King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 541 – Design and Analysis of Local Area Networks

Term 031

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Primer on Probability Theory

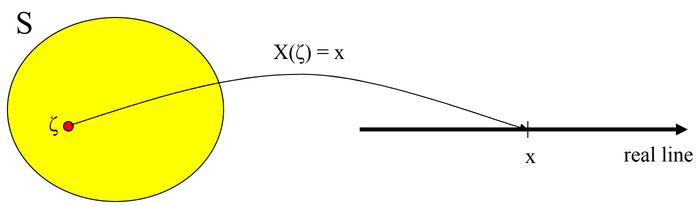
- **Source: Chapter 3 of:**

Alberto Leon-Garcia, Probability and Random Processes for Electrical Engineering, Addison Wisely

What is a Random Variable?

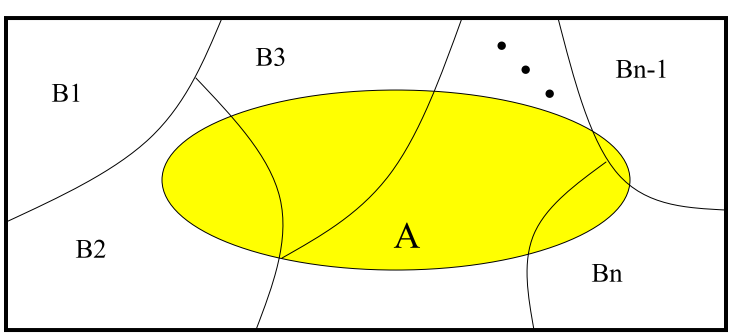
- **Random Experiment**
- **Sample Space**

- **Def: A random variable X is a function that assigns a number of X(ζ) to each outcome ζ in the sample space of S of the random experiment**



Bayes Rule

- **Let B1, B2, …, Bn be a partition of a sample space S. Suppose the event A occurs**



A Partition of S into n disjoint sets

Bayes Rule

- Theorem on total Probability:

$$P[A] = P[A/B1]P[B1] + P[A/B2]P[B2] + \ldots + P[A/Bn]P[Bn]$$

- What is the probability of the event Bj?

$$P[Bj/A] = \frac{P[A \wedge Bj]}{P[A]} = \frac{P[A/Bj]\,P[Bj]}{\sum P[A/Bk]\,P[Bk] \quad k=1,\ldots,n}$$

The Cumulative Distribution Function

- The cumulative distribution function (cdf) of a random variable X is defined as the probability of the event $\{X \leq x\}$:

$$F_X(x) = \text{Prob}\{X \leq x\} \quad \text{for } -\infty < x < \infty$$

i.e. it is equal to the probability the variable X takes on a value in the set $(-\infty, x]$

- A convenient way to specify the probability of all semi-infinite intervals

Properties of the CDF

- $0 \le F_X(x) \le 1$
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F_X(x) = 0$
- $F_X(x)$ is a nondecreasing function $\Rightarrow$ if $a < b$ $\Rightarrow$ $F_X(a) \le F_X(b)$
- $F_X(x)$ is continuous from the right $\Rightarrow$ for $h > 0$,
  $$F_X(b) = \lim_{h \to 0} F_X(b+h) = F_X(b^+)$$
- $\text{Prob}\,[a < X \le b] = F_X(b) - F_X(a)$
- $\text{Prob}\,[X = b] = F_X(b) - F_X(b^-)$

Example 1: Exponential Random Variable

- Problem: The transmission time X of a message in a communication system obey the exponential probability law with parameter λ, that is
  $$\text{Prob}\,[X > x] = e^{-\lambda x} \quad x > 0$$

Find the CDF of X. Find $\text{Prob}\,[T < X \le 2T]$ where $T = 1/\lambda$

Example 1: Exponential Random Variable – cont'd

- **Answer:**

The CDF of X is

$F_X(x) = \text{Prob}\{X \le x\} = 1 - \text{Prob}\{X > x\}$

$= 1 - e^{-\lambda x} \quad x \ge 0$

$= 0 \quad x < 0$

$\text{Prob}\{T < X \le 2T\} = F_X(2T) - F_X(T)$

$= 1 - e^{-2} - (1 - e^{-1})$

$= 0.233$

Example 2: Use of Bayes Rule

- **Problem: The waiting time W of a customer in a queueing system is zero if he finds the system idle, and an exponentially distributed random length of time if he finds the system busy. The probabilities that he finds the system idle or busy are p and 1-p, respectively. Find the CDF of W**

