

King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 202 – Fundamentals of Computer Engineering

Term 081

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1

Binary Logic

- Deals with *binary* variables that take one of two discrete values
- Values of variables are called by a variety of very different names
 - *high* or *low* based on voltage representations in electronic circuits
 - *true* or *false* based on their usage to represent logic states
 - *one* (1) or *zero* (0) based on their values in Boolean algebra
 - *open* or *closed* based on its operation in *gate* logic
 - *on* or *off* based on its operation in *switching* logic
 - *asserted* or *de-asserted* based on its effect in digital systems

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2

Basic Operations - AND

- Another Symbol is ".", e.g.

$$Z = X \text{ AND } Y \text{ or}$$

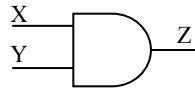
$$Z = X.Y \text{ or even}$$

$$Z = XY$$

- X and Y are inputs, Z is an output
- Z is equal to 1 if and only if $X = 1$ and $Y = 1$; $Z = 0$ otherwise (similar to the multiplication operation)

- Truth Table:

- Graphical symbol:



| X | Y | Z=XY |
|---|---|------|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

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3

Basic Operations - OR

- Another Symbol is "+", e.g.

$$Z = X \text{ OR } Y \text{ or}$$

$$Z = X + Y$$

- X and Y are inputs, Z is an output
- Z is equal to 0 if and only if $X = 0$ and $Y = 0$; $Z = 1$ otherwise (similar to the addition operation)

- Truth Table:

- Graphical symbol:



| X | Y | Z=X+Y |
|---|---|-------|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

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4

Basic Operations - NOT

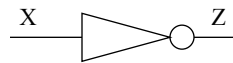
- Another Symbol is " $\bar{\quad}$ ", e.g.

$$Z = \overline{X} \text{ or } Z = X'$$

- X is the input, Z is an output
- Z is equal to 0 if X = 1; Z = 1 otherwise
- Sometimes referred to as the complement or invert operation
- Truth Table:

| X | Z=X' |
|---|------|
| 0 | 1 |
| 1 | 0 |

- Graphical symbol:

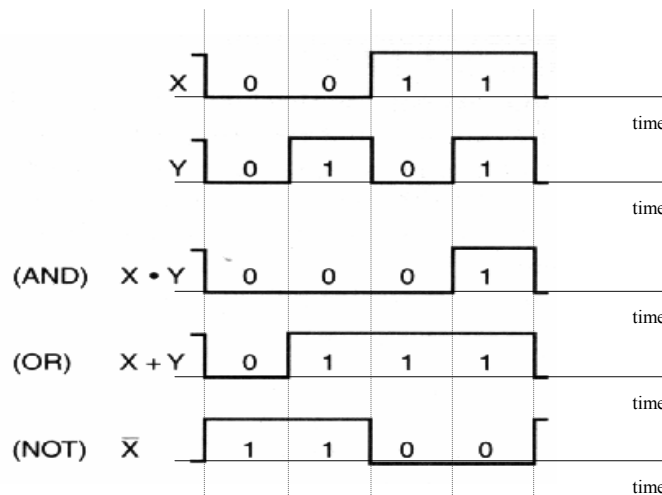


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5

Time Diagrams

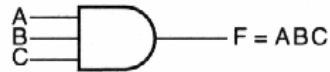


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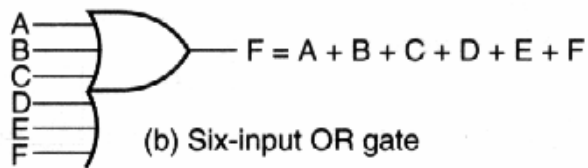
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6

Multiple Input Gates



(a) Three-input AND gate



(b) Six-input OR gate

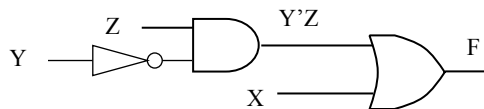
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7

