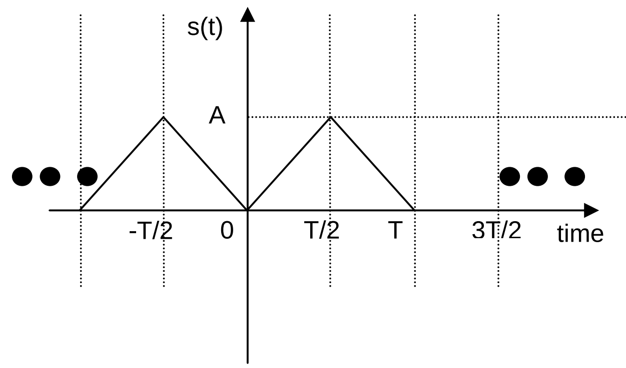
KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
COLLEGE OF COMPUTER SCIENCES & ENGINEERING
COMPUTER ENGINEERING DEPARTMENT

COE-341 – Data and Computer Communication

Handout #1: Fourier Series Expansion & Filtering

Consider the function shown in the figure.



a) Write a mathematical representation for s(t)?

b) What is the period of the function s(t)?

c) What is the fundamental frequency for s(t)?

d) Compute the DC component of s(t)?

e) Does it contain lower frequencies? What is(are) these?

f) Does it contain higher frequencies?

g) Compute the power of s(t)?

h) Find the Fourier series expansion of s(t)

i) What is the bandwidth of s(t)?

j) Specify the terms containing frequencies lower than the fundamental frequency and those containing frequencies higher than the fundamental frequency.

k) Compute the power using the Fourier Series expansion and show that is it equal to that obtained in part (g)

l) s(t) has infinite bandwidth (line spectrum) and it is required to truncate it such that it has a limited bandwidth but still has 95% of the original power. What terms of the original series expansion should be included?

m) What is the new bandwidth of the new truncated series?

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a) For the mathematical expression of s(t) it is enough to specify it in *any one period*; this is because s(t-T) = s(t) for any t. One can chose to write the expression describing the function for $t \in (-T/2, T/2)$, or alternatively for $t \in (0, T)$.

Writing the expression using $t \in (0, T)$

$$s(t) = \begin{cases} 2At/T & 0 < t < T/2 \\ 2A(1 - t/T) & T/2 < t < T \end{cases}$$

Verification:

It is always useful to verify your expression by substituting the end points and checking with the original graph of s(t)

s(t=0) = 0 ➔ matches the graph

s(t=T/2) = 2*A*(T/2)/T = A ➔ matches the graph

s(t=T/2) = 2*A*(1-(T/2)/T) = 2*A*(1-1/2) = A ➔ matches the graph

s(t=T) = 2*A*(1-T/T) = 0 ➔ matches the graph

Writing the expression using $t \in (-T/2, T/2)$

$$s(t) = \begin{cases} -2At/T & -T/2 < t < 0 \\ 2At/T & 0 < t < T/2 \end{cases}$$

Verification:

s(t=-T/2) = -2*A*(-T/2)/T = A ➔ matches the graph

s(t=0) = 0 ➔ matches the graph

s(t=T/2) = 2*A*(1-(T/2)/T) = 2*A*(1-1/2) = A ➔ matches the graph

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b) The period of the function s(t) is the time span after which the function repeats itself. For the given function, the period is equal to T (in time units)

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c) The fundamental frequency (f in Hz) is the reciprocal of the period duration (T in seconds). That is $f = 1/T$

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d) The DC component of the function s(t) is the given by averaging the function over one period. Hence, the DC component is equal to

$$DCcomponent = \frac{1}{T}\int_0^T s(t) \quad dt = \frac{1}{T}\int_{-T/2}^{T/2} s(t) \quad dt$$

Note that again once can choose whether to consider the period $t \in (0, T)$ or $t \in (-T/2, T/2)$. For the remaining of this exercise, we will consider the period $t \in (-T/2, T/2)$ because it will result in less integral terms when we are going to compute the Fourier series expansion.

The DC component is given by $\frac{1}{T}\int_{-T/2}^{T/2} s(t) \quad dt = \frac{2}{T}\int_0^{T/2} 2At/T \quad dt = \frac{4A}{T^2}\frac{t^2}{2}\Big|_{t=0}^{t=T/2} = \frac{A}{2}$. In other words, the DC component is equal to the area of curve for one period divided by the period duration (T).

