

**King Fahd University of
Petroleum & Minerals
Computer Engineering Dept**

**COE 341 – Data and Computer
Communications**

Term 082

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Lecture Contents

1. Fourier Analysis
 - a. Fourier Series Expansion
 - b. Fourier Transform
 - c. Ideal Low/band/high pass filters

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Signals

- **A signal is a function representing information**
 - Voice signal – microphone output
 - Video signal – camera output
 - Etc.
- **Types of Signals**
 - Analog – continuous-value continuous-time
 - Discrete – discrete-value continuous-time
 - Digital – predetermined discrete levels – much easier to reproduce at receiver with no errors
 - Binary – only two predetermined levels: e.g. 0 and 1

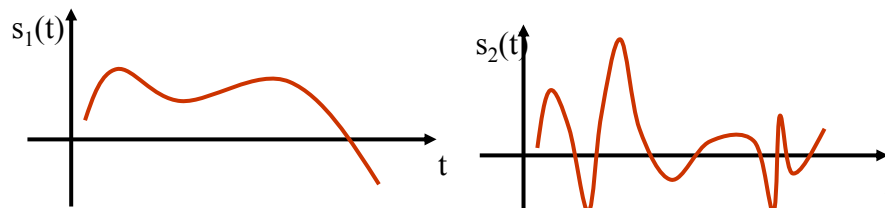
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Example of Continuous-Value Continuous-time signal

- $s_1(t)$ and $s_2(t)$ are two example of analog signals



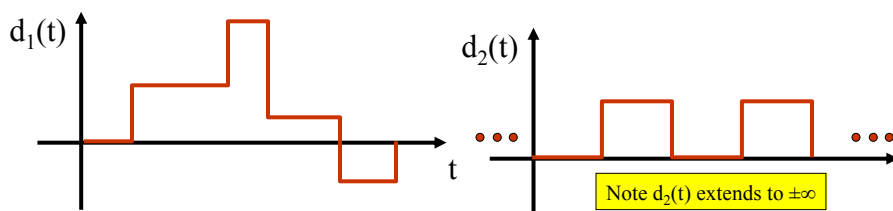
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Example of Discrete-Value Continuous-time signal

- $d_1(t)$ and $d_2(t)$ are two examples of discrete signals
 - $d_1(t)$ – takes more than two levels
 - $d_2(t)$ – takes only two levels - binary



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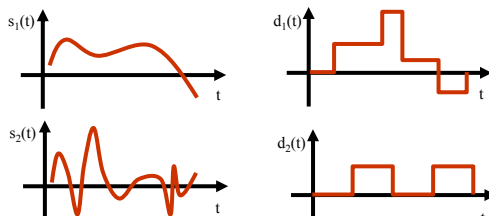
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Applies to BOTH analog and digital signals

Time Domain Representation

- Time domain representation – we plot value (voltage, current, electric field intensity, etc.) versus time
 - Can infer rate of change (speed or frequency) information – e.g. $s_2(t)$ seems faster than $s_1(t)$
 - Using calculus terms: *rate of change* for $s_2(t) >$ rate of change for $s_1(t)$



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Applies to BOTH analog and digital signals

Frequency - Bandwidth

- **$s_2(t)$ faster than $s_1(t)$ →**
 - **$s_2(t)$ contains higher frequencies than those contained in $s_1(t)$**
- **$s_1(t)$ and $s_2(t)$ contain more than one frequency**
 - **Minimum frequency = f_{\min}**
 - **Maximum frequency = f_{\max}**
- **Bandwidth = Range of frequencies contained in signal**
= $f_{\max} - f_{\min}$

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Applies to BOTH analog and digital signals

Frequency - Bandwidth (2)

- **For our example signals, assume:**
 - **$S_1(t)$: $f_{\min} = 10$ Hz, $f_{\max} = 500$ Hz**
 - **$S_2(t)$: $f_{\min} = 5$ Hz, $f_{\max} = 1000$ Hz**
- **This means:**
 - **BW for $s_1(t) = 500 - 10 = 490$ Hz**
 - **BW for $s_2(t) = 1000 - 5 = 995$ Hz**
- **Note that: because $s_2(t)$ is "faster than" $s_1(t)$ it should contain frequencies higher than those in $s_1(t)$**
 - **E.g. $s_2(t)$ contains frequencies (500,100] which do not exist in $s_1(t)$**

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Applies to BOTH analog
and digital signals

Frequency – Bandwidth (3)

- Consider the discrete signals $d_1(t)$ and $d_2(t)$
- The function plots have points of infinite slope
 - rate of change = $\infty \rightarrow$ frequency = ∞
- Therefore for signals that look like $d_1(t)$ and $d_2(t)$, $f_{\max} = \infty$
- Furthermore, $BW = \infty$
- Example:
 - $d_2(t)$ contains frequencies from some minimum f_{\min} Hz to $f_{\max} = \infty$ Hz

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Example of Signal BW

- Consider the human speech
- Typically $f_{\min} \sim 100\text{Hz}$
- $f_{\max} \sim 3500\text{ Hz}$
- BW of the human speech signal = 3100 Hz

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Bandwidth for Systems

- **For a system to respond (amplify, process, Tx, Rx, etc.) for a particular signal with all its details, the system should have an equal or greater bandwidth compared to that of the signal**
- **Example:**
 - **The system required to process $s_2(t)$ should have a greater bandwidth than the system required to process $s_1(t)$**

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Bandwidth for Systems (2)

- **Example 2: consider the human ear system**
 - **Responds to a range of frequencies only**
 - **$f_{min} = 20 \text{ Hz}$ $f_{max} = 20,000 \text{ Hz} \rightarrow \text{BW} = 19,980 \text{ Hz}$**
 - **It does not respond to sounds with frequencies outside this range**
- **Example 3: consider the copper wire**
 - **It passes (electric) signals only between a certain f_{min} and a certain f_{max}**
 - **The higher the quality of the wire – the wider the BW**
- **More on Systems BW later!**

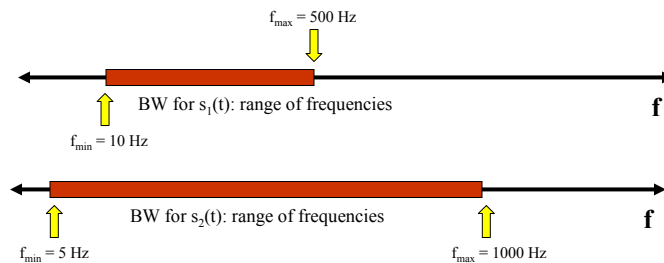
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Frequency Representation

- How to represent signals and indicate their frequency content?
- The X-axis: frequency (in Hertz or Hz)
- What is the Y-axis then? – the answer will be postponed!

