King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 541 – Design and Analysis of Local Area Networks

Term 031

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Random/Stochastic Processes

- **Consider a random experiment specified by the outcomes ζ from some sample space S, by the events defined on S, and by the probabilities on these events. Suppose that every outcome ζ in S, we assign a function of time according to some rule:**

X(t, ζ) = t in I

The graph of X(t, ζ) versus t, for ζ fixed, is call a REALIZATION or sample path of the random process

A stochastic process is said to be discrete-time if the index set I is a countable set

Random/Stochastic Processes – Example 4

- Let ζ be a number selected at random from the interval S =[0,1], and let b1, b2, … be the binary expansion of ζ:

$$\zeta = \sum_{i=1}^{\infty} b_i 2^{-i}$$

Define the discrete-time random process X(n, ζ) by

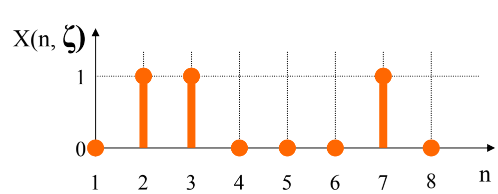
$$X(n, \zeta) = b_n \qquad \text{for } n = 1, 2, \ldots$$

Random/Stochastic Processes – Example 4

- Realizations of the random process

$$X(n, \zeta) = b_n \qquad \text{for } n = 1, 2, \ldots$$

For $\zeta = 2^{-2}+2^{-3}+2^{-7}$
$= 0.3828125$



For any ζ, you can produce a realization of X(n, ζ)

Random/Stochastic Processes – Example 5

- **Temperature recordings during day versus time**

Stationary Random Processes

- **Nature of randomness observed in the process does not change with time**

- **A discrete-time or continuous-time random process X(t) is stationary if the joint distribution of any set of sample does not depend on the placement of the time origin:**

Joints CDF of X(t1), X(t2), …, X(tk) is the same as joint CDF of X(t1+τ), X(t2+τ), …, X(tk+τ)

Wide-Sense Stationary Random Processes

- In many situations we can not determine whether a random process is stationary, but we can determine whether the mean is a constant:

$m_X(t) = m$ for all t

And whether the autocoverience (or autocorrelation) is a function of t1-t2 only:

$C_X(t1,t2) = C_X(t1-t2)$

➔ X(t) is a wide-sense stationary (WSS) process

Ergodic Processes

- Time averages = ensample average (expected value)
- Stats along the time access are the same as those resulting from different realizations

Markov Process

- **A random process X(t) is a Markov Process if the future of the process given the present is independent of the past.**

- **For arbitrary times: $t_1<t_2<\ldots<t_k<t_{k+1}$**

$$\text{Prob}[X(t_{k+1}) = x_{k+1}/X(t_k)=x_k, \ldots, X(t_1)=x_1]$$
$$= \text{Prob}[X(t_{k+1}) = x_{k+1}/X(t_k)=x_k]$$

Or (for discrete-valued)

$$\text{Prob}[a<X(t_{k+1})\leq b/X(t_k)=x_k, \ldots, X(t_1)=x_1]$$
$$= \text{Prob}[a<X(t_{k+1})\leq b/X(t_k)=x_k]$$

Markov Property

Markov Chain

- **An integer-valued Markov random process is called a Markov Chain**

- **The joint pmf for k+1 arbitrary time instances is given by:**

$$\text{Prob}[X(t_{k+1}) = x_{k+1}, X(t_k)=x_k, \ldots, X(t_1)=x_1]$$

$$= \text{Prob}[X(t_{k+1}) = x_{k+1}/X(t_k)=x_k] \times$$
$$\text{Prob}[X(t_k) = x_k/X(t_{k-1})=x_{k-1}] \times$$
$$\ldots$$
$$\text{Prob}[X(t_2) = x_2/X(t_1)=x_1] \times$$
$$\text{Prob}[X(t_1)=x_1]$$

transition probabilities

← pmf of the initial time

Discrete-Time Markov Chains

- **Let X_n be a discrete-time integer values Markov Chain that starts at n = 0 with pmf**