Example 2: cont'd

- **Answer:**

The CDF of W is found as follows:

$F_X(x) = \text{Prob}\{W \le x\}$

$= \text{Prob}\{W \le x/\text{idle}\}p + \text{Prob}\{W \le x/\text{busy}\}(1\text{-}p)$

Note $\text{Prob}\{W \le x/\text{idle}\} = 1$ **for any** $x > 0$

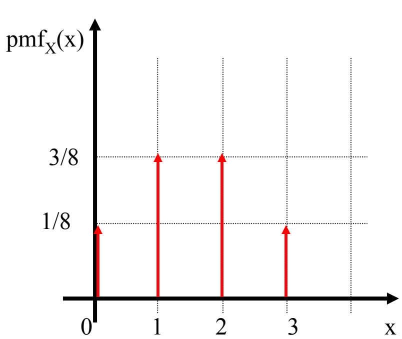
➔

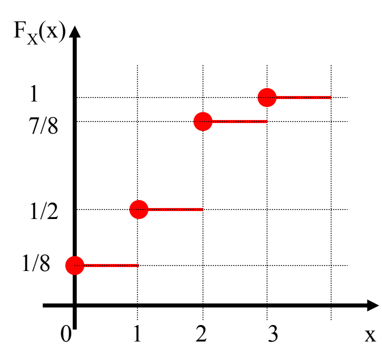
$F_X(x) = 0 \quad x < 0$

$= p+(1\text{-}p)(1\text{- }e^{-\lambda x}) \quad x \ge 0$

Types of Random Variables

- **(1) Discrete Random Variables**
  - **CDF is right continuous, staircase function of x, with jumps at countable set x0, x1, x2, …**





Pmf: probability mass function

Types of Random Variables

- **(2) Continuous Random Variables**
  - **CDF is contineous for all values of x → Prob { X = x} = 0 (recall the CDF properties)**
  - **Can be written as the integral of some non negative function**

$$F_X(x) = \int_{-\infty}^{\infty} f(t)dt$$

Or

$$f(t) = \frac{dF_X(x)}{dx}$$

f(t) is referred to as the probability density function or PDF

Types of Random Variables

- **(3) Random Variables of Mixed Types**

$$F_X(x) = p\,F_1(x) + (1-p)\,F_2(x)$$

Probability Density Function

- **The PDF of X, if it exists, is define as the derivative of CDF $F_X(x)$:**

$$f_x(x) = \frac{dF_X(x)}{dx}$$

Properties of the PDF

- **$f_x(x) \geq 0$**
- $$P\{a \leq x \leq b\} = \int_a^b f_x(x)dx$$
- $$F_X(x) = \int_{-\infty}^{x} f_x(t)dt$$
- $$1 = \int_{-\infty}^{\infty} f_x(t)dt$$

A valid pdf can be formed from any nonnegative, piecewise continuous function g(x) that has a finite integral:

$$\int_{-\infty}^{\infty} g(x)dx = c < \infty$$

By letting $f_X(x) = g(x)/c$, we obtain a function that satisfies the normalization condition.
This is the scheme we use to generate pdfs from simulation results!

Conditional PDFs and CDFs

- If some event A concerning X is given, then conditional CDF of X given A is defined by

$$F_X(x/A) = \frac{P\{[X \leq x] \wedge A\}}{P\{A\}} \quad \text{if } P\{A\} > 0$$

The conditional pdf of X given A is then defined by

$$f_X(x/A) = \frac{d}{dx} F_X(x/A)$$

Expectation of a Random Variable

- Expectation of the random variable X can be computed by

$$E[X] = \sum_{\forall i} x_i P[X = x_i]$$

for discrete variables, or

$$E[X] = \int_{-\infty}^{\infty} t f_x(t) dt$$

for continuous variables.

n^{th} Expectation of a Random Variable

- The n^{th} expectation of the random variable X can be computed by

$$E[X^n] = \sum_{\forall i} x^n{}_i P[X = x_i]$$

for discrete variables, or

$$E[X^n] = \int_{-\infty}^{\infty} t^n f_x(t)dt$$

for continuous variables.

