markov random process is called a Markov Chain**
- **The joint pmf for  $k+1$  arbitrary time instances is given by:**

$$\text{Prob}[X(t_{k+1}) = x_{k+1}, X(t_k) = x_k, \dots, X(t_1) = x_1]$$

$$\begin{aligned} = & \text{Prob}[X(t_{k+1}) = x_{k+1} / X(t_k) = x_k] \times \\ & \text{Prob}[X(t_k) = x_k / X(t_{k-1}) = x_{k-1}] \times \\ & \dots \\ & \text{Prob}[X(t_2) = x_2 / X(t_1) = x_1] \times \\ & \text{Prob}[X(t_1) = x_1] \end{aligned}$$

transition probabilities

← pmf of the initial time

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## Discrete-Time Markov Chains

- Let  $X_n$  be a discrete-time integer values Markov Chain that starts at  $n = 0$  with pmf

$$p_j(0) = \text{Prob}[X_0 = j] \quad j=0,1,2, \dots$$

$$\begin{aligned} & \text{Prob}[X_n=i_n, X_{n-1}=i_{n-1}, \dots, X_0=i_0] \\ &= \text{Prob}[X_n=i_n / X_{n-1}=i_{n-1}] \times \\ & \quad \text{Prob}[X_{n-1}=i_{n-1} / X_{n-2}=i_{n-2}] \times \\ & \quad \dots \\ & \quad \text{Prob}[X_1=i_1 / X_0=i_0] \times \\ & \quad \text{Prob}[X_0=i_0] \end{aligned}$$

Same as the previous slide  
but for discrete-time

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## Discrete-Time Markov Chains – cont'd (2)

- Assume the one-step state transition probabilities are fixed and do not change with time:

$$\text{Prob}[X_{n+1}=j / X_n=i] = p_{ij} \quad \text{for all } n$$

→  $X_n$  is said to be homogeneous in time

- The joint pmf for  $X_n, X_{n-1}, \dots, X_1, X_0$  is then given by

$$\begin{aligned} & P[X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0] \\ &= p_{i_{n-1}, i_n} \times p_{i_{n-2}, i_{n-1}} \times \dots \times p_{i_0, i_1} \times p_{i_0}(0) \end{aligned}$$

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## Discrete-Time Markov Chains – cont'd (3)

- Thus  $X_n$  is completely specified by the initial pmf  $p_i(0)$  and the matrix of one-step transition probabilities  $P$ :

$$P = \begin{bmatrix} P_{00} & P_{01} & \dots & P_{0i} & P_{0i+1} & \dots \\ P_{10} & P_{11} & \dots & P_{1i} & P_{1i+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ P_{i0} & P_{i1} & \dots & P_{ii} & P_{i,i+1} & \dots \\ P_{i+1,0} & P_{i+1,1} & \dots & P_{i+1,i} & P_{i+1,i+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

1-Step Transition Matrix, P  
(Transition Probabilities)

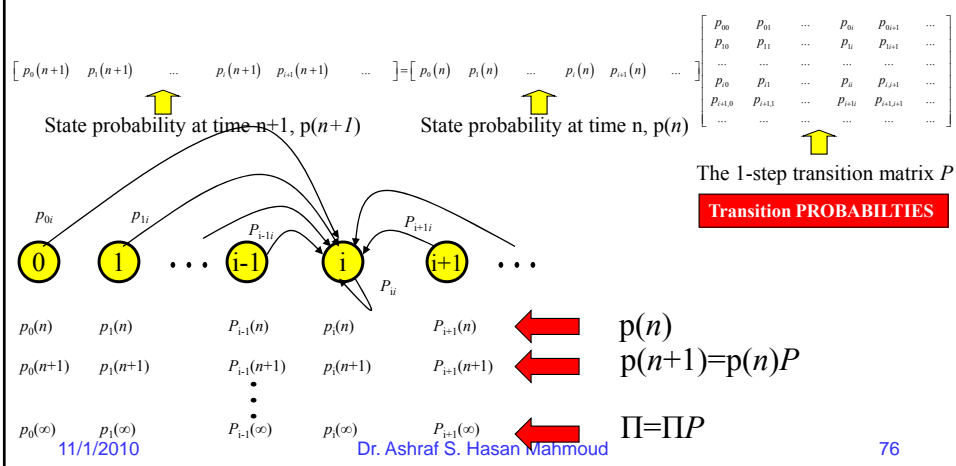
i.e. rows of P add to UNITY

$$1 = \sum_j P[X_{n+1} = j / X_n = i] = \sum_j p_{ij}$$

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## Discrete-Time Markov Chains – cont'd (4)