$p_j(0)$ = Prob[X_0 = j]   j=0,1,2, …

Prob[$X_n=i_n$, $X_{n-1}=i_{n-1}$,…,$X_0=i_0$]
= Prob[$X_n=i_n$/ $X_{n-1}=i_{n-1}$] X
Prob[$X_{n-1}=i_{n-1}$/ $X_{n-2}=i_{n-2}$] X
….
Prob[$X_1=i_1$/ $X_0=i_0$] X
Prob[$X_0=i_0$]

Same as the previous slide but for discrete-time

Discrete-Time Markov Chains – cont'd (2)

- **Assume the one-step state transition probabilities are fixed and do not change with time:**

Prob[X_{n+1}=j/X_n=i] = p_{ij} for all n

➔ **X_n is said to be homogeneous in time**

- **The joint pmf for X_n, X_{n-1}, …, X_1,X_0 is then given by**

$$P[X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0]$$
$$= p_{i_{n-1},i_n} \times p_{i_{n-2},i_{n-1}} \times \ldots \times p_{i_0,i_1} \times p_{i_0}(0)$$

Discrete-Time Markov Chains – cont'd (3)

- Thus X_n is completely specified by the initial pmf $p_i(0)$ and the matrix of one-step transition probabilities P:

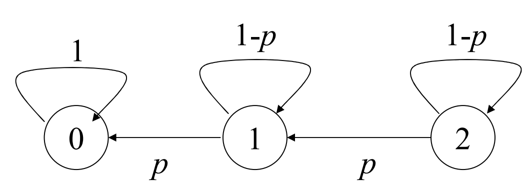
i.e. rows of P add to UNITY

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ . & . & . & . \\ p_{i0} & p_{i1} & p_{i2} & \cdots \\ . & . & . & . \end{bmatrix}$$

$$1 = \sum_j P[X_{n+1} = j / X_n = i] = \sum_j p_{ij}$$

Example 6: two-state Markov Chain

- On day 0 a house has two new light bulbs in reserve. The probability that the house will need a single new light bulb during day n is p and the probability that it will not need any is q = 1-p. Let Yn be the number of new light bulbs left in house at the end of day n.
- Yn is a Markov chain with state transition probability as shown



Example 6: two-state Markov Chain – cont'd

- **The state transition matrix P is given by**

$$
\begin{array}{c} Yn = 0 \quad 1 \quad 2 \\ \downarrow \quad \downarrow \quad \downarrow \end{array}
$$

$$
P = \begin{bmatrix} 1 & 0 & 0 \\ p & q & 0 \\ 0 & p & q \end{bmatrix}
$$

The n-step Transition Probabilities

- **Let $P(n) = \{p_{ij}(n)\}$ be the matrix of n-step transition probabilities, where**

$p_{ij}(n) = Prob[X_{n+k} = j / X_k = i] \quad n \geq 0; i,j \geq 0$

Note:

$Prob[X_{n+k} = j / X_k = i] = Prob[X_n = j / X_0 = i]$ for all n – why?

Transition probabilities do not depend on time (homogeneous)

It can be shown that:

$P(n) = \{p_{ij}(n)\} = P^n$ – where P is the 1-step transition probability matrix

The State Probabilities

- It can be shown that the state pmf at time n is obtained by multiplying the initial state pmf, p(0), by the n-step transition matrix, P(n), in other words

$$p(n) = p(0)\,P(n)$$
$$= p(0)\,P^{n}$$

Make a distinction between small p and capital P!