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e) Yes, the function s(t) contains a DC component (frequency equal to zero) which is lower than the fundamental frequency (f = 1/T)

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f) Yes, the function s(t) contains frequencies higher than the fundamental frequency; this is because of the sharp edges (at t = …, -T, -T/2, 0, T/2, T, …). If s(t) had *only* the one frequency f = 1/T, it would have been a continuous time sinusoid (such as A*cos(2pi*f*t) or A*sin(2pi*f*t)).

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g) For any periodic function s(t), the power, Ps, is given by

$$P_s = \frac{1}{T}\int_0^T |s(t)|^2 \quad dt = \frac{1}{T}\int_{-T/2}^{T/2} |s(t)|^2 \quad dt$$

for our example, $P_s = \frac{2}{T}\int_0^{T/2} (2At/T)^2 dt = \frac{8A^2}{3T^3} t^3 \Big|_{t=0}^{t=T/2} = \frac{A^2T^2}{12} = \frac{A^2}{3}$

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h) The Fourier Series Expansion of s(t) is given by:

$$s(t) = \frac{A_0}{2} + \sum_{n=1,2}^{\infty} A_n \cos(2\pi nft) + B_n \sin(2\pi nft)$$

where the coefficients are computed as

$$A_0 = \frac{2}{T} \int_{-T/2}^{T/2} s(t)dt$$

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} s(t)\cos(2\pi nft)dt \qquad n = 1,2,...$$

$$B_n = \frac{2}{T} \int_{-T/2}^{T/2} s(t)\sin(2\pi nft)dt \qquad n = 1,2,...$$

* The coefficient A_0 is equal to:

$$A_0 = \frac{2}{T} \int_{-T/2}^{T/2} s(t)dt = \frac{2}{T} \times 2 \int_0^{T/2} (2A/T)tdt = 4\frac{A}{T^2} t^2 \Big|_{t=0}^{t=T/2} = 4\frac{A}{T^2}\frac{T^2}{4} = A$$

Note that $A_0/2$ is the DC component, and it has been computed in part d)

* The coefficients A_n (n=1,2, 3, …) are computed as

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} s(t)\cos(2\pi nft)dt = \frac{2}{T} \times 2 \int_0^{T/2} (2A/T)t\cos(2\pi nft)dt = \frac{8}{T^2} A\left[\frac{\cos(2\pi nft)}{(2\pi nf)^2} - t\frac{\sin(2\pi nft)}{(2\pi nf)}\right]_{t=0}^{t=T/2}$$

$$= \frac{8A}{T^2}\frac{[\cos(\pi n) - 1]}{(2\pi nf)^2} = \frac{2A}{\pi^2 n^2}(\cos(n\pi) - 1) = \begin{cases} 0 & n = even \\ \dfrac{-4A}{\pi^2 n^2} & n = odd \end{cases}$$

Note that $s(t)\cos(2\pi nft)$ is an even function for $t \in (-T/2, T/2)$, since $s(-t)\cos(2\pi nf(-t)) = s(t)\cos(2\pi nft)$, and therefore

$$\int_{-T/2}^{T/2} s(t)\cos(2\pi nft)dt = \int_{-T/2}^{0} s(t)\cos(2\pi nft)dt + \int_0^{T/2} s(t)\cos(2\pi nft)dt = 2 \times \int_0^{T/2} s(t)\cos(2\pi nft)dt$$

* The coefficients B_n (n=1,2, 3, …) are computed as

$$B_n = \frac{2}{T} \int_{-T/2}^{T/2} s(t)\sin(2\pi nft)dt = \frac{2}{T} \times \left\{ \int_{-T/2}^{0} (2A/T)t\sin(2\pi nft)dt + \int_0^{T/2} (2A/T)t\sin(2\pi nft)dt \right\}$$

$$= \frac{2}{T} \times \left\{ -\int_0^{T/2} (2A/T)t\sin(2\pi nft)dt + \int_0^{T/2} (2A/T)t\sin(2\pi nft)dt \right\} = 0$$

Note that $s(t)\sin(2\pi nft)$ is an odd function for $t \in (-T/2, T/2)$, since $s(-t)\sin(2\pi nf(-t)) = -s(t)\sin(2\pi nft)$, and therefore

$$\int_{-T/2}^{T/2} s(t)\sin(2\pi nft)dt = \int_{-T/2}^{0} s(t)\sin(2\pi nft)dt + \int_0^{T/2} s(t)\sin(2\pi nft)dt$$