The Characteristic Function

- The characteristic function of a random variable X is defined by

$$\Phi_x(\omega) = E[e^{j\omega X}]$$
$$= \int_{-\infty}^{\infty} f_X(x)e^{j\omega X}dx$$

- Note that $\Phi_X(\omega)$ is simply the Fourier Transform of the PDF $f_X(x)$ (with a reversal in the sign of the exponent)
- The above is valid for continuous random variables only

The Characteristic Function (2)

- **Properties:**

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_x(\omega)\bigg|_{\omega=0}$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) e^{-j\omega x} d\omega$$

The Characteristic Function (3)

- **For discrete random variables,**

$$\Phi_x(\omega) = E[e^{j\omega X}]$$
$$= \sum_{\forall k} p_X(x_k) e^{j\omega x_k}$$

- **For integer valued random variables,**

$$\Phi_x(\omega) = \sum_{k=-\infty}^{\infty} p_X(k) e^{j\omega k}$$

The Characteristic Function (4)

- **Properties**

$$p_X(k) = \frac{1}{2\pi}\int_0^{2\pi} \Phi_x(\omega) e^{-j\omega k} d\omega$$

for k=0, ±1, ±2, ...

Expectation of a Function of the Random Variable

- **Let g(x) be a function of the random variable x, the expectation of g(x) is given by**

$$E[g(x)] = \sum_{\forall i} g(x_i) P[X = x_i]$$

for discrete variables, or

$$E[g(x)] = \int_{-\infty}^{\infty} g(t) f_x(t) dt$$

for continuous variables.

Probability Generating Function

- A matter of convenience – compact representation
- The same as the z-transform
- If N is a non-negative integer-valued random variable, the probability generating function is defined as

$$G_N(z) = E[z^N] = \sum_{k=0}^{\infty} p_N(k) z^k = p_N(0) + p_N(1)z + p_N(2)z^2 + \ldots.$$

Probability Generating Function (2)

- Properties:
- 1 $$p_N(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_N(z) \bigg|_{z=0}$$
- 2 $$E[N] = G'_N(1)$$
- 3 $$Var[N] = G''_N(1) + G'_N(1) - [G'_N(1)]^2$$

Probability Generating Function (3)

- **For non-negative continuous random variables, let us define the Laplace transform of the PDF**

$$X^*(s) = \int_0^\infty f_X(x)e^{-sx}dx$$

$$= E[e^{-sx}]$$

Properties:

$$E[X^n] = (-1)^n \frac{d^n}{ds^n} X^*(s)\Big|_{s=0}$$

Some Important Random Variables – Discrete Random Variables

- **Bernoulli**
- **Binomial**
- **Geometric**
- **Poisson**

Bernoulli Random Variable

- Let A be an event related to the outcomes of some random experiment. The indicator function for A is defined as

$$I_A(\zeta) = 0 \quad \text{if } \zeta \text{ not in A}$$
$$= 1 \quad \text{if } \zeta \text{ is in A}$$

- IA is random variable since it assigns a number to each outcome in S
- It is discrete r.v. that takes on values from the set {0,1}
- PMF is given by

$$p_I(0) = 1\text{-}p,\ p_I(1) = p$$

where P{A} = p

- Describes the outcome of a Bernoulli trial

- E[X] = p, VAR[X] = p(1-p)
- $G_x(z) = (1\text{-}p\text{+}pz)$

Binomial Random Variable

- Suppose a random experiment is repeated n independent times; let X be the number of times a certain event A occurs in these n trials

$$X = I1 + I2 + \ldots + In$$

i.e. X is the sum of Bernoulli trials (X's range = {0, 1, 2, …, n})

- X has the following pmf

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

for k=0, 1, 2, …, n

- E[X] = np, Var[X] = np(1-p)
- $G_X(z) = (1\text{-}p + pz)^n$

Geometric Random Variable

- Suppose a random experiment is repeated - We count the number of M of independent Bernoulli trials until the first occurrence of a success
- M is called geometric random variable
  - Range of M = 1, 2, 3, …
- X has the following pmf

$$P[X=k]=(1-p)^{k-1}p$$

for k=1, 2, 3, …

- E[X] = 1/p, Var[X] = $(1-p)/p^2$
- $G_X(z) = pz/(1-(1-p)z))$

Poisson Random Variable

- In many applications we are interested in counting the number of occurrences of an event in a certain time period
- The pmf is given by

$$P[X=k]=\frac{\alpha^k}{k!}e^{-\alpha}$$

For k=0, 1, 2, … ; α is the average number of event occurrences in the specified interval

- E[X] = α, Var[X] = α
- GX(z) = $e^{\alpha(z-1)}$

- Remember: time between events is exponentially distributed!
- Poisson is the limiting case for Binomial as n→∞, p →0, such that np = α