Example 7:

- Consider the problem given in Example 6 – find the n-step transition matrix and compute the state pmf p(n)

Example 7: cont'd

Answer: The n-step transition matrix can be found by multiplying P (the 1-step transition matrix) by itself n times or alternatively we can use:

$p_{22}(n)$ = Prob[no new light bulbs needed in n days] = q^n

$p_{21}(n)$ = Prob[1 light bulb needed in n days] = $n\,p\,q^{n-1}$

$p_{20}(n)$ = Prob[2 light bulbs needed in n days]

= $1 - p_{22}(n) - p_{21}(n)$

$p_{10}(n)$ = Prob[the one light bulb is not needed in n days] = $1 - q^n$

$p_{11}(n)$ = Prob[the one light bulb is not needed in n days] = q^n

$p_{12}(n) = 0$

$p_{00}(n) = 1$

$p_{01}(n) = 0$

$p_{02}(n) = 0$

Example 7: cont'd

- Therefore, the n-step transition matrix is given by

$$P^n = \begin{bmatrix} 1 & 0 & 0 \\ 1-q^n & q^n & 0 \\ 1-q^n-npq^n & npq^n & q^n \end{bmatrix}$$

Example 7: cont'd

- **Notes:**
  - **For all transition matrices, sum of any row SHOULD equal to ONE**
  - **For q = 1-p < 1 → as n→∞, then P^n limit is**

$$\lim_{n\to\infty} P^n \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Example 7: cont'd

- **Therefore, if we start with 2 light bulbs, then the state pmf p(n) approaches**

p(n) = p(0) P^n

$$p(n) \to [0 \quad 0 \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = [1 \quad 0 \quad 0]$$

Meaning – if n approaches ∞, then it is almost certain we will end up in the 0 (no light bulbs) state

Steady State Probabilities

- Some Markov chains settle into stationary behavior. As n → ∞, the n-step transition matrix approaches a matrix in which all rows are equal to the same pmf, that is

$$p_{ij}(n) \rightarrow \pi_j$$

Therefore,

$$p_j(n) \rightarrow \sum_i \pi_j \, p_i(0) = \pi_j$$

$$\Rightarrow \quad \pi_j = \sum_i p_{ij} \, \pi_j$$

Or in matrix form

$$\Pi = \Pi \, P \quad \text{- where } \Pi = \{\pi_j\}$$

In general the above formation has n-1 linearly independent equations – the additional equation required is provided by

$$\sum_i \pi_i = 1 \quad \text{or}$$

Steady State Probabilities – cont'd 2

- In other words:
- At steady state (n is very large) – the nth state pmf is the same as the n+1st state pmf
- Meaning the nth (n very large) state pmf is time invariant (steady state)

$$\Pi = \Pi \, P$$

Π → is the steady state pmf

P → is the 1-step transition matrix

Steady State Probabilities – cont'd 3

- Checking the dimensions:

Π ➔ is the steady state pmf of dimensions = 1Xk - assuming k states

= $[\pi_1\ \pi_2\ \pi_3\ \ldots\ \pi_k]$ where π_i $1 \le i \le k$ is the steady state probability for being in state i

P ➔ is the 1-step transition matrix of dimensions k X k

= $\{p_{ij}\}$ is the Probability of transitioning from state i to j

Recall that all rows of P sum to 1

Example: 8

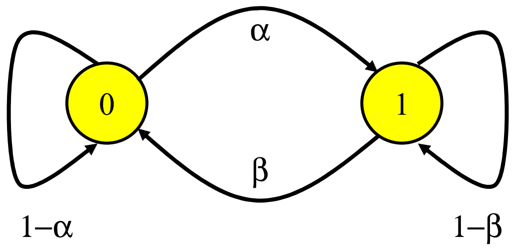
Problem: A Markov model for packet speech assumes that if the nth packet contains silence then the probability of silence in the next packet is $1-\alpha$ and the probability of speech activity is α. Similarly if the nth packet contains speech activity, then the probability of speech activity in next packet is $1-\beta$ and the probability of silence is β. Find the stationary state pmf.