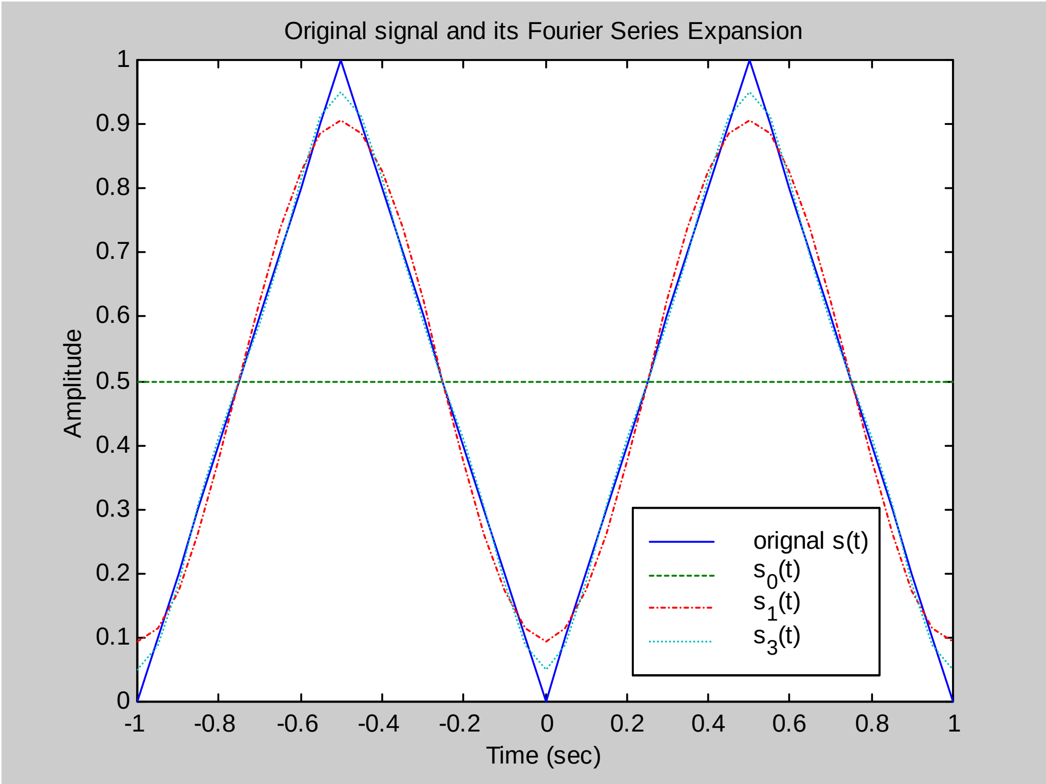
$$= -\int_0^{T/2} s(t)\sin(2\pi nft)dt + \int_0^{T/2} s(t)\sin(2\pi nft)dt = 0$$

Hence, using the computed coefficients, one can rewrite s(t) as

$$s(t) = \frac{A}{2} + \sum_{n=1,3,5}^{\infty} (\frac{-4A}{\pi^2 n^2})\cos(2\pi nft)\text{, or}$$

$$s(t) = \frac{A}{2} + (\frac{-4A}{\pi^2})\cos(2\pi ft) + (\frac{-4A}{\pi^2 3^2})\cos(2\pi \times 3\,ft) + (\frac{-4A}{\pi^2 5^2})\cos(2\pi \times 5\,ft) + ...$$

The original signal s(t) and its expansion are shown in the following figure



- The s0(t) is the expansion up to the zeroth term – i.e. including only the DC component

- s1(t) is the expansion up to the first harmonic – i.e. up to n = 1

- s3(t) is the expansion up to the second harmonic – i.e. up to n = 3

One can notice that as the number of harmonics increases, the closer we get to the original signal s(t) – See table 1 in part (i) for more details.

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i) The bandwidth of s(t):

fmin = 0 Hz (because of the DC or A/2 term)

fmax = infinite

Hence, the bandwidth is equal to fmax – fmin = infinite

This can be readily seen as the expansion of s(t) has harmonic terms with arbitrary high frequency

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j) The fundamental frequency is equal to f = 1/T

The term (A/2) is the DC term and its frequency is ZERO (lower than f)

The terms $\sum_{n=3,5}^{\infty}(\frac{-4A}{\pi^2 n^2})\cos(2\pi nft)$ have frequencies higher than f (3f, 5f, 7f, etc.)