- The state probability at time  $n+1$  is related to the state probabilities at time  $n$  as follows:



## Discrete-Time Markov Chains – cont'd (5)

- Therefore, the vector  $p(n)$  representing the state probabilities at  $n$  is given by

$$p(n) = p(n-1) P$$

Remember  $P$  is the 1-step transition matrix

- The above also means that one can write

$$p(n) = p(0) P^n$$

Where  $P^n$  ( $P$  raised to the power  $n$ ) is the  $n$ -step transition matrix

- Finally, the steady state distribution for the system,  $\Pi$ , is given by

$$\Pi = \Pi P$$

$\Pi \rightarrow$  is the steady state pmf

$P \rightarrow$  is the 1-step transition matrix

- This means at steady state – the state probabilities DO NOT change!

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## Markov Process versus Chains

- **Continuous-Time Markov Process**
- **Continuous-Time Markov Chain**
- **Discrete-Time Markov Process**
- **Discrete-Time Markov Chain**
  
- **Process versus Chain  $\rightarrow$  refers to the value of  $X(t)$**
- **Continuous-time versus Discrete-time  $\rightarrow$  refers to the instant when the variable (process)  $X(t)$  change**

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## Markov Chains Models

- **Model for an integer-values process**
  - Buffer size
  - No of customers
- **When change in process values occur at arbitrary (continuous) time values → continuous time Markov chains**
  - Length of queue at the bank teller – customer arrivals happen at any time instant
  - Size of input buffer of a router – packet arrivals at the port happen at any time instant
- **When change in process values occur at specific (discrete) time values → discrete-time Markov chains**
  - The buffer size of a TDM multi-channel multiplexer – packet arrivals are restricted to slots (i.e. time-axis is slotted)

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### Continuous-Time Markov Chain

## Examples 7: Continuous-Time Random Processes – Poisson Process

- **Problem:** Assume events (e.g. arrivals) occur at rate of  $\lambda$  events per second following a Poisson arrival process. Let  $N(t)$  be the number of occurrences in the interval  $[0,t]$
- Plot multiple realization of  $N(t)$
  - Write the pmf for  $N(t)$
  - Show that  $N(t)$  is a Markov chain

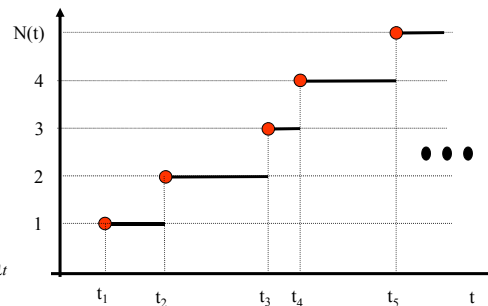
A)  $N(t)$  is non-decreasing integer-valued continuous-time random process – A plot for one realization is shown in figure – for other plots, choose different  $t_1, t_2, t_3, \dots$

Note the increments on Y-axis are in steps of 1 – while the arrival instants  $t_i, i=1, 2, \dots$  are random

B) pmf for  $N(t)$  is given by

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

for  $k=0,1, \dots$



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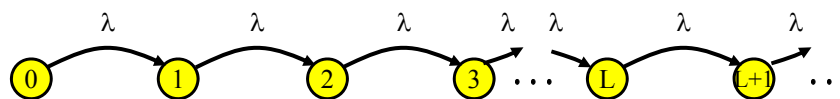
## Example 7: cont'd

- We can show that  $N(t)$  has:
  - Independent increments
  - Stationary increments – the distribution for the number of event occurrences in *ANY* interval of length  $t$  is given by the previous pmf.