Example: 8 – cont'd

Answer: The state diagram is as shown:

The 1-step transition probability, P, is given by:

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$



State 0: silence
State 1: speech

Example: 8 – cont'd 2

Answer: The steady state pmf $\Pi = [\pi_0 \ \pi_1]$ can be solved for using

$$\Pi = \Pi \mathbf{P}$$

Or

$$[\pi_0 \quad \pi_1] = [\pi_0 \quad \pi_1] \times \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Or

$$\pi_0 = (1-\alpha)\,\pi_0 + \beta\,\pi_1$$
$$\pi_1 = \alpha\,\pi_0 + (1-\beta)\,\pi_1$$

In addition to the constraint that $\pi_0 + \pi_1 = 1$

Example: 8 – cont'd 3

Answer: Therefore steady state pmf $\Pi = [\pi_0 \; \pi_1]$ is given by:

$\pi_0 = \beta / (\alpha + \beta)$

$\pi_1 = \alpha / (\alpha + \beta)$

Note that sum of all π_i's should equal to 1!!

For $\alpha = 1/10$, $\beta = 1/5 \rightarrow \Pi = [2/3 \;\; 1/3]$

Example: 8 – cont'd 4

Answer: Alternatively, one can find a general form for P^n and take the limit as $n \rightarrow \infty$.

P^n can be shown to be:

$$P^n = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} + \frac{(1-\alpha-\beta)^n}{\alpha+\beta}\begin{bmatrix} \alpha & -\alpha \\ -\beta & \beta \end{bmatrix}$$

Which clearly approaches:

$$\lim_{n\rightarrow\infty} P^n = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}$$

Example: 8 – cont'd 5

Answer: If the initial state pmf is $p_0(0)$ and $p_1(0) = 1-p_0(0)$

Then the nth state pmf ($n \to \infty$) is given by:

$$p(n) \text{ as } n \to \infty = [p_0(0) \;\; 1- p_0(0)]\, P^n$$

$$= [\beta/(\alpha+\beta) \quad \alpha/(\alpha+\beta)]$$

Same as the solution obtained using the 1-step transition matrix!!

Continuous-Time Markov Chains

- Back to the definition:

$$\text{Prob}[X(t_{k+1}) = x_{k+1}, X(t_k)=x_k, \ldots, X(t_1)=x_1]$$

$$= \text{Prob}[X(t_{k+1}) = x_{k+1}/X(t_k)=x_k] \times \text{Prob}[X(t_k) = x_k/X(t_{k-1})=x_{k-1}] \times \ldots \text{Prob}[X(t_2) = x_2/X(t_1)=x_1] \times \text{Prob}[X(t_1)=x_1]$$

- For continuous-time, the transition probability from an arbitrary time s to an arbitrary time s+t:

$$\text{Prob}[X(s+t) = j \,/\, X(s) = i] \qquad t \geq 0$$

Continuous-Time Markov Chains – cont'd

- For time-HOMOGENEOUS Markov chains:

  Prob[X(s+t) = j / X(s) = i]
  = Prob[X(t) = j / X(0) = i] $t \geq 0$

- Let $P(t) = \{p_{ij}(t)\}$ denote the matrix of transition probabilities in an interval of length t.
- Note: P(0) = I (identity matrix)

  since $p_{ii}(0) = 1$, and $p_{ij}(0) = 0$

  (in zero time if in state i, you will remain in i; and there is no chance in moving to state j)

Example 9: Poisson Process

- Consider a Poisson Process:

  Pij(t) = Prob[j-i events in t seconds]

  $= p_{0,j-i}(t)$

$$= \frac{(\alpha t)^{j-i}}{(j-i)!} e^{-\alpha t} \quad j \geq i$$

Therefore the transition matrix is given by

$$P(t) = \begin{matrix} e^{-\alpha t} & \alpha t e^{-\alpha t} & (\alpha t)^2/2! e^{-\alpha t} & . & . & . \\ 0 & e^{-\alpha t} & \alpha t e^{-\alpha t} & . & . & . \\ 0 & 0 & e^{-\alpha t} & \alpha t e^{-\alpha t} & . & . \\ . & . & & & & \\ . & . & & & & \end{matrix}$$

Example 9: Poisson Process – cont'd

- What is the dimension of the previous matrix?