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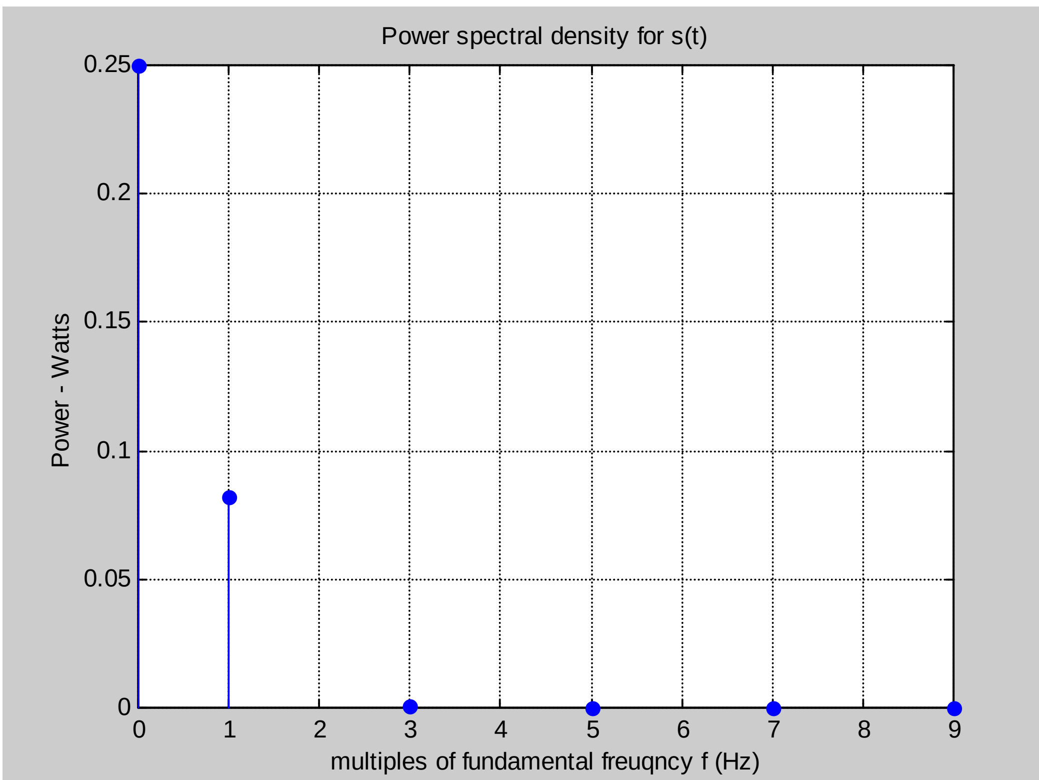
k) Computing the power of s(t) using the Fourier Series Expansion:

$$P_s = \left(\frac{A}{2}\right)^2 + 1/2 \times \sum_{n=1,3,5}^{\infty}\left[(\frac{-4A}{\pi^2 n^2})\right]^2 = A^2 \times \left(\frac{1}{4} + \frac{8}{\pi^4} \times \sum_{n=1,3,5}^{\infty}\left[\frac{1}{n^4}\right]\right)$$

For n=1, $P_s = A^2$X(0.3321), for up to n=3, $P_s = A^2$X(0.3331), and for up to n=5, $P_s = A^2$X(0.3333). As more terms are considered, the final answer approaches $A^2/3$ (same as in part (g)). The following figure shows the power contribution (the above equation plotted versus n) of the first 6 components (f = 0, f, 3f, 5f, 7f, and 9f). It can be seen that components with frequencies equal or higher than 3f have negligible contribution to the overall power.



i) To have at least 95% of the original power, it means:

$$A^2 \times \left(\frac{1}{4} + \frac{8}{\pi^4} \times \sum_{n=1,3,5}^{J}\left[\frac{1}{n^4}\right]\right) \geq 0.95 \times \frac{A^2}{3}$$

This translates to $\sum_{n=1,3,5}^{J}\left[\frac{1}{n^4}\right] \geq \frac{\pi^4}{8} \times \left[0.95 \times \frac{1}{3} - \frac{1}{4}\right] = 0.8117$