Boolean Algebra

- Consider the following *function*, F
$$F = X + Y'Z$$
- The function F is referred to as a BOOLEAN FUNCTION
- F has two terms: X and $Y'Z$
- The circuit diagram for F is as shown below



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8

Boolean Algebra – cont'd

- The truth table for F is as follows

| X | Y | Z | F |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 |

- Note:

- In general, a truth table for an n-variable function, has 2^n rows to cover all possible input combinations
- The table covers all possible combinations of the inputs
- To arrive at the F's column one could use an Y'Z column as follows

| X | Y | Z | Y'Z | F |
|---|---|---|-----|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 |

The Y'Z column is computed using the Y and Z columns and then using the columns X and Y'Z, the column F is computed
The column Y'Z is **not** an essential part of truth table

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9

Basic Identities

- For the AND operation
- For the OR operation

$$X \cdot 1 = X$$

$$X + 0 = X$$

$$X \cdot 0 = 0$$

$$X + 1 = 1$$

$$X \cdot X = X$$

$$X + X = X$$

$$X \cdot X' = 0$$

$$X + X' = 1$$

- For the NOT operation

$$X'' = X$$

Notice the **duality**: start with one identity
-Replace the AND by OR
-Replace the 1 by 0 and vice versa
You end up with an identity from the other group

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10

Basic Identities (2)

- For the AND operation

| | |
|---|--------------------|
| <u>Commutative</u> : $X.Y=Y.X$ | OR Operation |
| <u>Associative</u> : $X(YZ)=(XY)Z$ | $X+Y=Y+X$ |
| <u>Distributive</u> : $X+YZ=(X+Y)(X+Z)$ | $X+(Y+Z)=(X+Y)+Z$ |
| <u>DeMorgan's</u> : $(X.Y)'=X'+Y'$ | $X(Y+Z)=(XY)+(XZ)$ |
| | $(X+Y)'=X'.Y'$ |

- All above properties can be generalized to $n > 2$ variables: e.g:
 - $(X_1+X_2+\dots+X_n)' = X_1'.X_2'. \dots .X_n'$, or
 - $(X_1.X_2. \dots .X_n)' = X_1'+X_2'+ \dots +X_n'$

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11

Verifying Basic Identities

- Any identity (not only the basic ones) can be verified using the truth table
- Example: verify that $(X+Y)' = X'.Y'$

| X | Y | X+Y | $(X+Y)'$ | X' | Y' | $X'.Y'$ |
|---|---|-----|----------|----|----|---------|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |

ALL possible combinations of inputs

Some columns to aid in calculations

The two quantities in question

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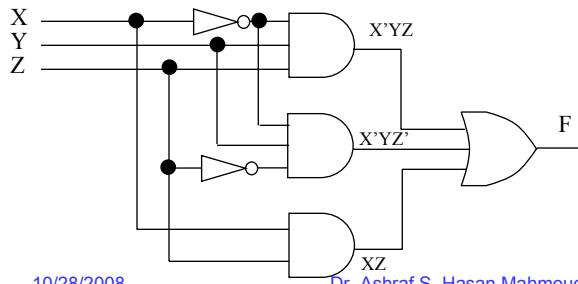
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12

Algebraic Manipulation - Example

- Consider the following function, F

$$F = X'YZ + X'YZ' + XZ$$
- The function can be implemented using above expressions as in



We need:
 -2 inverters
 -3 AND gates
 -1 OR gate

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13

Algebraic Manipulation – Example – cont'd

- The function

$$F = X'YZ + X'YZ' + XZ$$

can be simplified "ALGEBRAICALLY" as follows:

$$F = X'YZ + X'YZ' + XZ$$

$$= XY(Z + Z') + XZ \quad \rightarrow \text{by the distributive property}$$

$$= XY(1) + XZ \quad \rightarrow \text{by the properties of the OR operation}$$

$$= XY + XZ \quad \rightarrow \text{by the properties of the AND operation}$$

- Therefore F can be written as

$$F = XY + XZ$$

- Using this simpler form, one can implement the function as

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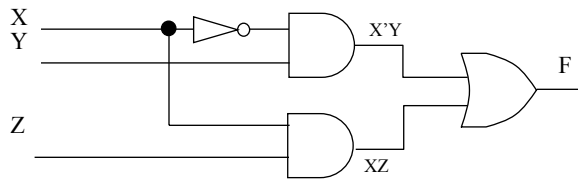
14