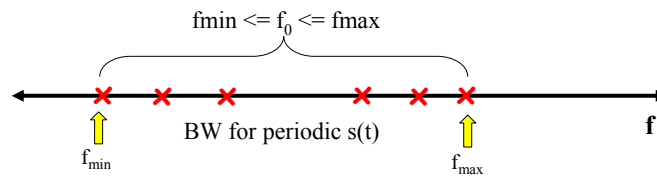


Periodic Signals

- A periodic signal repeats itself every T seconds
 - Period \rightarrow T seconds
- In calculus terms:
 - $S(t)$ is periodic if $s(t) = s(t+T)$ for any $-\infty < t < \infty$
- For previous examples: $s_1(t)$, $s_2(t)$, and $d_1(t)$ are not periodic – however, $d_2(t)$ is periodic

Periodic Signals (2)

- A periodic signal has a **FUNDAMENTAL FREQUENCY** – f_0
 - $f_0 = 1 / T$ – where T is the period
- A periodic signal may also has frequencies other than the fundamental frequency f_0



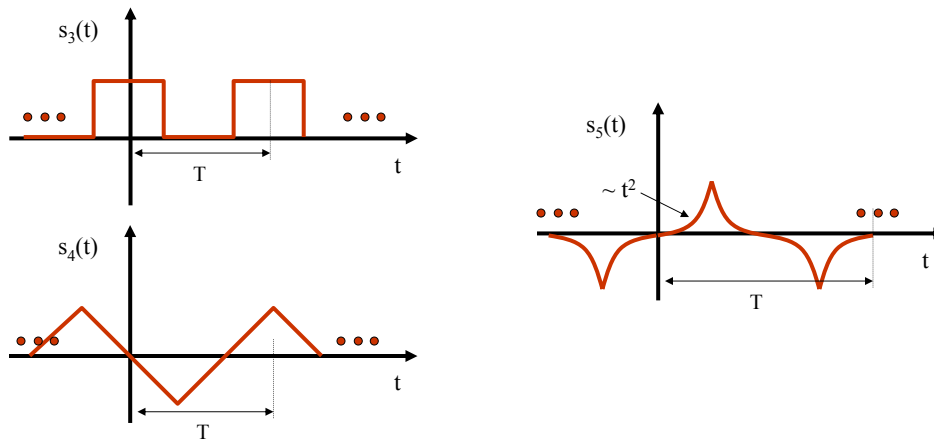
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Periodic Signals (3)

- **Examples of other periodic signals:**



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Applies to BOTH analog and digital signals

Energy/Power of Signals

- **Energy for any signal is defined as**

$$E_s = \int |s(t)|^2 dt$$

where the integral is carried over ALL range of t

- **In other words, E_s is the area under the absolute squared of the signal**
- **The unit of energy is Joules**

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Applies to BOTH analog and digital signals

Energy/Power of Signals (2)

- **Note that for periodic signal E_s is equal to infinity since it is defined on $(-\infty, \infty)$**
 - **However power is FINITE for these type of signals**

- **Power is defined as the average of the absolute squared of the signal, i.e.**

$$P_s = \frac{1}{T} \int_0^T |s(t)|^2 dt$$

Note the integral can be performed on $[0, T]$, $[-T/2, T/2]$, or any continuous interval of length T

- **The unit of power is Joules/sec or Watt**

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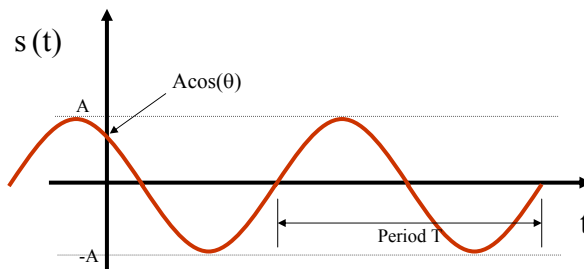
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A VERY SPECIAL Analog Signal

- **A function of the form**

$$s(t) = A \cos(2\pi ft + \theta)$$



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Characteristics of COSINE

- **Completely specified by:**
 - **Amplitude – A**
 - **Phase - θ**
 - **Frequency – f**
- **$s(t = 0) = A \cos(\theta)$**
- **Periodic signal – repeats itself every T seconds**
 - **$T = 1 / f$**
- **Time to review your trigonometry !!**
 - **E.g. $\sin(x) = \cos(x - \pi/2)$**

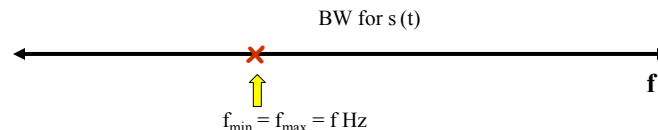
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Characteristics of COSINE (2)

- Energy for this signal, $E_s = \text{infinity}$
- Power for this signal, $P_g = A^2/2$
 - Note P_g is dependent only on the amplitude A
 - **Exercise: Verify the above results using the power formula**
- It contains **ONLY ONE** frequency f
 - The "purest" form of analog signals
- Frequency representation:



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Characteristics of COSINE (3)

- **Very Useful Properties ($f = 1/T$)**

$$\int_0^T \cos(2\pi ft + \theta) dt = 0 \qquad \frac{1}{T} \int_0^T \cos^2(2\pi ft + \theta) dt = 1/2$$

$$\int_0^T \cos(2\pi mft + \theta) dt = 0 \qquad \frac{1}{T} \int_0^T \cos^2(2\pi mft + \theta) dt = 1/2$$

$$\frac{1}{T} \int_0^T \cos(2\pi nft) \cos(2\pi mft) dt = \begin{cases} 0 & n \neq m \\ 1/2 & n = m \end{cases}$$

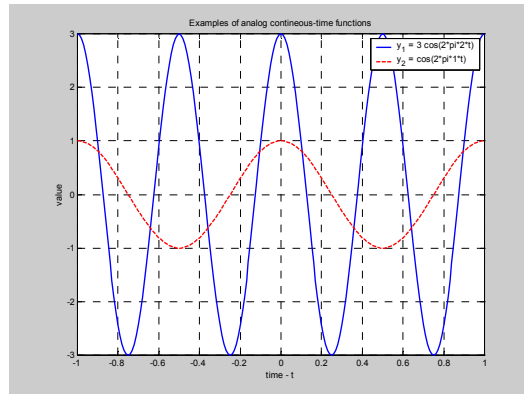
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Example of Cosine Functions

- $Y_1(t)$ – has
 - a frequency f of 2 Hz ($T = 1/2$ sec)
 - An amplitude of 3
 - $P_{Y1} = 3^2/2 = 4.5$ Watts
- $Y_2(t)$ - has
 - a frequency f of 1 Hz ($T = 1/1 = 1$ sec)
 - An amplitude of 1
 - $P_{Y2} = 1^2/2 = 0.5$ Watts



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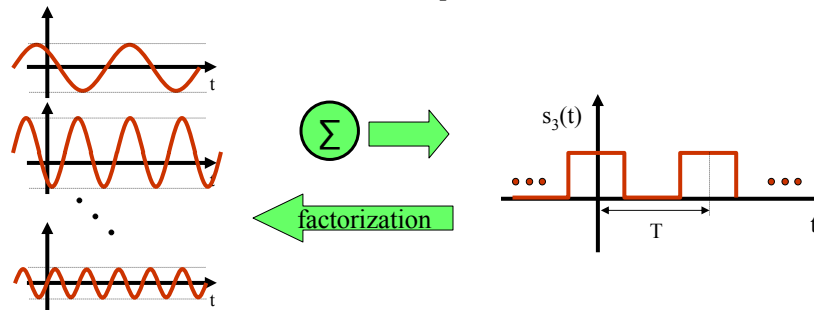
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ONLY FOR PERIODIC SIGNALS

Fourier Series Expansion

- Can we use the basic cosine functions to represent periodic signals?
- YES – Fourier Series Expansion



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Fourier Series Expansion (2)

- For a periodic signal $s(t)$ can be represented as a sum of sinusoidal signals as in

$$s(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)]$$

where the coefficients are computed using:

$$A_0 = \frac{2}{T} \int_0^T s(t) dt$$

f_0 is the fundamental frequency of $s(t)$ and is equal to $1/T$

$$A_n = \frac{2}{T} \int_0^T s(t) \cos(2\pi n f_0 t) dt \quad B_n = \frac{2}{T} \int_0^T s(t) \sin(2\pi n f_0 t) dt$$

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Fourier Series Expansion (3)

- Another form for the series:

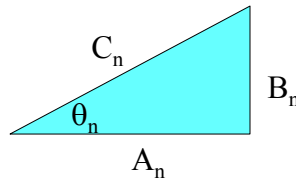
$$s(t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(2\pi n f_0 t + \theta_n)$$

where the coefficients are computed using:

$$C_0 = A_0$$

$$C_n = \sqrt{A_n^2 + B_n^2}$$

$$\theta_n = \tan^{-1} \left(\frac{-B_n}{A_n} \right)$$



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Notes on Fourier Series Expansion

- **The representation (the sum of sinusoids) is completely identical and equivalent to the original specification of $s(t)$**
- **It applies to any periodic signal – analog or digital!**

Very powerful tool - it reveals all frequencies contained in the original periodic signal $s(t)$

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Notes on Fourier Series Expansion (2)

- **In general, $s(t)$ contains**
 - **DC term – the zero frequency term = $A_0/2$**
 - **A (possibly infinite) number of harmonics (or sinusoids) at multiples of the fundamental frequency, f_0**
- **The contribution of a harmonic with frequency nf_0 is proportional to $|A_n^2 + B_n^2|$ or C_n^2**
 - **E.g. if $C_n^2 \sim 0$, then we say the harmonic at nf_0 (or higher) does not contribute significantly towards building $s(t)$ – more on this when we discuss total power!**

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Notes on Fourier Series Expansion (3)

- A harmonic with frequency equal to nf_0 ($n > 0$), has a period of $1/(nT)$
- In general the series expansion of $s(t)$ contains INFINITE number of terms (harmonics)
- However for less than 100% accurate representation one can ignore higher terms – terms with frequencies greater than certain $n \cdot f_0$

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Notes on Fourier Series Expansion (4)

- Lets define the following function:

$$s_e(n=k)$$

To be the series expansion of $s(t)$ up to and including the $n = k$ term

It should be noted that $s_e(n=k)$ is periodic with period T

- Examples:

$$s_e(n=0) = A_0 / 2$$

$$\begin{aligned} s_e(n=1) &= A_0 / 2 + A_1 \cos(2\pi f_0 t) + B_1 \sin(2\pi f_0 t) \\ &= A_0 / 2 + C_1 \cos(2\pi f_0 t + \theta_1) \end{aligned}$$

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Notes on Fourier Series Expansion (5)

- **Examples – cont'd:**

$$\begin{aligned} s_{-e}(n=2) &= A_0/2 + A_1 \cos(2\pi f_0 t) + B_1 \sin(2\pi f_0 t) \\ &\quad + A_2 \cos(2\pi \times 2 f_0 t) + B_2 \sin(2\pi \times 2 f_0 t) \\ &= A_0/2 + C_1 \cos(2\pi f_0 t + \theta_1) + C_2 \cos(2\pi \times 2 f_0 t + \theta_2) \end{aligned}$$

⋮

$$\begin{aligned} s_{-e}(n=\infty) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)] \\ &= A_0/2 + \sum_{n=1}^{\infty} C_n \cos(2\pi n f_0 t + \theta_n) \end{aligned}$$

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Notes on Fourier Series Expansion (6)

- **It is obvious that $s(t)$ is 100% represented by $s_{-e}(n=\infty)$**
- **$s_{-e}(n = n^* < \infty)$ produces a less than 100% accurate representation of the original $s(t)$**
- **For most practical periodic signals $s_{-e}(n=10)$ provides a more than enough accuracy in representing $s(t)$**
 - **No need for infinite number of terms**

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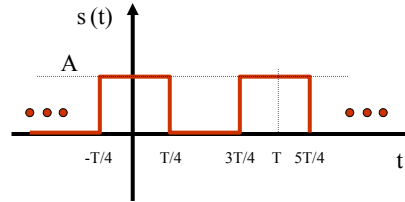
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Example 1:

- Consider the following $s(t)$

- Over one period, the signal is defined as

$$s(t) = \begin{cases} A & -T/4 < t \leq T/4 \\ 0 & T/4 < t \leq 3T/4 \end{cases}$$



- Finding the Series Expansion:

- The DC term A_0

$$\begin{aligned} A_0 &= \frac{2}{T} \int_{-T/4}^{T/4} s(t) dt = \frac{2}{T} \times \frac{T}{2} \times A \\ &= A \end{aligned}$$

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Example 1: cont'd

- The term A_n :

$$\begin{aligned} A_n &= \frac{2}{T} \int_{-T/4}^{T/4} s(t) \cos(2\pi f_0 t) dt = \frac{2A}{T} \int_{-T/4}^{T/4} \cos(2\pi f_0 t) dt \\ &= \frac{2A}{2\pi f_0 T} \sin(2\pi f_0 t) \Big|_{t=-T/4}^{t=T/4} = \frac{A}{\pi n} \times 2 \times \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

$$= \begin{cases} 0 & n = 2, 4, 6, \dots \\ \frac{2A}{\pi n} & n = 1, 5, 9, \dots \\ -\frac{2A}{\pi n} & n = 3, 7, 11, \dots \end{cases}$$

Remember

- $f_0 = 1/T$
- $\text{Int}(\cos(ax)) = 1/a \sin(ax)$
- $\sin(n\pi) = 0$ for integer n
- $\sin(n\pi/2) = 1$ for $n=1, 5, 9, \dots$
- $\sin(n\pi/2) = -1$ for $n=3, 7, 11, \dots$

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Example 1: cont'd

- Therefore A_n is given by:

$$= \begin{cases} 0 & n = 2, 4, 6, \dots \\ (-1)^{(n-1)/2} \times \frac{2A}{\pi n} & n = 1, 3, 5, 7, \dots \end{cases}$$

Remember

$$\begin{aligned} (-1)^{(n-1)/2} &= 1 \text{ for } n = 1, 5, 9, \dots \\ &= -1 \text{ for } n = 3, 7, 11, \dots \end{aligned}$$

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Example 1: cont'd

- The term B_n :

$$\begin{aligned} B_n &= \frac{2}{T} \int_{-T/4}^{T/4} s(t) \sin(2\pi f_0 t) dt = \frac{2A}{T} \int_{-T/4}^{T/4} \sin(2\pi f_0 t) dt \\ &= \frac{-2A}{2\pi f_0 T} \cos(2\pi f_0 t) \Big|_{t=-T/4}^{t=T/4} = \frac{-2A}{\pi n} \times \left\{ \cos\left(\frac{n\pi}{2}\right) - \cos\left(-\frac{n\pi}{2}\right) \right\} \\ &= 0 \end{aligned}$$

Remember

1. $\int \cos(ax) dx = 1/a \sin(ax)$
2. $\cos(x) = \cos(-x)$

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Example 1: cont'd

- Therefore, the overall series expansion is given by

$$s(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \times \cos(2\pi n f_0 t)$$

$$s(t) = \frac{A}{2} + \frac{2A}{\pi} \times \cos(2\pi f_0 t) - \frac{2A}{3\pi} \cos(2\pi \times 3 f_0 t) \\ + \frac{2A}{5\pi} \times \cos(2\pi \times 5 f_0 t) - \frac{2A}{7\pi} \cos(2\pi \times 7 f_0 t) + \dots$$

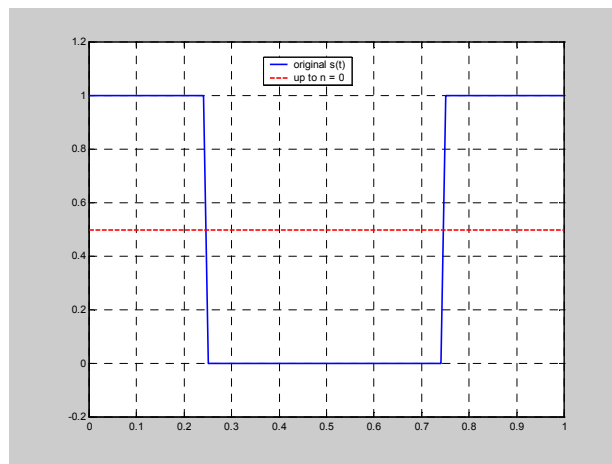
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Example 1: cont'd