One can note that for J=1, the summation term is equal to 1 which is greater than 0.8117. Hence, it is enough to have the first term (n=1) to contain 95% of the power. The truncated series is given by:

$$\hat{s}(t) = \frac{A}{2} + (\frac{-4A}{\pi^2 n^2})\cos(2\pi n f t)\Big|_{n=1} = \frac{A}{2} - (\frac{4A}{\pi^2})\cos(2\pi f t)$$

The bandwidth of the truncated signal is given by

fmin = 0 Hz (still has the DC value)

fmax = f (the fundamental frequency)

Therefore, the bandwidth is fmax – fmin = f Hz

- If the question was find the truncated function $\hat{s}(t)$ that contains 99.9% of the power, then by following the same procedure:

$$A^2 \times \left(\frac{1}{4} + \frac{8}{\pi^4} \times \sum_{n=1,3,5}^{J} \left[\frac{1}{n^4}\right]\right) \geq 0.999 \times \frac{A^2}{3}$$

which simplifies to

$$\sum_{n=1,3,5}^{J} \left[\frac{1}{n^4}\right] \geq \frac{\pi^4}{8} \times \left(0.999 \times \frac{1}{3} - \frac{1}{4}\right) = 1.0106$$

- For J=1, the summation term is equal to 1.0 < 1.0106;
- For J=3, the summation term is equal to 1 + 1/9 = 1.111 >= 1.0106

Hence, J is equal to 3. The truncated function containing at least 99.9% of the original power is given by

$$\hat{s}(t) = \frac{A}{2} - (\frac{4A}{\pi^2})\cos(2\pi f t) - (\frac{4A}{\pi^2 3^2})\cos(2\pi \times 3 f t)$$

The bandwidth of the truncated signal is given by

fmin = 0 Hz (still has the DC value)

fmax = 3f

Therefore, the bandwidth is fmax – fmin = 3f Hz

Table 1: shows that as higher frequency terms are included in the truncated expression of s(t), the difference (error) between the truncated signal and the original function decreases. If we include all the terms (J = infinity), then the truncated expression $\hat{s}_J(t)$ is identical to s(t)

Table 1:Truncated Fourier Series Expansion for s(t)

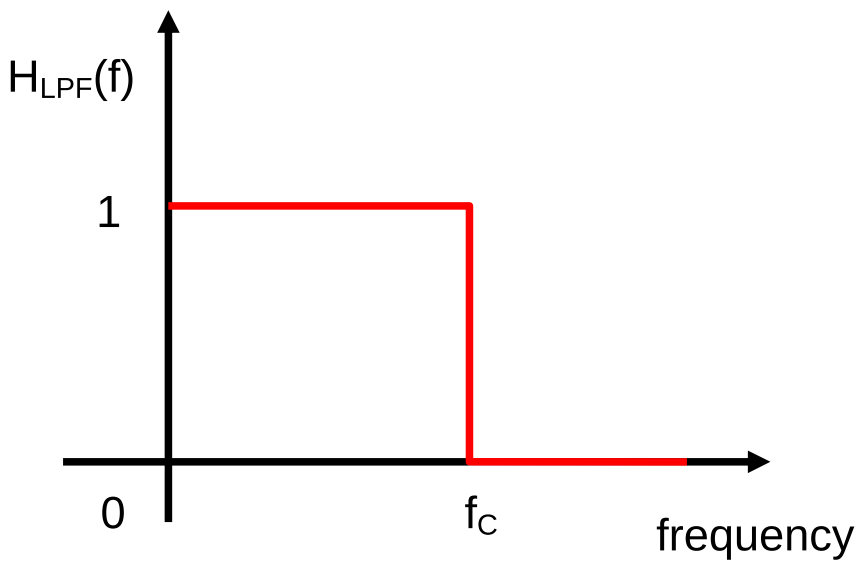
Truncated function ($\hat{s}_J(t)$)		f_{min}	f_{max}	BW	$P_{\hat{s}}$
$\hat{s}_0(t)$	$\frac{A}{2}$	0	0	0	$\frac{A^2}{4}$
$\hat{s}_1(t)$	$\frac{A}{2} - (\frac{4A}{\pi^2})\cos(2\pi f t)$	0	f	f	$\left(\frac{1}{4} + \frac{8}{\pi^4}\right)A^2$ =0.332127A^2
$\hat{s}_3(t)$	$\frac{A}{2} - (\frac{4A}{\pi^2})\cos(2\pi f t) - (\frac{4A}{\pi^2 3^2})\cos(2\pi \times 3 f t)$	0	3f	3f	$\left(\frac{1}{4} + \frac{8}{\pi^4} \times \left\{1 + \frac{1}{3^4}\right\}\right)A^2$ = 0.333141 A^2