Algebraic Manipulation – Example – cont'd

- Therefore F can be written as

$$F = X'Y + XZ$$

- Using this simpler form, one can implement the function as



We need:
-1 inverters
-2 AND gates
-1 OR gate



Reduced hardware cost

- One can use the truth table method to show that $F = X'YZ + X'YZ' + XZ$ is indeed equal to $X'Y + XZ$

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15

More Notes on Function

- The function

$$F = X'Y + XZ$$

Can be written as

$$F(X,Y,Z) = X'Y + XZ$$

These are referred to as literals

These are referred to as terms

This is to emphasize the fact that the function has three inputs or variables

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16

More Identities

- Page ?? in the text → VERY IMPORTANT – make sure you can prove/verify all of these identities
- Listing
 1. $X + XY = X$
 2. $XY + XY' = X$
 3. $X + X'Y = X + Y$
 4. $X(X + Y) = X$
 5. $(X + Y)(X + Y') = X$
 6. $X(X' + Y) = XY$
 7. $XY + X'Z + YZ = XY + X'Z$ (the consensus theorem)

The proof/verification of these is in the textbook

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17

More Identities - continued

- Using the duality principle (refer to slide XX) there are other equivalent 7 identities
- Example: The proof of the consensus theorem is as follows

$$\begin{aligned}\text{The RHS} &= XY + X'Z + YZ \\ &= XY + X'Z + YZ(X + X') \\ &= XY + X'Z + XYZ + X'YZ \\ &= XY + XYZ + X'Z + X'YZ \\ &= XY(1+Z) + X'Z(1+Y) \\ &= XY + X'Z \\ &= \text{LHS}\end{aligned}$$

- The dual of the consensus theorem is given by
 $(X+Y)(X'+Z)(Y+Z) = (X+Y)(X'+Z)$

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18

Complement of a Function

- Using the truth table – complementing F means replacing every 0 with 1 and every 1 with 0 in the F column
- Algebraically, complementing F one can use DeMorgan's rule or the duality principle
- To use the duality principle
 - Replace Each AND with an OR and each OR with an AND
 - Complement each variable and constant

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19

Example

- **Problem:** Find the complement of each of the following two functions $F_1 = X'YZ' + X'Y'Z$, and $F_2 = X(Y'Z' + YZ)$

- **Solution:**

For F_1 , applying DeMorgan's rule as many times as necessary

$$\begin{aligned}F_1' &= (X'YZ' + X'Y'Z)' \\ &= (X'YZ')' \cdot (X'Y'Z)' \\ &= (X + Y' + Z) \cdot (X + Y + Z')\end{aligned}$$

Similarly for F_2 :

$$\begin{aligned}F_2' &= (X(Y'Z' + YZ))' \\ &= X' + (Y'Z' + YZ)' \\ &= X' + (Y'Z')' \cdot (YZ)' \\ &= X' + (Y + Z) \cdot (Y' + Z')\end{aligned}$$

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20

Examples

- **Problem 2-2:** Prove the identity of each of the following Boolean equations, using algebraic manipulations.

a) $X'Y' + X'Y + XY = X' + Y$

b) $A'B + B'C' + AB + B'C = 1$

- **Solution:**

a) LHS $= X'Y' + X'Y + XY$
 $= X'Y' + X'Y + X'Y + XY$
 $= X'(Y'+Y) + Y(X + X')$
 $= X' + Y$

$= \text{RHS}$

b) LHS $= A'B + B'C' + AB + B'C$
 $= (A'+A)B + B'(C'+C)$
 $= B + B'$

$= 1$

$= \text{RHS}$

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21

Examples

- **Problem 2-6:** Simplify the following Boolean expressions to a minimum number of literals:

a) $ABC + ABC' + A'B$

e) $(A+B'+AB')(AB+A'C+BC)$

- **Solution:**

a) Expression $= ABC + ABC' + A'B$
 $= AB(C + C') + A'B$
 $= (A+A')B$
 $= B$

e) Expression $= (A+B'+AB')(AB+A'C+BC)$
 $= (A+(1+A)B')(AB + A'C)$
 $= (A+B')(AB+A'C)$
 $= A(AB+A'C) + B'(AB+A'C)$
 $= AB + A'B'C$