- Original $s(t)$ and the series up to and including $n = 0$
- i.e. Comparing:
 $s(t)$
vs.
 $s_e(n=0) = A/2$



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Example 1: cont'd

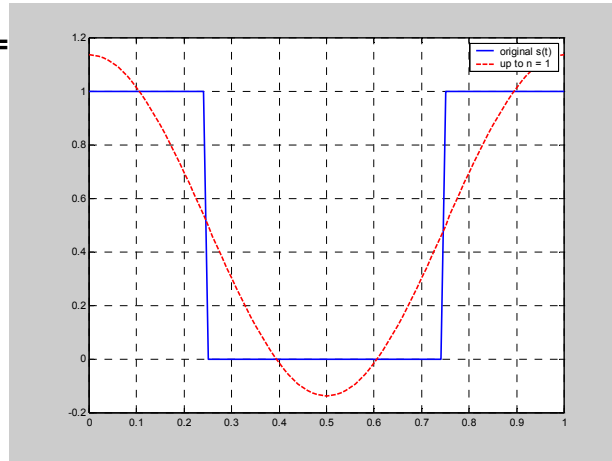
- Original $s(t)$ and the series up to and including $n = 1$

- i.e. Comparing:

$s(t)$

vs.

$$s_e(n=1) = \frac{A}{2} + \frac{2A}{\pi} \cos(2\pi f_0 t)$$



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Example 1: cont'd

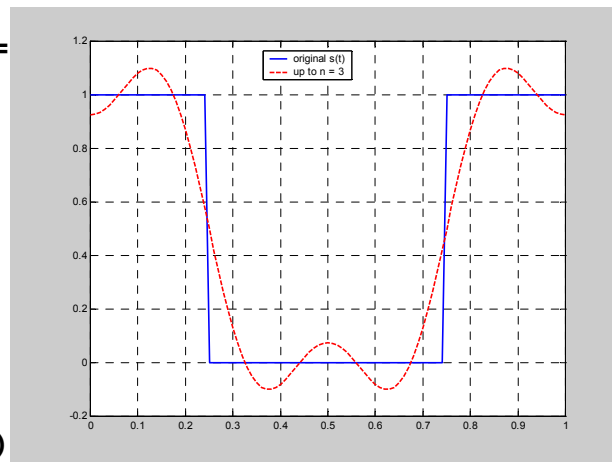
- Original $s(t)$ and the series up to and including $n = 3$

- i.e. Comparing:

$s(t)$

vs.

$$s_e(n=3) = \frac{A}{2} + \frac{2A}{\pi} \cos(2\pi f_0 t) - \frac{2A}{3\pi} \cos(2\pi 3f_0 t)$$



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Example 1: cont'd

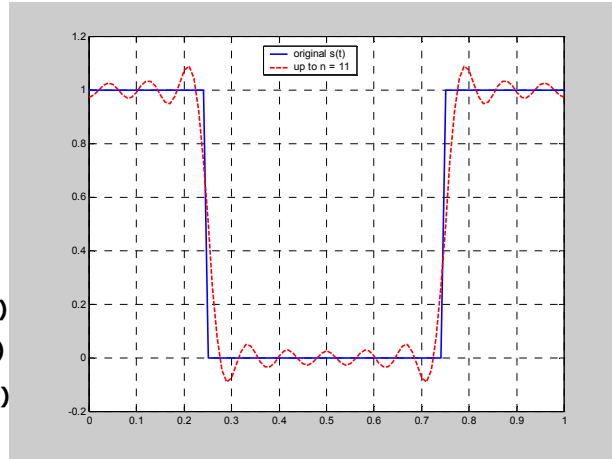
- Original $s(t)$ and the series up to and including $n = 11$

- i.e. Comparing:

$s(t)$

vs.

$$s_e(n=11) = \frac{A}{2} + \frac{2A}{\pi} \cos(2\pi f_0 t) - \frac{2A}{3\pi} \cos(2\pi 3f_0 t) + \frac{2A}{5\pi} \cos(2\pi 5f_0 t) - \frac{2A}{7\pi} \cos(2\pi 7f_0 t) + \frac{2A}{11\pi} \cos(2\pi 11f_0 t)$$



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Example: cont'd

```
clear all

T = 1;
A = 1;
t = -1:0.01:1;
n_max = 11;

s = (A*square(2*pi/T*(t+T/4))+A)/2;

figure(1)
plot(t, s);
grid
axis([0 1 -0.2 1.2]);

s_e = A/2*ones(size(t));

for n=1:2:n_max
    s_e = s_e + (-1)^((n-1)/2) * 2*A/(n*pi) * cos(2*pi*n/T*t);
end

figure(2)
plot(t, s, 'b-', t, s_e, 'r--');
axis([0 1 -0.2 1.2]);
legend('original s(t)', 'up to n = 11');
grid
```

- The matlab code for plotting and evaluating the Fourier Series Expansion
- This code builds the series incrementally using the “for” loop
- Make sure you study this code!!

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Notes Previous Example

- **The more terms included in the series expansion → the closer the representation to the original $s(t)$**
- i.e. comparing $s(t)$ with $s_e(n=n^*)$, the greater the n^* the closer the representation is
- **How to measure "closeness"?**
- **Answer: Let's use power!!**

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Power Calculation Using Fourier Series Expansion

- **Rule: if $s(t)$ is represented using Fourier Series expansion, then its power can be calculated using:**

$$\begin{aligned} P_s &= \frac{1}{T} \int_0^T |s(t)|^2 dt = \frac{A_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} [A_n^2 + B_n^2] \\ &= \frac{A_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2 \end{aligned}$$

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Power Calculation Using Fourier Series Expansion (2)

- The previous result is based on the following two facts:
 - (1) For $f(t) = \text{constant}$
→ power of $f(t) = \text{constant}^2$

Proof:

$$\begin{aligned}\text{power} &= 1/T \times \int_0^T \text{constant}^2 dt \\ &= 1/T \times \text{constant}^2 \times T \\ &= \text{constant}^2 \text{ Watts}\end{aligned}$$

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Power Calculation Using Fourier Series Expansion (3)

- The previous result is based on the following facts (continued):
 - (2) For $f(t) = A \cos(2\pi n f_0 t + \theta)$
→ power of $f(t) = A^2/2$

Proof:

$$\begin{aligned}P_f &= \frac{1}{T} \int_0^T |f(t)|^2 dt = \frac{A^2}{T} \int_0^T \cos^2(2\pi n f_0 t + \theta) dt \\ &= \frac{A^2}{T} \int_0^T \left[\frac{1}{2} + \frac{1}{2} \cos(4\pi n f_0 t + 2\theta) \right] dt \\ &= \frac{A^2}{T} \left[\frac{T}{2} + 0 \right] = \frac{A^2}{2}\end{aligned}$$

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Example 2:

- **Problem:** What is the power of the signal $s(t)$ used in previous example? And find n^* such that the power contained in $s_e(n=n^*)$ is 95% of that existing in $s(t)$?