- For a very small interval $t = \delta$, $e^{-\alpha\delta} \approx 1-\alpha\delta$

Therefore for a small interval, the transition matrix is given by

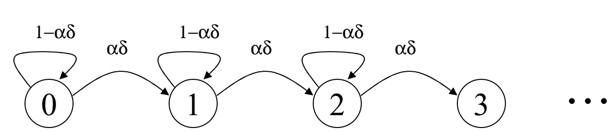
$$P(\delta) = \begin{matrix} 1-\alpha\delta & \alpha\delta & 0 & . & . & . \\ 0 & 1-\alpha\delta & \alpha\delta & . & . & . \\ 0 & 0 & 1-\alpha\delta & \alpha\delta & . & . \\ . & . & & & & \\ . & . & & & & \end{matrix}$$

Where terms including δ^2 or higher have been neglected (i.e. the probability of more than on transition in a very short time interval is negligible)

Example 9: Poisson Process – cont'd

- This is referred to as a pure birth process!

- State variable: number of events (arrivals) in δ seconds



State Occupancy Time

- How much time does a process spends in a particular state?

- **For all continuous-time Markov Chains, X(t) remains at a given value (state) for an exponentially-distributed random time**

- Why?

State Occupancy Time – cont'd

- The time spent in state i, T_i, is an exponential r.v. with some mean $1/v_i$:

$$\text{Prob}\,[\,T_i > t] = e^{-v_i t}$$

Therefore, the mean state occupancy time is $1/v_i$ – usually different for each state

Transition Rates and Time-Dependent State Probabilities

- **Consider the transition probabilities in a very short time duration δ seconds**

The probability the process remains in state i during the interval is:

$$Prob[T_i > \delta] = e^{-v_i \delta}$$
$$= 1 - v_i\delta/1! + v_i\, \delta^2/2! - \ldots$$
$$= 1 - v_i\delta + O(\delta)$$

Where $O(\delta)$ denotes terms that become negligible relative to δ as δ approaches zero

Transition Rates and Time-Dependent State Probabilities – cont'd

- **The exponential distribution of the state occupancy time, T_i, implies that it is highly unlikely that the process will make more than one transition from the ith state →**

Possibilities: **either remain in state i** — Zero transitions
or
leave state i — One transition

$$p_{ii}(\delta) = Prob[T_i > \delta]$$
$$= 1 - v_i\delta + O(\delta)$$

Or

$1 - p_{ii}(\delta) = v_i\delta$ **→ The rate at which process leaves state i is equal to v_i**

Transition Rates and Time-Dependent State Probabilities – cont'd 2

- Once the process leaves state i, it enters state j with probability q_{ij}
- Therefore,

$$p_{ij}(\delta) = (1-p_{ii}(\delta))\, q_{ij}$$
$$= v_i\, q_{ij}\, \delta + O(\delta)$$
$$= \gamma_{ij}\, \delta + O(\delta)$$

γ_{ij} is the rate at which the process X(t) enters state j from state i

Hence, $\gamma_{ii} = -v_i$!! Or

$$1 - p_{ii}(\delta) = \gamma_{ii}\, \delta$$

Transition Rates and Time-Dependent State Probabilities – cont'd 3

- To summarize:

Prob[leaves state i to state j in δ seconds]

$$= p_{ij}(\delta)$$
$$= \gamma_{ij}\, \delta + O(\delta)$$

Leaving state i to state j

And

$$1 - p_{ii}(\delta) = \gamma_{ii}\, \delta + O(\delta)$$

Leaving state i

Transition Rates and Time-Dependent State Probabilities – cont'd 4

- **Let's divide by δ and take the limit as δ goes to zero → to find the instantaneous rates of transition**