$\hat{s}_5(t)$	$\frac{A}{2}-(\frac{4A}{\pi^2})\cos(2\pi ft)$ $-(\frac{4A}{\pi^2 3^2})\cos(2\pi\times 3ft)$ $-(\frac{4A}{\pi^2 5^2})\cos(2\pi\times 5ft)$	0	5f	5f	$\left(\frac{1}{4}+\frac{8}{\pi^4}\times\left\{1+\frac{1}{3^4}+\frac{1}{5^4}\right\}\right)A^2$ $= 0.333273\ A^2$
$\hat{s}_\infty(t)=s(t)$	$\frac{A}{2}+\sum_{n=1,3,5}^{\infty}(\frac{-4A}{\pi^2 n^2})\cos(2\pi nft)$	0	∞	∞	$\left(\frac{1}{4}+\frac{8}{\pi^4}\times\sum_{n=1,3,5}^{\infty}\left[\frac{1}{n^4}\right]\right)\times A^2$ $= A^2/3$

These functions are shown graphically (T = 1 second, A = 1) in the following figures.

Filtering and Amplification/Attenuation

- If we pass the signal through an *ideal* low pass filter (LPF) whose cut-off frequency is f_C, then we assume all frequency components lower or equal to f_C pass unaffected, while those components with frequencies higher than f_C are rejected (are not passed). The transfer function of the ideal low pass filter is shown in the following figure.



Ideal Low Pass Filter (LPF)

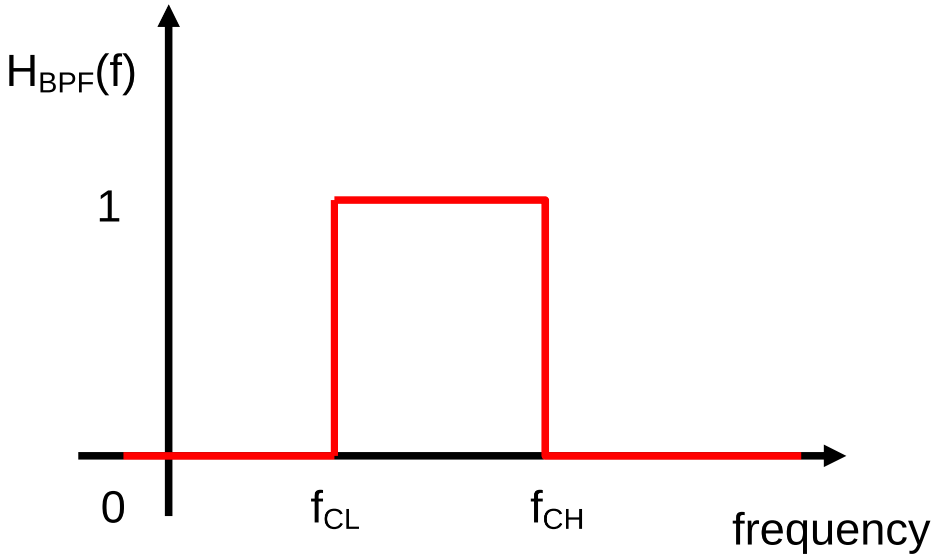
Example 1:

If we pass s(t) through a LPF whose cut-off frequency is f_C is 2Xf – what is the output signal?

Ans: Since only frequencies equal or lower than 2Xf are passed, then the output signal is equal to $s_{LPF}(t)=\frac{A}{2}-(\frac{4A}{\pi^2})\cos(2\pi ft)$

Note that frequencies 3f, 5f, … were rejected.

- If we pass the signal through an *ideal* band pass filter (BPF) whose lower cut-off frequency is f_{CL} and higher cutt-off frequency is f_{CH}, then we assume all frequency components greater or equal to f_{CL} and lower or equal to f_{CH} pass unaffected, while those components with frequencies lower than f_{CL} and those higher than f_{CH} are rejected (are not passed). The following figure shows the transfer function of an ideal band pass filter.