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22

Standard Forms of a Boolean Function

- A Boolean function can be written algebraically in a variety of ways
- Standard form: is an algebraic expression of the function that facilitates simplification procedures and frequently results in more desirable logic circuits (e.g. less number of gates)
- Standard form: contains product terms and sum terms
 - Product term: $X'Y'Z$
 - Sum term: $X + Y + Z'$

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23

Standard Forms of a Boolean Function – cont'd

- A minterm: a product term in which all variables (or literals) of the function appear exactly once
- A maxterm: a sum term in which all the variables (or literals) of the function appear exactly once
- Example: for the function $F(X,Y,Z)$,
 - the term $X'Y$ is not a minterm, but XYZ' is a minterm
 - The term $X'+Z$ is not a maxterm, but $X+Y'+Z'$ is maxterm

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24

More on Minterms and Maxterms

- A function of n variables – have 2^n possible minterms and 2^n possible maxterms
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25

Naming Convention for Minterms

- Consider a function $F(X, Y)$

| X | Y | Product Terms | Symbol | m_0 | m_1 | m_2 | m_3 |
|---|---|---------------|--------|-------|-------|-------|-------|
| 0 | 0 | $X'Y'$ | m_0 | 1 | 0 | 0 | 0 |
| 0 | 1 | $X'Y$ | m_1 | 0 | 1 | 0 | 0 |
| 1 | 0 | XY' | m_2 | 0 | 0 | 1 | 0 |
| 1 | 1 | XY | m_3 | 0 | 0 | 0 | 1 |



Variable complemented if 0
Variable not complemented if 1

m_i indicated the i^{th} minterm
For each binary combination of X and Y there is a minterm
The index of the minterm is specified by the binary combination
 m_i is equal to 1 for ONLY THAT combination

Naming Convention for Maxterms

- Consider a function $F(X, Y)$

| X | Y | Sum Terms | Symbol | M_0 | M_1 | M_2 | M_3 |
|---|---|-----------|--------|-------|-------|-------|-------|
| 0 | 0 | $X+Y$ | M_0 | 0 | 1 | 1 | 1 |
| 0 | 1 | $X+Y'$ | M_1 | 1 | 0 | 1 | 1 |
| 1 | 0 | $X'+Y$ | M_2 | 1 | 1 | 0 | 1 |
| 1 | 1 | $X'+Y'$ | M_3 | 1 | 1 | 1 | 0 |



Variable complemented if 1
Variable not complemented if 0

M_i indicated the i^{th} maxterm
For each binary combination of X and Y there is a maxterm
The index of the maxterm is specified by the binary combination
 M_i is equal to 0 for ONLY THAT combination

More on Minterms and Maxterms

- In general, a function of n variables has
 - 2^n minterms: $m_0, m_1, \dots, m_{2^n-1}$
 - 2^n maxterms: $M_0, M_1, \dots, M_{2^n-1}$

- $m_i' = M_i$ or $M_i' = m_i$

Example: for $F(X, Y)$:

$$m_2 = XY' \rightarrow m_2' = X'+Y = M_2$$

More on Minterms and Maxterms – cont'd

- A Boolean function can be expressed algebraically from a give truth table by forming the logical sum of ALL the minterms that produce 1 in the function