$p_{ij}(\delta) / \delta \rightarrow \gamma_{ij}$

$(1 - p_{ii}(\delta)) / \delta \rightarrow \gamma_{ii}$

Note that $O(\delta)/ \delta \rightarrow 0$ as $\delta \rightarrow 0$

Transition Rates and Time-Dependent State Probabilities – cont'd 5

- **Let's define $p_j(t)$ = Prob[X(t)=j]**

Then for $\delta > 0$, one can write:

Adding over all possible routes to state j

$p_j(t+\delta)$ = Prob[X($t+\delta$) = j]

$= \sum_i$ Prob[X($t+\delta$)=j/X(t) = i] Prob[X(t) = i]

$= \sum_i p_{ij}(\delta)\, p_i(t)$

Transition Rates and Time-Dependent State Probabilities – cont'd 6

Now subtract $p_j(t)$ from both sides

$$p_j(t+\delta) - p_j(t) = \sum_{i \neq j} p_{ij}(\delta)\, p_i(t) + (p_{jj}(\delta) - 1)p_j(t)$$

divide by δ and take the limit as $\delta \rightarrow 0$

$$p'_j(t) = \sum_{i} \gamma_{ij}\, p_i(t)$$

Note: if $\Gamma = \{\gamma_{ij}\}$ is the matrix of transition rates from state i to state j, then the rows of Γ add to zeros

This is a form of the Chapman-Kolmogrov Equations

Example 10:

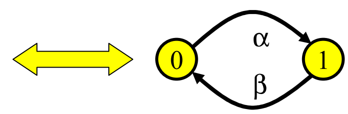
Problem: Consider a queueing system that alternates between two states. In state 0, the system is idle and waiting from a customer to arrive. This idle time is an exponential r.v. with mean $1/\alpha$. In state 1, the system is busy servicing a customer. The time in the busy state is an exponential r.v. with mean $1/\beta$. Find the state probabilities $p_0(t)$ and $p_1(t)$ in terms of the initial state probabilities $p_0(0)$ and $p_1(0)$

Example 10: cont'd

<u>**Answer**</u>**: The system moves from state 0 to state 1 at rate of α, and from state 1 to state 0 at rate β.**

Therefore: $\gamma_{00} = -\alpha \qquad \gamma_{01} = \alpha$

$\gamma_{10} = \beta \qquad \gamma_{11} = -\beta$



Using C-K equations

$$p'_0(t) = -\alpha\, p_0(t) + \beta\, p_1(t)$$
$$p'_1(t) = \alpha\, p_0(t) - \beta\, p_1(t)$$

In addition to the constraint $p_0(t) + p_1(t) = 1$

Example 10: cont'd

<u>**Answer**</u>**: Solving these differential equations, yields,**

$$p_0(t) = \beta/(\alpha+\beta) + (p_0(0) - \beta/(\alpha+\beta))\, e^{-(\alpha+\beta)t}$$

$$p_1(t) = \alpha/(\alpha+\beta) + (p_1(0) - \alpha/(\alpha+\beta))\, e^{-(\alpha+\beta)t}$$

The above specify the probabilities at any instant t!

where $p_0(0)$ and $p_1(0)$ are the initial conditions needed to determine the constants in the differential equations solutions.

Example 10: cont'd

Answer: The steady state distribution can be obtained if we let $t \rightarrow \infty$

$$p_0(t) = \beta/(\alpha+\beta) \quad \text{as } t \rightarrow \infty$$

$$p_1(t) = \alpha/(\alpha+\beta) \quad \text{as } t \rightarrow \infty$$

Note this steady state distribution is independent of t and also independent of the initial state probabilities $p_0(0)$ and $p_1(0)$.