Ideal Band Pass Filter (BPF)

Example:

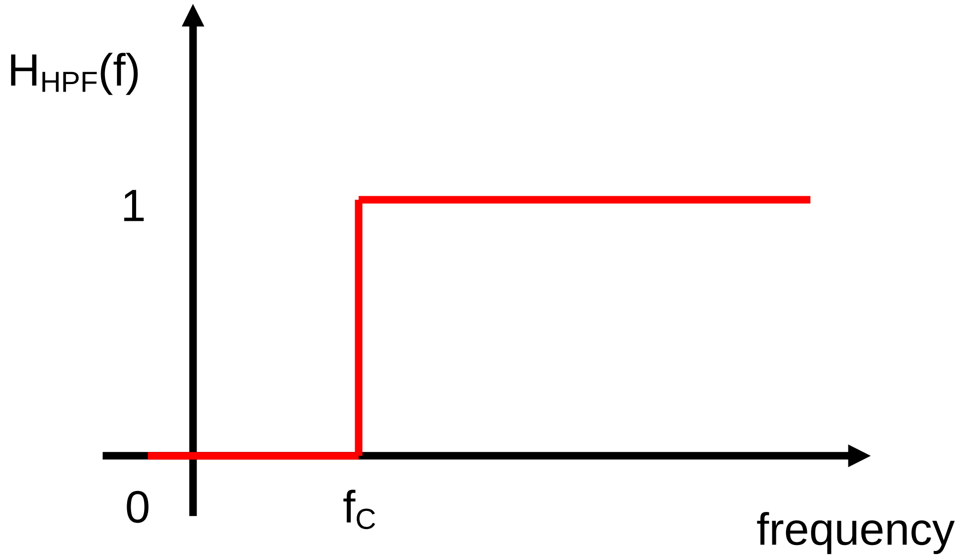
If we pass s(t) through a band pass filter whose f_{CL} and f_{CH} are equal to 2Xf and 6Xf respectively. What is the output signal?

Ans: The output signal is given by

$$s_{BPF}(t) = (\frac{-4A}{\pi^2 3^2})\cos(2\pi \times 3ft) + (\frac{-4A}{\pi^2 5^2})\cos(2\pi \times 5ft)$$

Note that the DC term and the 1Xf term were rejected. Also terms 7f, 9f, and higher were also rejected.

- If we pass the signal through an *ideal* high pass filter (HPF) whose cut-off frequency is f_C, then we assume all frequency components greater or equal to f_C pass unaffected, while those components with frequencies lower than f_C are rejected (are not passed). The following figure shows the transfer function of an ideal high pass filter.



Ideal High Pass Filter (HPF)

Example:

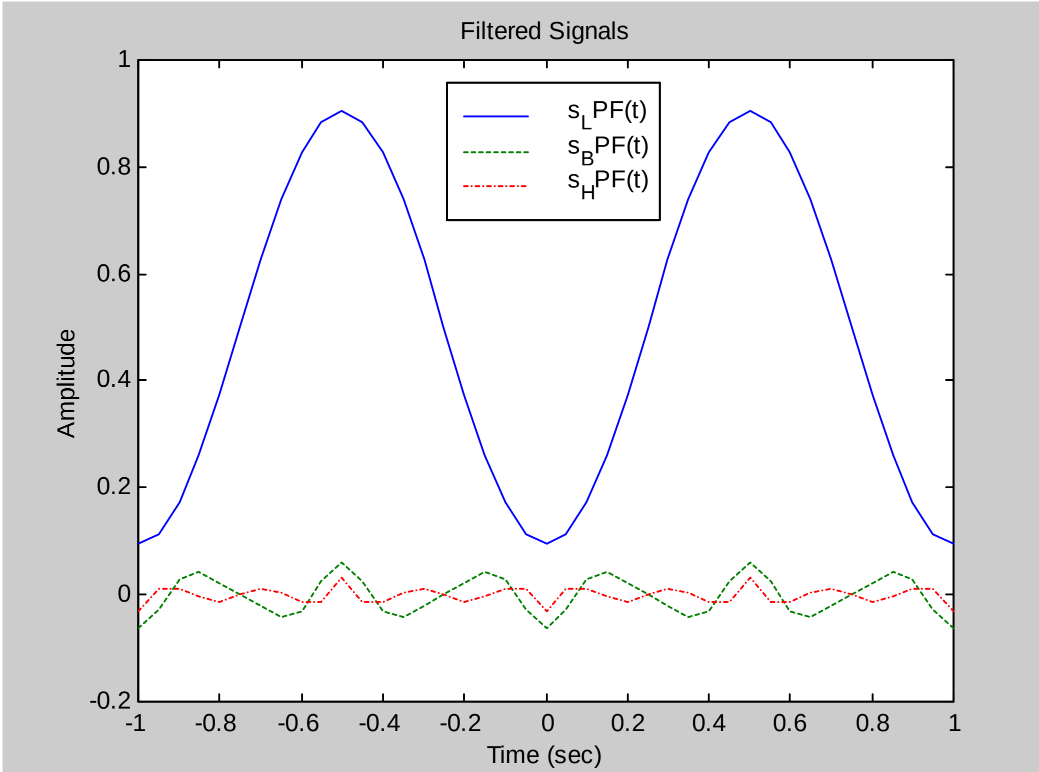
If the signal s(t) is passed through a high pass filter whose f_C is equal to 4Xf, what is the output signal?