Example:  $P(N(t_1)=i, N(t_2)=j) =$   
 $= P(N(t_1)=i)P(N(t_2)-N(t_1)=j-i)$   
 $= P(N(t_1)=i)P(N(t_2-t_1)=j-i)$

$$= \frac{(\lambda t_1)^i e^{-\lambda t_1} (\lambda(t_2-t_1))^{j-i} e^{-\lambda(t_2-t_1)}}{i! (j-i)!}$$

- If we select the value of  $N(t)$  as the STATE variable – one can draw the equivalent Markov model (below) – pure birth process



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## Example 8: Poisson Arrival Process is EQUIVALENT to Exponential Interarrival Times

- Problem:**  $N(t)$  is a Poisson arrival process – Show that  $T$ , the time between event occurrences is exponentially distributed

- Solution:**  
pmf of  $N(t)$  is given by

$$P(N(t)=k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad k = 0, 1, \dots$$

$$P(T > t) = P(\text{ZERO events in } t \text{ seconds})$$

$$= e^{-\lambda t}$$

Therefore,  $P(T \leq t) = F_T(t) = 1 - e^{-\lambda t}$  – i.e.  $T$  is exponentially distributed with mean  $1/\lambda$

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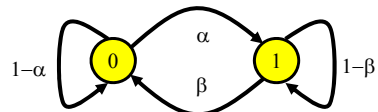
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## Example 9: Speech Activity Model

**Problem:** A Markov model for packet speech assumes that if the  $n$ th packet contains silence then the probability of silence in the next packet is  $1-\alpha$  and the probability of speech activity is  $\alpha$ . Similarly if the  $n$ th packet contains speech activity, then the probability of speech activity in next packet is  $1-\beta$  and the probability of silence is  $\beta$ . Find the stationary state pmf.

**Solution:**

The state diagram is as shown:



State 0: silence  
State 1: speech

The 1-step transition probability,  $P$ , is given by:

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

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## Example 9: cont'd 2

**Answer:** The steady state pmf  $\Pi = [\pi_0 \ \pi_1]$  can be solved for using

$$\Pi = \Pi P$$

Or

$$[\pi_0 \ \pi_1] = [\pi_0 \ \pi_1] \times \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Or

$$\begin{aligned} \pi_0 &= (1-\alpha) \pi_0 + \beta \pi_1 \\ \pi_1 &= \alpha \pi_0 + (1-\beta) \pi_1 \end{aligned}$$

In addition to the constraint that  $\pi_0 + \pi_1 = 1$

Therefore steady state pmf

$\Pi = [\pi_0 \ \pi_1]$  is given by:

$$\begin{aligned} \pi_0 &= \beta / (\alpha + \beta) \\ \pi_1 &= \alpha / (\alpha + \beta) \end{aligned}$$

Note that sum of all  $\pi_i$ 's should equal to 1!!

For  $\alpha = 1/10$ ,  $\beta = 1/5 \rightarrow \Pi = [2/3 \ 1/3]$  – Refer to the matlab code to check convergence!!

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## Example 9: cont'd

**Answer:** Alternatively, one can find a general form for  $P^n$  and take the limit as  $n \rightarrow \infty$ .

$P^n$  can be shown to be:

$$P^n = \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + \frac{(1 - \alpha - \beta)^n}{\alpha + \beta} \begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix}$$

Which clearly approaches:

$$\lim_{n \rightarrow \infty} P^n = \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}$$

If the initial state pmf is  $p_0(0)$  and  $p_1(0) = 1 - p_0(0)$

Then the  $n^{\text{th}}$  state pmf ( $n \rightarrow \infty$ ) is given by:

$$p(n) \text{ as } n \rightarrow \infty = [p_0(0) \ 1 - p_0(0)] P^n \\ = [\beta / (\alpha + \beta) \ \alpha / (\alpha + \beta)]$$

Same as the solution obtained using the 1-step transition matrix!!

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## Example 9: cont'd

- This shows a simple Matlab code to determine  $p(n)$  for  $n=1, 2, 3, \dots$  given the 1-step probability matrix  $P$  and the initial condition  $p(0)$
- Student must be convinced that the steady state distribution, if it exists, does not depend on  $p(0)$  but solely on  $P$ .