Steady State Probabilities and Global Balance Equations

- Remember C-K equations:

$$p'_j(t) = \sum_i \gamma_{ij} \, p_i(t) \quad \text{for all } j$$

If equilibrium exists, then $p'_j(t) = 0$ (i.e. no change in the state probabilities with time)

Therefore, at steady state (if it exists), the following holds:

$$0 = \sum_i \gamma_{ij} \, p_i(t) \quad \text{for all } j$$

These are referred to as the GLOBAL BALANCE EQUATIONS!!

All flows (rate X probability) algebraically added for any state j equal to ZERO

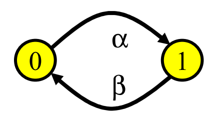
Example 11:

- **Problem: Consider the queueing system in Example 10 – find the steady state probabilities.**

- **Answer:**

$\gamma_{00} = -\alpha \qquad \gamma_{01} = \alpha$

$\gamma_{10} = \beta \qquad \gamma_{11} = -\beta$



Applying the global balance equations, yields

$\alpha\pi_0 = \beta\pi_1$ **and** $\beta\pi_1 = \alpha\pi_0$

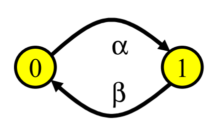
In addition to the constraints that: $\pi_0 + \pi_1 = 1$

Example 11: cont'd

- **Answer: Solving the previous simple equations leads to:**

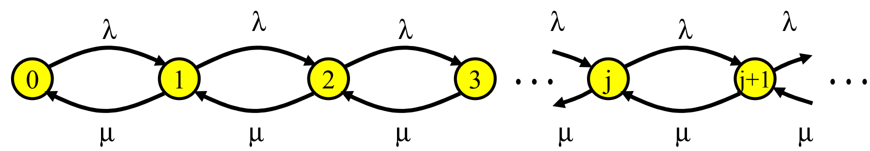
$\pi_0 = \beta/(\alpha+\beta)$

$\pi_1 = \alpha/(\alpha+\beta)$



Example 12:

- **<u>Problem</u>: The M/M/1 single-server queueing system**



Example 12: cont'd

- **<u>Answer</u>: The state transition rates:**
  - **Customers arrive with rate λ ➔ $\gamma_{i,i+1} = \lambda$ for i = 0, 1, 2, …**
  - **When system is not empty, customers depart at rate μ ➔ $\gamma_{i,i-1} = \mu$ for i = 1, 2, 3, …**

- **The global balance equations:**

$$\lambda p_0 = \mu p_1 \quad \text{for } j = 0$$

$$(\lambda + \mu)p_j = \lambda p_{j-1} + \mu p_{j+1} \quad \text{for } j=1, 2, \ldots$$

➔ $\lambda p_j - \mu p_{j+1} = \lambda p_{j-1} - \mu p_j$ **for j=1,2, …**

$= \text{constant}$

Example 12: cont'd

- **Answer**:

For j = 1, we have

$$\lambda p_0 - \mu p_1 = \text{constant}$$

Therefore the constant is equal to zero.

Hence,

$$\lambda p_{j-1} = \mu p_j \text{ or}$$
$$p_j = (\lambda/\mu)\, p_{j-1} \quad \text{for } j=1,2, \ldots$$

By simple induction:

$$p_j = \rho^j p_0$$

where $\rho = \lambda/\mu$

Example 12: cont'd

- **Answer**:

To obtain p_0, we use the fact that

$$1 = \sum_j p_j = (1+\rho+\rho^2+\ldots)p_0$$

note the above series converges only for $\rho < 1$ or equivalently $\lambda < \mu$

Therefore, $p_0 = 1-\rho$

In general, the steady state pmf for the M/M/1 queue is given by

$$p_j = (1-\rho)\rho^j$$

References

- **Alberto Leon-Garcia, Probability and Random Processes for Electrical Engineering, Addison Wesley, 1989**

- **L. Kleinrock. *Queueing Theory*. Wiley, New York, 1975**