- **Solution:**

Let the power of $s(t)$ be given by P_s

$$P_s = \frac{1}{T} \int_0^T |s(t)|^2 dt = \frac{1}{T} \times A^2 \times \frac{T}{2} = \frac{A^2}{2} = 0.5A^2$$

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Example 2: cont'd

- **Now it is desired to compute the power using the Fourier Series Expansion**
- **What is the power in $s_e(n=0) = A/2$?**
- **Ans: we apply the power formula:**

$$\begin{aligned} P_{s_e(n=0)} &= \frac{1}{T} \int_0^T |s_e(n=0)|^2 dt \\ &= \frac{1}{T} \times \frac{A^2}{4} \times T = \frac{A^2}{4} = 0.25A^2 \end{aligned}$$

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Example 2: cont'd

- What is the power in
 $s_{-e}(n=1) = A/2 + 2A/\pi \cos(2\pi f_0 t)$
- Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:

$$P_{s_{-e}(n=1)} = \frac{1}{T} \int_0^T |s_{-e}(n=1)|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2}$$
$$= \left(\frac{1}{4} + \frac{2}{\pi^2} \right) A^2 = 0.4526 A^2$$

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Example 2: cont'd

- What is the power in
 $s_{-e}(n=3) = A/2 + 2A/\pi \cos(2\pi f_0 t) - 2A/(3\pi) \cos(2\pi 3f_0 t)$
- Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:

$$P_{s_{-e}(n=3)} = \frac{1}{T} \int_0^T |s_{-e}(n=3)|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2} + \frac{2A^2}{9\pi^2}$$
$$= \left(\frac{1}{4} + \frac{2}{\pi^2} + \frac{2}{9\pi^2} \right) A^2 = 0.4752 A^2$$

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Example 2: cont'd

- What is the power in

$$s_{-}e(n=5) = A/2 + 2A/\pi \cos(2\pi f_0 t) - 2A/(3\pi) \cos(2\pi 3f_0 t) + 2A/(5\pi) \cos(2\pi 5f_0 t)$$

- Ans: we can use the result on slide **Power Calculation Using Fourier Series Expansion:**

$$P_{s_{-}e(n=5)} = \frac{1}{T} \int_0^T |s_{-}e(n=5)|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2} + \frac{2A^2}{9\pi^2} + \frac{2A^2}{25\pi^2}$$

$$= \left(\frac{1}{4} + \frac{2}{\pi^2} + \frac{2}{9\pi^2} + \frac{2}{25\pi^2} \right) A^2 = 0.4833A^2$$

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Example 2: cont'd

- What is the power in

$$s_{-}e(n=\infty) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \times \cos(2\pi n f_0 t)$$

- Ans: we can use the result on slide **Power Calculation Using Fourier Series Expansion:**

$$P_{s_{-}e(n=\infty)} = \frac{1}{T} \int_0^T |s_{-}e(n=\infty)|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \left(\frac{1}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right) A^2 = 0.5A^2$$

This the EXACT SAME power contained in $s(t)$ -
This is expected since $s(t)$ is 100% represented by $s_{-}e(n=\infty)$

Example 2: cont'd

$s_e(n=k)$	Expression	Power	% Power ⁺
$k = 0$	$A/2$	$0.25 A^2$	$(0.25A^2)/(0.5A^2) = 50\%$
$k = 1$	$A/2 + 2A/\pi\cos(2\pi f_0 t)$	$0.4526 A^2$	$(0.4526A^2)/(0.5A^2) = 90.5\%$
$k = 2^*$	$A/2 + 2A/\pi\cos(2\pi f_0 t)$	$0.4526 A^2$	90.5%
$k = 3$	$A/2 + 2A/\pi\cos(2\pi f_0 t) - 2A/(3\pi)\cos(2\pi 3f_0 t)$	$0.4752 A^2$	95.0%
$k = 5$	$A/2 + 2A/\pi\cos(2\pi f_0 t) - 2A/(3\pi)\cos(2\pi 3f_0 t) + 2A/(5\pi)\cos(2\pi 5f_0 t)$	$0.4833 A^2$	96.7%

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⁺ % power = power of $s_e(n=k)$ relative to original power in $s(t)$ which is equal to $0.5A^2$
^{*} For $k = 2$, the expression $s_e(n=k)$ is the same as that for $s_e(k=1)$. Why?

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Example 2: cont'd

- Therefore, $s_e(n=n^*)$ such that 95% of power is contained $\rightarrow n^* = 3$

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Power Spectral Density Function

- **Fourier Series Expansion:**
 - Specifies all the basic harmonics contained in the original function $s(t)$
 - $C_n^2/2 = (A_n^2 + B_n^2) / 2$ determines the power contribution of the n th harmonic with frequency nf_0
- The power Spectral Density function is a function specifying: how much power is contributed by a given frequency

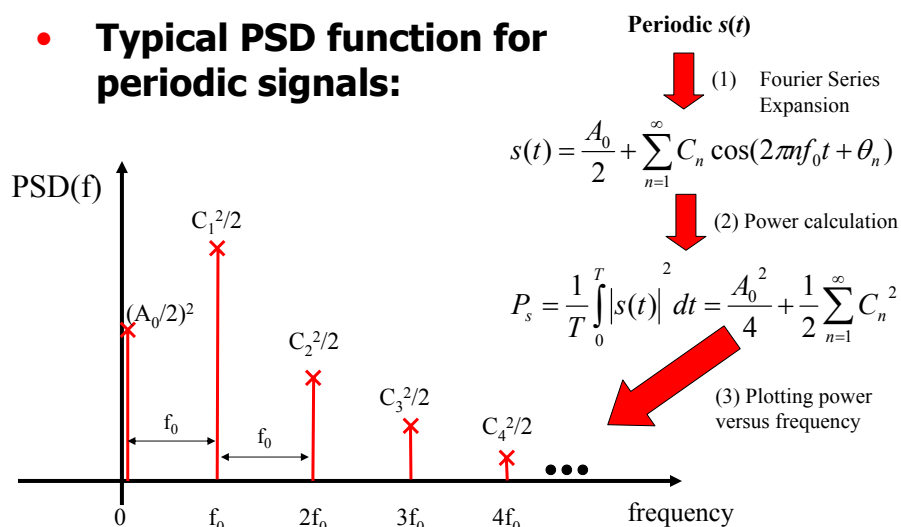
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Power Spectral Density Function (2)

- **Typical PSD function for periodic signals:**



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Power Spectral Density Function (3)

- A mathematical expression for PSD(f) can be written as

$$PSD(f) = \begin{cases} A_0^2/4 & f = 0 \\ C_n^2/2 & f = n \times f_0 \\ 0 & \text{otherwise} \end{cases}$$

- Another way (more compact) of writing PSD(f) is as follows:

$$PSD(f) = \frac{A_0^2}{4} \times \delta(f) + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2 \times \delta(f - nf_0)$$

where $\delta(f)$ is defined by

$$\delta(f) = \begin{cases} 1 & f = 0 \\ 0 & f \neq 0 \end{cases}$$

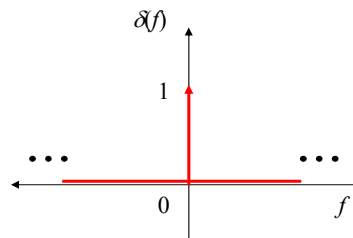
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Power Spectral Density Function (4)

- $\delta(f)$ is referred to as the dirac function or unit impulse function



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Note on the PSD Function

- PSD function has units of Watts/Hz
- For periodic signals → PSD is a discrete function - defined for integer multiples of the fundamental frequency
 - Specifies the power contribution of every harmonic component $C_n^2/2 \leftrightarrow nf_0$
- The separation between the discrete components is at least f_0
 - It is exactly f_0 if all C_n 's are not zeros
 - E.g. for the previous $s(t)$ example, $C_n=0$ for even n → separation = $2f_0$

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Note on the PSD Function (2)

- To calculate the total power of signal → Integrate PSD over all contained frequencies
 - For discrete PSD: integration = summation
- Therefore total power of $s(t)$,

$$P_s = (A_n/2)^2 + \sum C_n^2/2 \text{ in Watts}$$

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Example 3:

- Find the PSD function of the periodic signal $s(t)$ considered in Example 1.
- From Example 1, $s(t)$ is given by

$$s(t) = \frac{A}{2} + \frac{2A}{\pi} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \times \cos(2\pi n f_0 t)$$