The rest is for initialization and presentation

```
0001 clear all
0002 LineWidth = 2; MarkerSize = 8;
0003 FontSize = 14;
0004 % 1-step probability transition matrix P
0005 Alpha = 1/10; Beta = 1/5;
0006 P = [ 1-Alpha Alpha; Beta 1-Beta];
0007 % Set initial probability state distribution p(0)
0008 p_0 = [1 0]; % i.e. system starts in 0
0009 N = 11; % how many steps to predict (simulate)
0010 p_n = zeros(N,2); % p_n evolution of distribution
0011 p_n(1,:) = p_0; % insert initial condition
0012 for n=2:N % the main loop in the code
0013 % find the state probability distribution after 1-step
0014 p_n(n,:) = p_n(n-1,:) * P;
0015 end
0016 % compare with analytical result - refer to class slides
0017 Pi_vector = [Beta Alpha]/(Beta + Alpha);
0018 % Show results graphically - JUST FOR PRESENTATION
0019 n = 0:N-1; % define the x-axis for plotting
0020 figure(1);
0021 h = plot(n, p_n(:,1),'-xb', ... % for state 0
0022 n, p_n(:,2),'-dr', ... % for state 1
0023 n, Pi_vector(1)*ones(size(n)), '-b', ...
0024 n, Pi_vector(2)*ones(size(n)), '-r');
0025 set(h, 'LineWidth', LineWidth, 'MarkerSize', MarkerSize);
0026 set(gca, 'FontSize', FontSize);
0027 title(['Two state on/off discrete-time Markov chain']);
0028 ['\alpha = ' num2str(Alpha) ' and \beta = ' num2str(Beta) ...
0029 ' Initial condition p(0) = [' num2str(p_0(1)) ' ', ' num2str(p_0(2)) ' ]'];
0030 xlabel('time index, n');
0031 ylabel('state probability distribution');
0032 hl = legend('state 0 evolution', 'state 1 evolution', ...
0033 'state 0 steady state', 'state 1 steady state');
0034 grid; set(hl, 'FontSize', 10); % for legend only
```

- Program Execution and results:

```
>> SimpleONOFFMarkovChain
>> p_n

p_n =

    1.0000    0.0000
    0.9000    0.1000
    0.8300    0.1700
    0.7810    0.2190
    0.7467    0.2533
    0.7227    0.2773
    0.7059    0.2941
    0.6941    0.3059
    0.6859    0.3141
    0.6801    0.3199
    0.6761    0.3239

>> Pi_vector

Pi_vector =

    0.6667    0.3333

>>
```

You can see that p(n) converges!

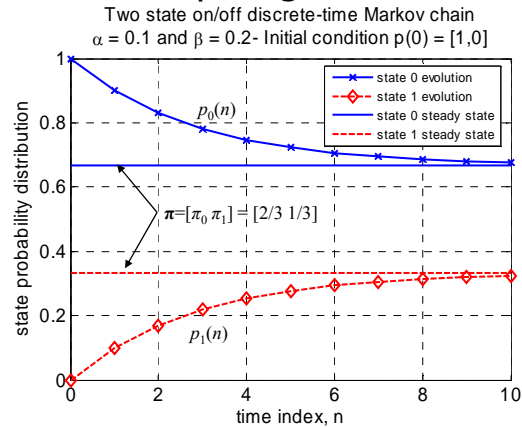
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## Example 9: cont'd

- Plotting the state probability distribution  $p(n)$  as a function of time & comparing with analytical result



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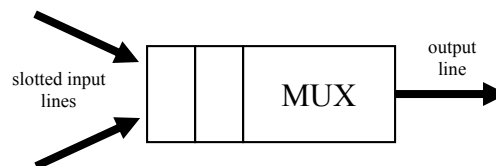
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## Discrete-Time Markov Chain

## Example 10: Multiplexer

**Problem:** Data in the form of fixed-length packets arrive in slots on both of the input lines of a multiplexer. A slot contains a packet with probability  $p$ , independent of the arrivals during other slots or on the other line. The multiplexer transmits one packet per time slot and has the capacity to store two messages only. If no room for a packet is found, the packet is dropped.