Ans: The output signal is given by:

$$s_{HPF}(t) = \sum_{n=5,7,9}^{\infty} \left(\frac{-4A}{\pi^2 n^2}\right)\cos(2\pi nft)$$

Note that frequencies lower than 4Xf (such 0, 1Xf, 3Xf) are rejected.

The following figure shows graphically the signals $s_{LPF}(t)$, $s_{BPF}(t)$, and $s_{HPF}(t)$.


Filtered Signals
s_L PF(t)
s_B PF(t)
s_H PF(t)
Amplitude
Time (sec)
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Appendix A: The matlab code used to generate the figures:

```
clear all

A = 1;   % this is amplitude of the traingular signal
T = 1;   % this is the period of the traingular signal

%
% define the time axis (vector)
t = -1:0.05:1;

%
% Calculate the function s(t) using its math expression
t_period = t./T - floor(t./T);
for i =1:length(t)
   if (t_period(i) < T/2)
      s(i) = 2*A/T *t_period(i);
   else
      s(i) = 2*A*(1-t_period(i)/T);
   end
end

%
% This is the fundamental frequency
f = 1/T;

%
% The following are the first four terms of the Fourier Series
% Expansion
s0 = A/2*ones(size(t));

s1 = A/2 + (-4*A)/(pi*pi)*cos(2*pi*f*t);

s3 = A/2 + (-4*A)/(pi*pi)*cos(2*pi*f*t) + (-4*A)/(pi*pi*3*3) * cos(2*pi*3*f*t);

s5 = A/2 + (-4*A)/(pi*pi)*cos(2*pi*f*t) + (-4*A)/(pi*pi*3*3) * cos(2*pi*3*f*t) + ...
                                          (-4*A)/(pi*pi*5*5) * cos(2*pi*5*f*t);
%
% Calculate the power of the signal using the Fourier Series Expansion
n  = 1:2:9;
sp  = 1./(n.^4);
P  = 8/(pi^4) * sp;

n  = [0 n];     % add in the power of the DC component at the zeroth index
P  = [A^2/4 P];

%
% plot the power versus n - the order of the harmonic
figure(1)
stem(n, P,'filled'), grid;
title('Power spectral density for s(t)');
xlabel('Multiples of fundamental freuqncy f (Hz)');
ylabel('Power - Watts');

%
% plot the signal and its Fourier Series Expansion
figure(2)
plot(t, s, '-', t, s0, '--', t, s1, '-.', t, s3, ':');
legend('orignal s(t)', 's_0(t)', 's_1(t)', 's_3(t)');
title('Original signal and its Fourier Series Expansion');
xlabel('Time (sec)');
ylabel('Amplitude');
grid


%
% Compute the filtered signals
s_L = s1;

s_B = -4*A/(pi*pi) *(cos(2*pi*3*f*t)/9 + cos(2*pi*5*f*t)/25);

s_H = -4*A/(pi*pi) *(cos(2*pi*5*f*t)/25 + cos(2*pi*7*f*t)/49 + ...
                     cos(2*pi*9*f*t)/81 + cos(2*pi*11*f*t)/121) ;
%
% Plot the filtered signals
figure(3)
plot(t, s_L, '-', t, s_B, '--', t, s_H, '-.');
legend('s_LPF(t)', 's_BPF(t)', 's_HPF(t)');
title('Filtered Signals');
xlabel('Time (sec)');
```

```
ylabel('Amplitude');
axis([-1 1 -0.2 1.0]);
```