- Using Example 2:
 - Power at the zero frequency = $(A/2)^2 = A^2/4$
 - Power at the n th harmonic (n odd) is equal to $2A^2/(n\pi)^2$
 - Power at the n th harmonic (n even) is zero
 - Therefore the PSD function is given by

$$PSD(f) = \frac{A^2}{4} \times \delta(f) + \frac{2A^2}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \times \delta(f - n f_0)$$

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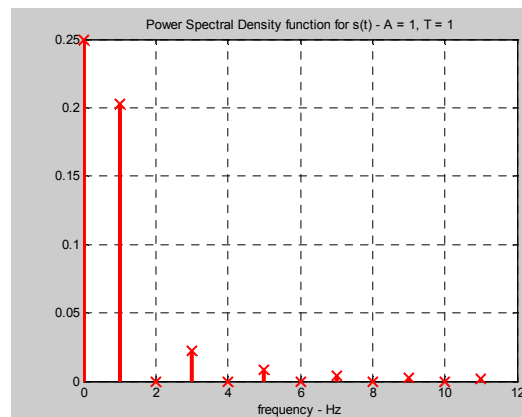
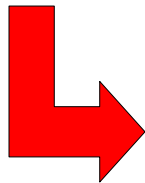
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Example 3: cont'd

- The PSD is plotted as shown ($A = 1, T = 1$)

$$PSD(f) = \frac{A^2}{4} \times \delta(f) + \frac{2A^2}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \times \delta(f - n f_0)$$



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Example 3: cont'd

- **Matlab Code to plot PSD**

```
clear all

T = 1;
A = 1;
t = -1:0.01:1;
n_max = 11;

Frequency = [0:1:n_max];
PwrSepctralD = zeros(size(Frequency));

% Record the DC term power at f = 0
PwrSepctralD(1) = (A/2)^2;

% Record the nth harmonic power at f = nf0
for n=1:2:n_max
    PwrSepctralD(n+1) = (2*A/(n*pi))^2 / 2;
end

figure(1)
stem(Frequency, PwrSepctralD,'rx');
title('Power Spectral Density function for s(t) - A = 1, T = 1');
xlabel('frequency - Hz');
grid
```

The “stem” function is typically used to plot discrete functions

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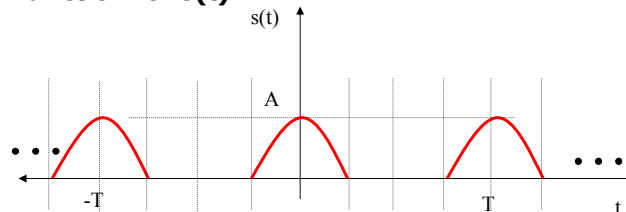
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Example 4:

This is a typical exam question

- **Problem: Consider the periodic half-wave rectified signal $s(t)$ depicted in figure.**
 - Write a mathematical expression for $s(t)$
 - Calculate the Fourier Series Expansion for $s(t)$
 - Calculate the total power for $s(t)$
 - Find n^* such that $s_e(n^*)$ has 95% of the total power
 - Determine the PSD function for $s(t)$
 - Plot the PSD function for $s(t)$



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Example 4: cont'd

- Answer:

(a) To write a mathematical expression for $s(t)$, remember that the general form of a sinusoidal function is given by

$$A \cos(2\pi \times \text{Freq} \times t), \text{ or} \\ A \cos(2\pi / \text{Period} \times t)$$

Therefore $s(t)$ is given by

$$s(t) = A \cos(2\pi/T t) \quad -T/4 < t \leq T/4 \\ = 0 \quad T/4 < t \leq 3T/4$$

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Example 4: cont'd

- Answer:

(b) The F.S.E of $s(t)$: $s(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)]$

The DC term is given by

$$A_0 = \frac{2}{T} \int_{-T/4}^{T/4} s(t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi t / T) dt \\ = \frac{A}{\pi} \times \sin(2\pi t / T) \Big|_{t=-T/4}^{t=T/4} = \frac{A}{\pi} [\sin(\pi / 2) - \sin(-\pi / 2)] \\ = \frac{2A}{\pi}$$

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Example 4: cont'

- **Answer:**

The An term is given by (remember $1/T = f_0$)

$$A_n = \frac{2}{T} \int_{-T/4}^{T/4} s(t) \cos(2\pi f_0 t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi t / T) \cos(2\pi f_0 t) dt$$

$$= \frac{2A}{T} \times \left[\frac{\sin(2\pi(n+1)f_0 t)}{4\pi(n+1)f_0} + \frac{\sin(2\pi(n-1)f_0 t)}{4\pi(n-1)f_0} \right] \Bigg|_{t=-T/4}^{t=T/4} \quad \text{For } n \neq 1$$

$$= \frac{A}{\pi} \times \left[\frac{\cos(n\pi/2)}{(n+1)} + \frac{-\cos(n\pi/2)}{(n-1)} \right] \quad \text{For } n \neq 1$$

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This means: the n=1 should be special!

Remember:

$$\int [\cos(ax)\cos(bx)] dx = \frac{\sin(ax+bx)}{2(a+b)} + \frac{\sin(ax-bx)}{2(a-b)}$$

for $a \neq b$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

Example 4: cont'd

But

$$\cos(n\pi/2) = 0 \quad n = \text{odd}, n \neq 1$$

$$= (-1)^{(1+n/2)} \quad n = \text{even}$$

Therefore

$$A_n = \frac{A}{\pi} \times \left[\frac{(-1)^{(1+n/2)}}{(n+1)} + \frac{(-1)(-1)^{(1+n/2)}}{(n-1)} \right] \quad \text{For } n \text{ even}$$

$$= 0 \quad \text{For } n \text{ odd}, n \neq 1$$

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Example 4: cont'd

The expression for A_n (for even n) can be further simplified to

$$\begin{aligned}
 A_n &= \frac{A}{\pi} \times \left[\frac{(-1)^{(1+n/2)}}{(n+1)} + \frac{(-1)(-1)^{(1+n/2)}}{(n-1)} \right] \\
 &= \frac{A}{\pi} \times \left[\frac{(-1)^{(1+n/2)}(n-1) + (-1)(-1)^{(1+n/2)}(n+1)}{(n+1)(n-1)} \right] \\
 &= \frac{A}{\pi(n^2-1)} \times \left[(-1)^{(1+n/2)}(n-1) - (-1)^{(1+n/2)}(n+1) \right] \\
 &= \frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)}
 \end{aligned}$$

For n even

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Example 4: cont'd

An is still not completely specified – we still need to calculate it for $n=1$; in other words we need to calculate A_1 :

$$A_{n=1} = \frac{2}{T} \int_{-T/4}^{T/4} s(t) \cos(2\pi \times 1 \times f_0 t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi t / T) \cos(2\pi f_0 t) dt$$

Therefore:

$$\begin{aligned}
 A_1 &= \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos^2(2\pi f_0 t) dt \\
 &= \frac{2A}{T} \times \left[\frac{t}{2} + \frac{1}{4 \times 2\pi f_0} \sin(4\pi f_0 t) \right] \Bigg|_{t=-T/4}^{t=T/4} = \frac{2A}{T} \times \left[\frac{T}{4} + \frac{\sin(\pi) - \sin(-\pi)}{8\pi f_0} \right] \\
 &= \frac{A}{2}
 \end{aligned}$$

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Example 4: cont'd

This mean A_n is equal to the following:

$$A_n = \begin{cases} 2A/\pi & n = 0 \\ 0 & n \text{ odd, } n \neq 1 \\ A/2 & n = 1 \\ \frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)} & n = 2, 4, 6, \dots \end{cases}$$