- Draw the state diagram and define the matrix  $P$
- Compute the throughput of the multiplexer for  $p = 0.3$



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## Example 10: Multiplexer – cont'd

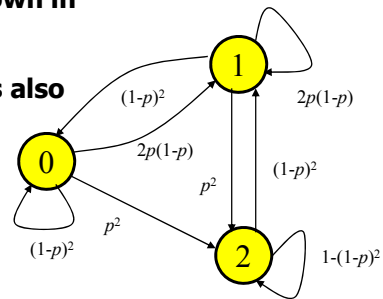
**Solution:** In any slot time, the arrivals pmf is given by

$$P(j \text{ cells arrive}) = \begin{matrix} (1-p)^2 & j=0 \\ 2p(1-p) & j=1 \\ p^2 & j=2 \end{matrix}$$

Let the state be the number of packets in the buffer, then the state diagram is shown in figure.

The corresponding transition matrix is also given below

$$P = \begin{bmatrix} (1-p)^2 & 2p(1-p) & p^2 \\ (1-p)^2 & 2p(1-p) & p^2 \\ 0 & (1-p)^2 & 1-(1-p)^2 \end{bmatrix}$$



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## Example 10: Multiplexer – cont'd

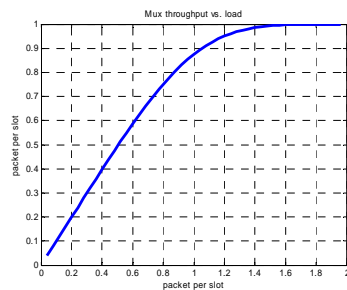
**Solution-cont'd:**

**Load:** average arrivals =  $2p$  packets/slot

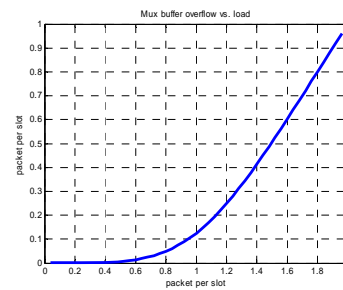
**Throughput:**  $\pi_1 + \pi_2$

**Buffer overflow** = Prob(two packet arrivals while in state 2)  
 = Prob(two arrivals)  $\times \pi_2$   
 =  $p^2 \pi_2$

The graphs below show the relation of load versus –throughput and buffer overflow for the MUX



S.



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## Example 10: Multiplexer – cont'd

### Solution-cont'd:

The matlab code used for plotted previous results is shown below.

Make sure you understand the matrix formulation and the solution for the steady state probability vector  $\mathbf{n}$

```
clear all
Step = 0.02;
ArrivalProb = [Step:Step:1-Step];
A = zeros(4,3);
E = zeros(4,1);
E(4) = 1;
for i=1:length(ArrivalProb)
    p = ArrivalProb(i);
    P = [(1-p)^2 2*p*(1-p) p^2; ...
         (1-p)^2 2*p*(1-p) p^2; ...
         0 (1-p)^2 1-(1-p)^2];
    A(1:3,:) = (P - eye(3))';
    A(4,:) = ones(1,3);
    E(4) = 1;
    SteadyStateP = A\E;
    % Prob(packet is lost) = Prob(2 arrivals) X
    % Prob(being in state 2);
    DropProb(i) = p^2*SteadyStateP(3);
    Throughput(i) = sum(SteadyStateP(2:3));
end
```

```
% matlab code continued
figure(1),
h = plot(2*ArrivalProb, Throughput);
set(h, 'LineWidth', 3);
title('Mux throughput vs. load');
ylabel('packet per slot');
xlabel('packet per slot');
grid
figure(2),
h = plot(2*ArrivalProb, DropProb);
set(h, 'LineWidth', 3);
title('Mux buffer overflow vs. load');
ylabel('packet per slot');
xlabel('packet per slot');
grid
```

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## Continuous-Time Markov Chains -Steady State Probabilities and Global Balance Equations