Some Important Random Variables – Continuous Random Variables

- Uniform
- Exponential
- Gaussian (Normal)
- Rayleigh
- Gamma
- …

Uniform Random Variables

- Realizations of the r.v. can take values from the interval [a, b]
- PDF $f_X(x) = 1/(b-a) \qquad a \le x \le b$
- $E[X] = (a+b)/2, \quad Var[X] = (b-a)^2/12$
- $\Phi_X(\omega) = [e^{j\omega b} - e^{j\omega a}]/(j\omega(b-a))$

Exponential Random Variables

- The exponential r.v. X with parameter λ has pdf

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

- And CDF given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

- Range of X: [0, ∞)
- $E[X] = 1/\lambda$, $Var[X] = 1/\lambda^2$
- $\Phi_X(\omega) = \lambda/(\lambda - j\omega)$

Exponential Random Variables – cont'd

- The exponential r.v. is the only r.v. with the memoryless property!!
- Memoryless Property:

  $P[X>t+h/ X>t] = P[X>h]$

i.e. the probability of having to wait h additional seconds given that one has already been waiting t second IS EXACTLY equal to the probability of waiting h seconds when one first begins to wait

Gaussian (Normal) Random Variable

- **Rises in situations where a random variable X is the sum of a large number of "small" random variables – central limit theorem**
- **PDF** $$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-m)^2/(2\sigma^2)}$$

For -∞<x< ∞; m and σ > 0 are real numbers

- **E[X] = m, Var[X] = σ**
- $\Phi_X(\omega) = e^{jm\omega - \sigma^2\omega^2/2}$
- **Under wide range of conditions X can be used to approximate the sum of a large number of independent random variables**

Rayleigh Random Variable

- **Rises in modeling of mobile channels**
- **Range: [0, ∞)**
- **PDF:** $$f_X(x) = \frac{x}{\alpha^2} e^{-x^2/(2\alpha^2)}$$
- **For x ≥ 0, α > 0**
- **E[X] = α√(π/2), Var[X] = (2-π/2)α²**

Gamma Random Variable

- Versatile distribution ~ appears in modeling of lifetime of devices and systems
- Has two parameters: $\alpha > 0$ and $\lambda > 0$
- PDF:

$$f_X(x) = \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

- For $0 < x < \infty$
- The quantity $\Gamma(z)$ is the gamma function and is specified by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

- The gamma function has the following properties:
  - $\Gamma(1/2) = \sqrt{\pi}$
  - $\Gamma(z+1) = z\Gamma(z)$ for $z>0$
  - $\Gamma(m+1) = m!$ For m nonnegative integer
- $E[X] = \alpha/\lambda$, $Var[X] = \alpha/\lambda^2$
- $\Phi_X(\omega) = 1/(1-j\omega/\lambda)^a$

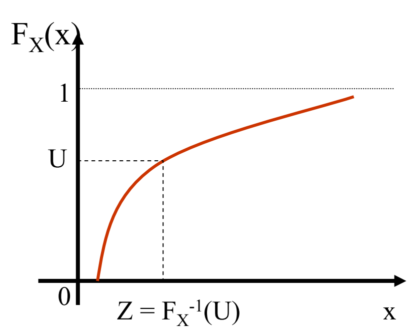
If $\alpha = 1$ ➔ gamma r.v. becomes exponential

Computer Methods for Generating Random Variables

(1) The transformation method

Procedure:

a. Obtain $F_X(x)$
b. Generate U ~ uniform between 0 and 1
c. Find $Z = F_X^{-1}(U)$ – Z follows the distribution specified by $f_X(x)$



Computer Methods for Generating Random Variables – Example 3

Problem: Generating exponential random variables with parameter λ

Answer:

To generate an exponentially distributed r.v. X with parameter λ (i.e. its mean is 1/λ), we need to find $F_X(x)$ and invert it.

$F_X(x) = 1 – e^{-\lambda x}$ (see example 1)

Therefore, $F_X^{-1}(x)$ is equal to

$$X = -(1/\lambda)\ \ln(1-U)$$

where ln(t) is the natural logarithm of t while U is a uniform r.v. between 0 and 1. Note that the above expression can be simplified to be

$$X = -(1/\lambda)\ \ln(U)$$

This is because 1-U is also a uniform random r.v. between 0 and 1

Computer Methods for Generating Random Variables

(2) Rejection Method

See references for details

Transformation method is sufficient for simulations required in this course

References

- Alberto Leon-Garcia, Probability and Random Processes for Electrical Engineering, Addison Wesley, 1989

- L. Kleinrock. *Queueing Theory*. Wiley, New York, 1975