The above expression specifies A_n for ALL POSSIBLE values of $n \rightarrow$ specification is complete

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Example 4: cont

Remember:

$$\int [\sin(ax)\cos(bx)] dx = \frac{-\cos(ax+bx)}{2(a+b)} - \frac{\cos(ax-bx)}{2(a-b)} \quad \text{for } a \neq b$$

We still need to compute B_n :

$$\begin{aligned} B_n &= \frac{2}{T} \int_{-T/4}^{T/4} s(t) \sin(2\pi n f_0 t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi t / T) \sin(2\pi n f_0 t) dt \\ &= \frac{2A}{T} \times \left[\frac{\cos(2\pi(n+1)f_0 t)}{4\pi(n+1)f_0} - \frac{\cos(2\pi(n-1)f_0 t)}{4\pi(n-1)f_0} \right] \Bigg|_{t=-T/4}^{t=T/4} \quad \text{For } n \neq 1 \\ &= \frac{A}{2\pi} \times \left[\frac{-\cos(\pi/2(n+1)) + \cos(-\pi/2(n+1))}{(n+1)} - \frac{\cos(\pi/2(n-1)) - \cos(-\pi/2(n-1))}{(n-1)} \right] \\ &= 0 \quad \text{For } n \neq 1 \end{aligned}$$

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This means: the $n=1$ should be special!

Remember:

$$\sin(2ax) = 2 \cos(ax) \sin(ax)$$

Example 4: cont'd

B_n is still NOT completely specified – we still need to calculate it for $n=1$; in other words we need to calculate B_1 :

$$B_{n=1} = \frac{2}{T} \int_{-T/4}^{T/4} s(t) \sin(2\pi \times 1 \times f_0 t) dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi t / T) \sin(2\pi f_0 t) dt$$

Therefore:

$$B_1 = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi f_0 t) \sin(2\pi f_0 t) dt = \frac{A}{T} \times \int_{-T/4}^{T/4} \sin(4\pi f_0 t) dt$$

$$= \frac{-A}{4\pi} \times \cos(4\pi f_0 t) \Big|_{t=-T/4}^{t=T/4} = \frac{-A}{4\pi} \times [\cos(\pi) - \cos(-\pi)]$$

$$= 0$$

→ This means $B_n = 0$ for all n

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Example 4: cont'd

- To summarize:

$$A_n = \begin{cases} 2A/\pi & n = 0 \\ 0 & n \text{ odd, } n \neq 1 \\ A/2 & n = 1 \\ \frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)} & n = 2, 4, 6, \dots \end{cases}$$

And

$$B_n = 0 \quad \text{for all } n$$

- Having computed A_n and B_n we are now in a position to write the Fourier Series Expansion for $s(t)$

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Example 4: cont'd

- The Fourier Series Expansion for $s(t)$ is given by

$$s(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t)]$$

$$= \frac{A}{\pi} + \frac{A}{2} \cos(2\pi f_0 t) + \frac{2A}{\pi} \sum_{n=2,4,6}^{\infty} \frac{(-1)^{(1+n/2)}}{n^2 - 1} \cos(2\pi n f_0 t)$$

The C_n terms (**there is a typo in the textbook**) are as follows:

$$C_0 = A/\pi$$

$$C_1 = A/2$$

$$C_n = \begin{cases} \frac{2A(-1)^{(1+n/2)}}{\pi(n^2 - 1)}, & n = 2, 4, 6, \dots \\ 0, & n \text{ odd, } n \neq 1 \end{cases}$$

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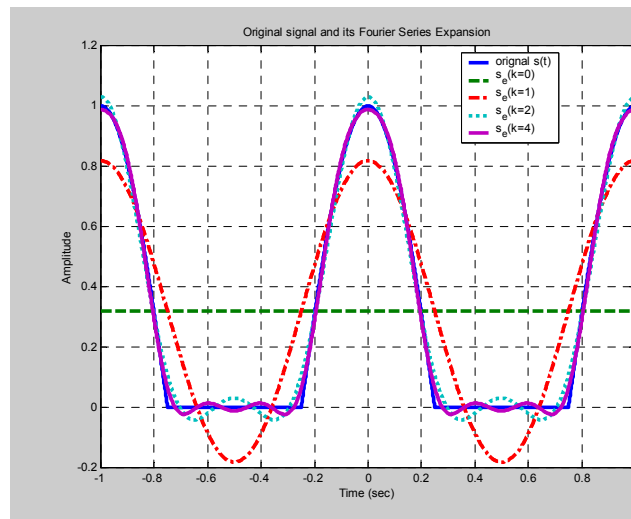
Example 4: cont'd

Remember:

$$\sin(2ax) = \frac{1}{2} [\cos(ax) \sin(ax) + \sin(ax) \cos(ax)]$$

Plot for $s(t)$ and the Fourier Series Expansion for $k=0, 1, 2,$ and 4

Note: As k increases $s_e(n=k)$ approaches the original $s(t)$



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Example 4: cont'd

- The total power of $s(t)$ is given by:

$$\begin{aligned}
 P_s &= \frac{1}{T} \int_{-T/4}^{3T/4} |s(t)|^2 dt = \frac{A^2}{T} \times \int_{-T/4}^{T/4} \cos^2(2\pi t / T) \\
 &= \frac{A^2}{T} \times \left[\frac{t}{2} + \frac{\sin(4\pi t / T)}{8\pi / T} \right] \Bigg|_{t=-T/4}^{t=T/4} \\
 &= \frac{A^2}{4}
 \end{aligned}$$

Therefore total power of $s(t) = 0.25 A^2$

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Example 4: cont'd

- To find n^* such that power of $s_e(n=n^*) = 95\%$ of total power:

$s_e(n=k)$	Expression	Power	% Power ⁺
$k = 0$	A/π	$0.1013 A^2$	$(0.1013A^2)/(0.25 A^2) = 40.5\%$
$k = 1$	$A/\pi + A/2 \cos(2\pi f_0 t)$	$0.2263 A^2$	$(0.2262A^2)/(0.25A^2) = 90.5\%$
$k = 2$	$A/\pi + A/2 \cos(2\pi f_0 t) + 2A/(3\pi) \cos(2\pi 2f_0 t)$	$0.2488 A^2$	$(0.2488A^2)/(0.25A^2) = 99.5\%$

Therefore $n^* = 2 \rightarrow$ power of $s_e(n=2) = 0.2488 A^2$ which is 99.5% of total power of $s(t)$

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Example 4: cont'd

- The PSD function for $s(t)$ is as follows:
 - Power for DC term = $(A/\pi)^2$
 - Power for harmonic at $f = f_0$: $(A/2)^2/2 = A^2/8$
 - Power for harmonic at $f = nf_0$ ($n=2,4,6, \dots$): $[2A/(\pi(n^2-1))]^2/2 = 2A^2/(\pi(n^2-1))^2$

- Therefore PSD function equals to

$$PSD(f) = \left(\frac{A}{\pi}\right)^2 \delta(f) + \frac{A^2}{8} \delta(f - f_0) + \frac{2A^2}{\pi^2} \sum_{n=2,4,6}^{\infty} \frac{\delta(f - nf_0)}{(n^2 - 1)^2}$$