- Example:**

Consider the function defined by the truth table

$F(X,Y,Z) \rightarrow 3$ variables $\rightarrow 8$ minterms

F can be written as

$$\begin{aligned} F &= X'Y'Z' + X'YZ' + XY'Z + XYZ, \text{ or} \\ &= m_0 + m_2 + m_5 + m_7 \\ &= \Sigma m(0,2,5,7) \end{aligned}$$

| X | Y | Z | m | F |
|---|---|---|-------|---|
| 0 | 0 | 0 | m_0 | 1 |
| 0 | 0 | 1 | m_1 | 0 |
| 0 | 1 | 0 | m_2 | 1 |
| 0 | 1 | 1 | m_3 | 0 |
| 1 | 0 | 0 | m_4 | 0 |
| 1 | 0 | 1 | m_5 | 1 |
| 1 | 1 | 0 | m_6 | 0 |
| 1 | 1 | 1 | m_7 | 1 |

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29

More on Minterms and Maxterms – cont'd

- A Boolean function can be expressed algebraically from a give truth table by forming the logical product of ALL the maxterms that produce 0 in the function

- Example:**

Consider the function defined by the truth table

$F(X,Y,Z) \rightarrow$ in a manner similar to the previous example, F' can be written as

$$\begin{aligned} F' &= m_1 + m_3 + m_4 + m_6 \\ &= \Sigma m(1,3,4,6) \end{aligned}$$

Now apply DeMorgan's rule

$$\begin{aligned} F &= F' = [m_1 + m_3 + m_4 + m_6]' \\ &= m_1' \cdot m_3' \cdot m_4' \cdot m_6' \\ &= M_1 \cdot M_3 \cdot M_4 \cdot M_6 \\ &= \Pi M(1,3,4,6) \end{aligned}$$

| X | Y | Z | M | F | F' |
|---|---|---|-------|---|----|
| 0 | 0 | 0 | M_0 | 1 | 0 |
| 0 | 0 | 1 | M_1 | 0 | 1 |
| 0 | 1 | 0 | M_2 | 1 | 0 |
| 0 | 1 | 1 | M_3 | 0 | 1 |
| 1 | 0 | 0 | M_4 | 0 | 1 |
| 1 | 0 | 1 | M_5 | 1 | 0 |
| 1 | 1 | 0 | M_6 | 0 | 1 |
| 1 | 1 | 1 | M_7 | 1 | 0 |

Note the indices in this list are those that are missing from the previous list in $\Sigma m(0,2,5,7)$

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30

Summary

- A Boolean function can be expressed algebraically as:
 - The logical sum of minterms
 - The logical product of maxterms
- Given the truth table, writing F as
 - Σm_i – for all minterms that produce 1 in the table, or
 - ΠM_i – for all maxterms that produce 0 in the table
- Another way to obtain the Σm_i or ΠM_i is to use ALGEBRA – see next example

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31

Example:

- Write $E = Y' + X'Z'$ in the form of Σm_i and ΠM_i ?

- Solution: **Method1**

First construct the Truth Table as shown

Second:

$E = \Sigma m(0,1,2,4,5)$, and

$E = \Pi M(3,6,7)$

| X | Y | Z | m | M | E |
|---|---|---|-------|-------|---|
| 0 | 0 | 0 | m_0 | M_0 | 1 |
| 0 | 0 | 1 | m_1 | M_1 | 1 |
| 0 | 1 | 0 | m_2 | M_2 | 1 |
| 0 | 1 | 1 | m_3 | M_3 | 0 |
| 1 | 0 | 0 | m_4 | M_4 | 1 |
| 1 | 0 | 1 | m_5 | M_5 | 1 |
| 1 | 1 | 0 | m_6 | M_6 | 0 |
| 1 | 1 | 1 | m_7 | M_7 | 0 |

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32

Example: cont'd

- Solution: **Method2 a**

$$\begin{aligned}
 E &= Y' + X'Z' \\
 &= Y'(X+X')(Z+Z') + X'Z'(Y+Y') \\
 &= (XY'+X'Y')(Z+Z') + X'YZ'+X'Z'Y' \\
 &= XY'Z+X'Y'Z+XY'Z'+X'Y'Z'+ \\
 &\quad X'YZ'+X'Z'Y' \\
 &= m_5 + m_1 + m_4 + m_0 + m_2 + m_0 \\
 &= m_0 + m_1 + m_2 + m_4 + m_5 \\
 &= \Sigma m(0,1,2,4,5)
 \end{aligned}$$