- What relation govern the state probabilities for continuous-time Markov chains?
  - Remember that probability for state  $i$ ,  $p_i(t)$ , is now a function of time!
- The answer is given by Remember Chapman-Kolmogorov equations:

$$p'_j(t) = \sum_i \gamma_{i,j} p_i(t) \quad \text{for all } j$$

Or in matrix form:  $P'(t) = P(t)\Gamma$ ,  
where  $P(t) = [p_0(t), p_1(t), \dots, p_j(t), \dots]$

$$\Gamma = \begin{bmatrix} \gamma_{0,0} & \gamma_{0,1} & \dots & \gamma_{0,j} & \gamma_{0,j+1} & \dots \\ \gamma_{1,0} & \gamma_{1,1} & \dots & \gamma_{1,j} & \gamma_{1,j+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{j,0} & \gamma_{j,1} & \dots & \gamma_{j,j} & \gamma_{j,j+1} & \dots \\ \gamma_{j+1,0} & \gamma_{j+1,1} & \dots & \gamma_{j+1,j} & \gamma_{j+1,j+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} p'_0(t) & p'_1(t) & \dots & p'_j(t) & p'_{j+1}(t) & \dots \end{bmatrix} = \begin{bmatrix} p_0(t) & p_1(t) & \dots & p_j(t) & p_{j+1}(t) & \dots \end{bmatrix} \begin{bmatrix} \gamma_{0,0} & \gamma_{0,1} & \dots & \gamma_{0,j} & \gamma_{0,j+1} & \dots \\ \gamma_{1,0} & \gamma_{1,1} & \dots & \gamma_{1,j} & \gamma_{1,j+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{j,0} & \gamma_{j,1} & \dots & \gamma_{j,j} & \gamma_{j,j+1} & \dots \\ \gamma_{j+1,0} & \gamma_{j+1,1} & \dots & \gamma_{j+1,j} & \gamma_{j+1,j+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

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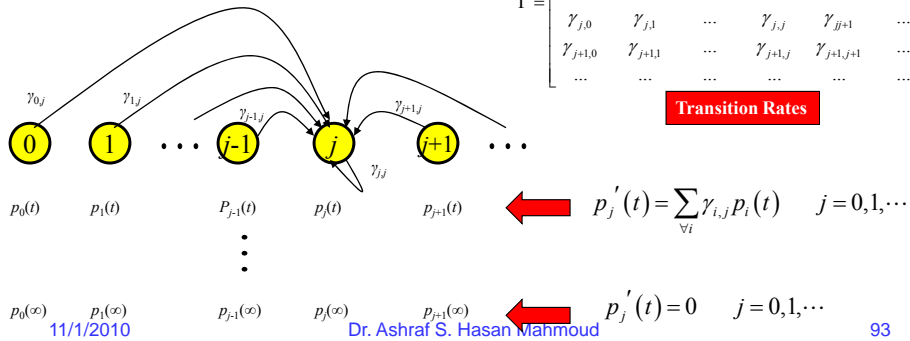
## Continuous-Time Markov Chains -Steady State Probabilities and Global Balance Equations (2)

- What relation govern the state probabilities for continuous-time Markov chains?
  - Remember that probability for state  $i$ ,  $p_i(t)$ , is now a function of time!
- The answer is given by Remember Chapman-Kolmogorov equations:

$$p'_j(t) = \sum_i \gamma_{ij} p_i(t) \quad \text{for all } j$$

Or in matrix form:  $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{\Gamma}$ ,  
 where  $\mathbf{P}(t) = [p_0(t), p_1(t), \dots, p_j(t), \dots]$

$$\mathbf{\Gamma} = \begin{bmatrix} \gamma_{0,0} & \gamma_{0,1} & \dots & \gamma_{0,j} & \gamma_{0,j+1} & \dots \\ \gamma_{1,0} & \gamma_{1,1} & \dots & \gamma_{1,j} & \gamma_{1,j+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{j,0} & \gamma_{j,1} & \dots & \gamma_{j,j} & \gamma_{j,j+1} & \dots \\ \gamma_{j+1,0} & \gamma_{j+1,1} & \dots & \gamma_{j+1,j} & \gamma_{j+1,j+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$



## Continuous-Time Markov Chains -Steady State Probabilities and Global Balance Equations (3)

- If equilibrium exists, then  $p'_j(t) = 0$  (i.e. no change in the state probabilities with time)
- Therefore, at steady state (if it exists), the following holds:

$$0 = \sum_i \gamma_{ij} p_i(t) \quad \text{for all } j$$

- These are referred to as the **GLOBAL BALANCE EQUATIONS!!**
- All flows (rate X probability) algebraically added for any state  $j$  equal to ZERO

- It is given that  $\gamma_{01}$  is  $\alpha$ , since the sum of row entries should be 0  $\rightarrow \gamma_{00}$  is  $-\alpha$   
 - Same for the row corresponding to state 1.  
**By Definition:**  
 $\gamma_{ii}$  = - sum of all exit rates from state  $i$

## Example 11:

- **Problem:** Consider the queuing system in Example 9 – find the steady state probabilities.