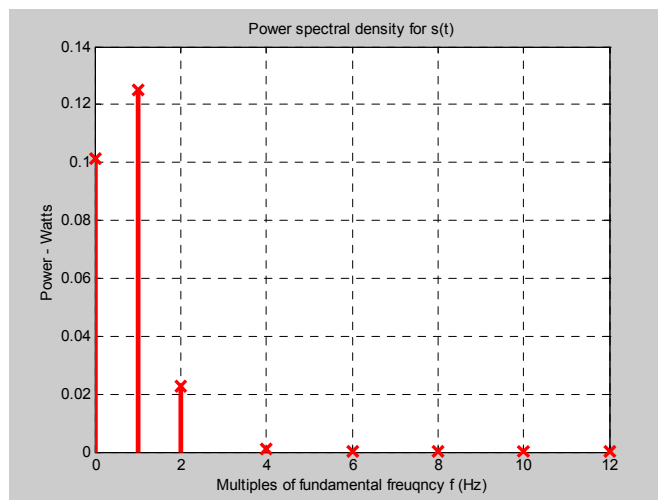
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Example 4: cont'd

- Plot of The PSD function for $s(t)$



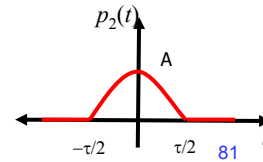
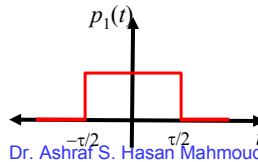
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Fourier Transform

- **Fourier Series Expansion analysis is applicable for PERIODIC signals ONLY**
- **There are important signals that are not periodic such as**
 - **Your voice waveform**
 - **Pulse signal $p(t)$ – used for modulation and transmission**
 - **Examples: $p_1(t)$ and $p_2(t)$**



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Fourier Transform (2)

- **How to find the frequency content of such signals?**
- **Use FOURIER TRANSFORM**

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi jft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{2\pi jft} df$$

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Notes on Fourier Transform

- F.T describes a two-way transformation

$$x(t) \quad \leftarrow \rightarrow \quad X(f)$$

where $x(t)$ is the time representation of the signal, while $X(f)$ is the frequency representation of the signal

- $X(f)$ is defined on a continuous range of frequencies
 - All frequencies within the range of $X(f)$ where $X(f)$ is not zero contribute towards building $x(t)$

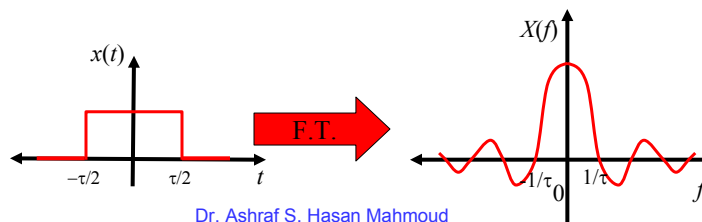
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Notes on Fourier Transform (2)

- The magnitude of the contribution of a particular frequency f^* in $x(t)$ is proportional to $|X(f^*)|^2$
- Example: Consider the F.T. pair shown below – clearly frequencies belonging to $(-1/\tau, 1/\tau)$ contribute significantly more compared to frequencies belonging to $(1/\tau, \infty)$ or $(-\infty, -1/\tau)$



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Properties of Fourier Transform

- If $x(t)$ is time-limited $\rightarrow X(f)$ is not frequency-limited
 - i.e. the range of $X(f) = (-\infty, \infty)$
- If $x(t)$ is a real-valued symmetric $\rightarrow X(f)$ is real-valued

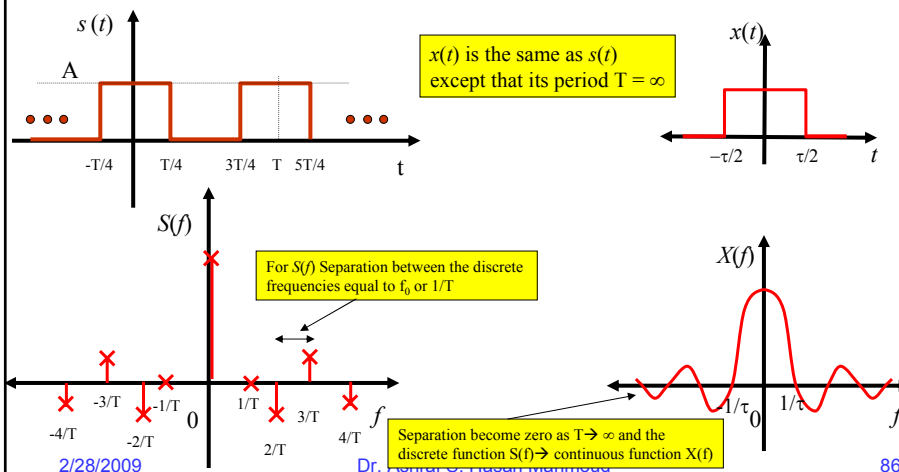
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Relation between Fourier Series Expansion and Fourier Transform

- Consider the following two signals:



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Relation between Fourier Series Expansion and Fourier Transform (2)

- **The separation between spectral lines for a periodic signal is $1/T$**
- **As $T \rightarrow$ infinity and $s(t)$ becomes non periodic \rightarrow the separation between spectral lines \rightarrow zero (i.e. it becomes continuous)**

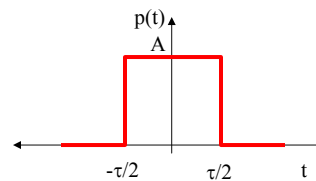
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Example 5:

- **Problem: Consider the square pulse function shown in figure:**
 - **Write a mathematical expression for $p(t)$**
 - **Find the Fourier transform for $p(t)$**
 - **Plot $P(f)$**



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Example 5: cont'd

- Answer: $p(t)$ can be expressed as

$$p(t) = A \quad |t| \leq \tau/2 \\ = 0 \quad \text{otherwise}$$

The F.T. for $p(t)$, $P(f)$ is given by

$$P(f) = \int_{-\infty}^{\infty} p(t)e^{-2\pi jft} dt$$

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Example 5: cont'd

- Which is equal to

$$\begin{aligned} P(f) &= \int_{-\infty}^{\infty} p(t)e^{-2\pi jft} dt = \int_{-\tau/2}^{\tau/2} Ae^{-2\pi jft} dt \\ &= \frac{A}{-2\pi jf} \int_{-\tau/2}^{\tau/2} e^{-2\pi jft} dt = -\frac{A}{2\pi jf} \times (e^{-\pi jf\tau} - e^{\pi jf\tau}) \\ &= \frac{A}{\pi f} \times \frac{(e^{\pi jf\tau} - e^{-\pi jf\tau})}{2j} \\ &= A\tau \frac{\sin(\pi f\tau)}{\pi f\tau} \end{aligned}$$

Remember: Euler identity :
 $e^{jx} = \cos(x) + j \sin(x)$, OR
 $\cos(x) = (e^{jx} + e^{-jx})/2$
 $\sin(x) = (e^{jx} - e^{-jx})/(2j)$

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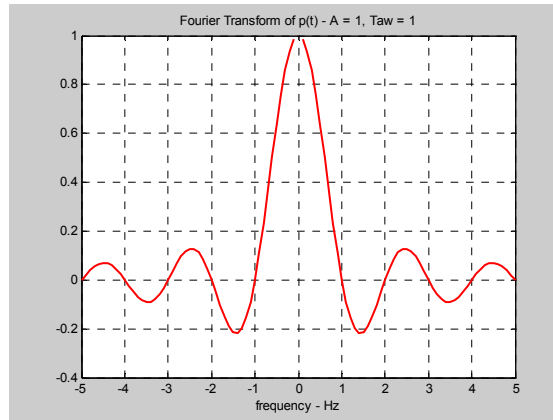
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Example 5: cont'd

- **P(f) plot for A = 1 and $\tau = 1$**

- **Note:**

- **P(f) is define on $(-\infty, \infty)$**
- **P(f) is continuous**
- **P(f) = ZERO for $f = n/\tau$**
- **For practical pulses – P(f) approaches zero as $f \rightarrow \pm\infty$**
- **Most of the energy of p(t) is contained in the period of $(-1/\tau, 1/\tau)$**



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