To find the form ΠM_i , consider the remaining indices

$$E = \Pi M(3,6,7)$$

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- Solution: **Method2 b**

$$\begin{aligned}
 E &= Y' + X'Z' \\
 E' &= Y(X+Z) \\
 &= YX + YZ \\
 &= YX(Z+Z') + YZ(X+X') \\
 &= XYZ+XYZ'+X'YZ \\
 E &= (X'+Y'+Z')(X'+Y'+Z)(X+Y'+Z') \\
 &= M_7 \cdot M_6 \cdot M_3 \\
 &= \Pi M(3,6,7)
 \end{aligned}$$

To find the form Σm_i , consider the remaining indices

$$E = \Sigma m(0,1,2,4,5)$$

33

Exercise

- What is $G(X,Y) = \Sigma m(0,1,2,3)$ equal to?

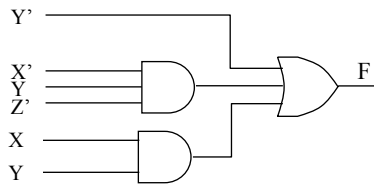
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34

Implementation – Sum of Products

- Consider $F = Y' + X'YZ' + XY$
 - Three products: Y' (one literal), $X'YZ'$ (three literals), and XY (two literals)
- The logic diagram



- Two-level implementation:
 - AND-OR
 - Each product term requires an AND gate (except one literal terms)
 - Logic diagram requires ONE OR gate

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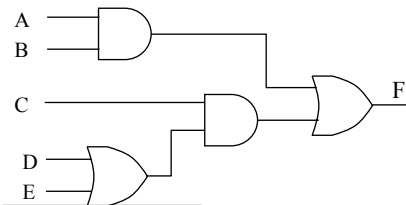
35

Implementation – Sum of Products – cont'd

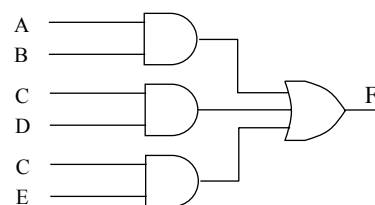
- Consider $F = AB + C(D+E)$
- This expression is NOT in the sum-of-products form
- Use the identities/algebraic manipulation to convert to a standard form (sum of products), as in

$$F = AB + CD + CE$$

- Logic Diagrams:



3-level circuit



2-level circuit

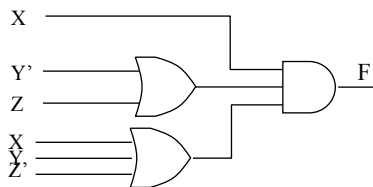
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36

Implementation – Product of Sums

- Consider $F = X(Y'+Z)(X+Y+Z')$
- This expression is in the product-of-sums form:
 - Three summation terms: X (one literal), $Y'+Z$ (two literals), and $X+Y+Z'$ (three literals)
- Logic Diagrams:
 - Two-level implementation:
 - OR-AND
 - Each sum term requires an OR gate (except one literal terms)
 - Logic diagram requires ONE AND gate



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37

Examples:

- Problem 2-10b: Obtain the truth table of the following function and express each function in sum-of-minterms and product-of-maximterms form: $(A'+B)(B'+C)$

- Solution:

$$\text{Let } F(A,B,C) = (A'+B)(B'+C)$$

The truth table is as shown in figure

$$F(A,B,C) = A'B'C' + A'B'C + A'BC + ABC \\ = \Sigma m(0,1,3,7)$$

$$F(A,B,C) = (A+B'+C)(A'+B+C)(A'+B+C')(A'+B'+C) \\ = \Pi M(2,4,5,6)$$

| A | B | C | F |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

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38