- **Answer:**

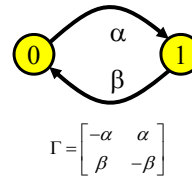
$$\begin{aligned} \gamma_{00} &= -\alpha & \gamma_{01} &= \alpha \\ \gamma_{10} &= \beta & \gamma_{11} &= -\beta \end{aligned}$$

$$[0 \ 0] = [\pi_0 \ \pi_1] \times \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

Applying the global balance equations, yields

$$\alpha\pi_0 = \beta\pi_1 \quad \text{and} \quad \beta\pi_1 = \alpha\pi_0$$

In addition to the constraints that:  $\pi_0 + \pi_1 = 1$



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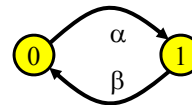
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## Example 11: cont'd

- **Answer:** Solving the previous simple equations leads to:

$$\pi_0 = \beta / (\alpha + \beta)$$

$$\pi_1 = \alpha / (\alpha + \beta)$$



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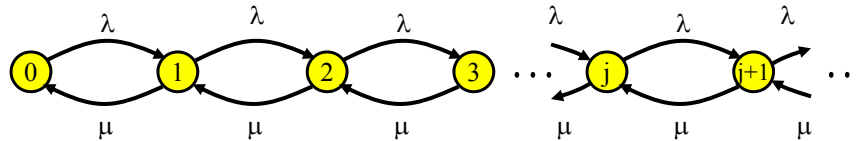
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## Example 12:

- **Problem:** The M/M/1 single-server queueing system



The corresponding rate transition matrix is given by

$$\Gamma = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -(\lambda + \mu) & \lambda & 0 & \dots \\ 0 & \mu & -(\lambda + \mu) & \lambda & \dots \\ 0 & 0 & \mu & -(\lambda + \mu) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

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## Example 12: cont'd

- **Answer:** The state transition rates:
  - Customers arrive with rate  $\lambda \rightarrow \gamma_{i,i+1} = \lambda$  for  $i = 0, 1, 2, \dots$
  - When system is not empty, customers depart at rate  $\mu \rightarrow \gamma_{i,i-1} = \mu$  for  $i = 1, 2, 3, \dots$
- The global balance equations:
 
$$\lambda p_0 = \mu p_1 \quad \text{for } j = 0$$

$$(\lambda + \mu)p_j = \lambda p_{j-1} + \mu p_{j+1} \quad \text{for } j = 1, 2, \dots$$
- $\rightarrow \lambda p_j - \mu p_{j+1} = \lambda p_{j-1} - \mu p_j$  for  $j = 1, 2, \dots$   
 $\quad \quad \quad = \text{constant}$

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## Example 12: cont'd

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- **Answer:**

For  $j = 1$ , we have

$$\lambda p_0 - \mu p_1 = \text{constant}$$

Therefore the constant is equal to zero.

Hence,

$$\begin{aligned} \lambda p_{j-1} &= \mu p_j \text{ or} \\ p_j &= (\lambda/\mu) p_{j-1} \text{ for } j=1,2, \dots \end{aligned}$$

By simple induction:

$$p_j = \rho^j p_0$$

where  $\rho = \lambda/\mu$

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## Example 12: cont'd

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- **Answer:**

To obtain  $p_0$ , we use the fact that

$$1 = \sum_j p_j = (1 + \rho + \rho^2 + \dots) p_0$$

note the above series converges only for  $\rho < 1$  or equivalently  $\lambda < \mu$

Therefore,  $p_0 = 1 - \rho$

In general, the steady state pmf for the M/M/1 queue is given by

$$p_j = (1 - \rho) \rho^j$$

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## References

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- **Alberto Leon-Garcia, Probability and Random Processes for Electrical Engineering, Addison Wesley, 1989**