King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 541 – Design and Analysis of Local Area Networks

Term 031

Dr. Ashraf S. Hasan Mahmoud

Rm 22-148-3

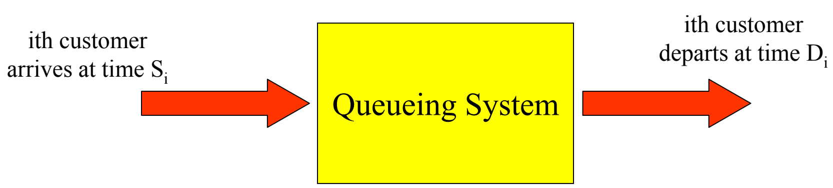
Ext. 1724

Email: ashraf@ccse.kfupm.edu.sa

Queuing Model

- **Consider the following system:**

A(t) $\quad$ $N(t) = A(t) - D(t)$ $\quad$ D(t)



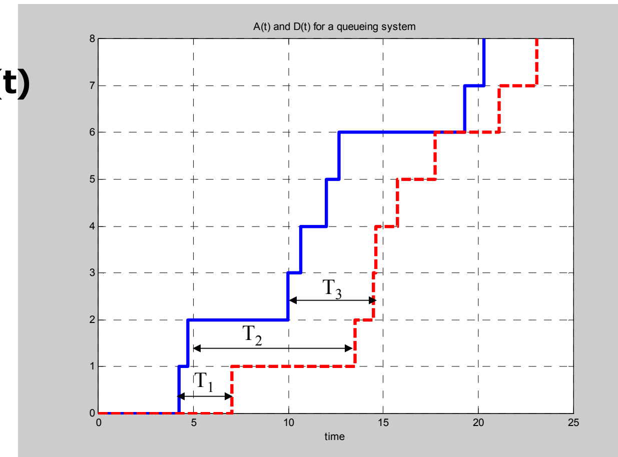
$$T_i = D_i - S_i$$

A(t) – number of arrivals in (0, t]
D(t) – number of departures in (0, t]
N(t) – number of customers in system in (0,t]
T_i – duration of time spent in system for ith customer

Number of Customers in System

- **Blue curve: A(t)**
- **Red curve: D(t)**

- **Total time spent in the system for all customers = area in between two curves**



Little's Formula

- **Consider the time average of the number of customers in the system N(t) during (0,t],**

$$\langle N\rangle_t = \frac{1}{t}\int_0^t N(\tau)d\tau$$

i.e. average area under the curve for N(t)

$<N>_t$ is also given by

$$\langle N\rangle_t = \frac{1}{t}\sum_{i=1}^{A(t)} T_i$$

Little's Formula – cont'd

- The average arrival rate $<\lambda>_t$ is given by

$$\langle \lambda \rangle_t = \frac{A(t)}{t}$$

- Combining the previous equations we get:

$$\langle N \rangle_t = \langle \lambda \rangle_t \frac{1}{A(t)} \sum_{i=1}^{A(t)} T_i$$

- Let the quantity $<T>_t$ be the average time a customer spends in the system, then

$$\langle T \rangle_t = \frac{1}{A(t)} \sum_{i=1}^{A(t)} T_i$$

Little's Formula – cont'd

- Combining the last two equations:

$$\langle N \rangle_t = \langle \lambda \rangle_t \langle T \rangle_t$$

- Which relates the time averages of the arrival rate, the number of customers in the system and the average time spent in the system
- Let t →∞, then one can write:

$$E[N] = \lambda E[T]$$

Little's Formula – cont'd

- **Little's formula:**

$$E[N] = \lambda E[T]$$

Holds for many service disciplines and for systems with arbitrary number of servers. It holds for many interpretations of the system as well

Example 1:

- **<u>Problem</u>: Let Ns(t) be the number of customers being served at time t, and let τ denote the service time. If we designate the set of servers to be the "system"m then Little's formula becomes:**

$$E[Ns] = \lambda E[\tau]$$

Where E[Ns] is the average number of busy servers for a system in the steady state.

Example 1: cont'd

Note: for a single server Ns(t) can be either 0 or 1 ➔ E[Ns] represents the portion of time the server is busy. If p_0 = Prob[Ns(t) = 0], then we have

$$1 - p_0 = E[Ns] = \lambda E[\tau], \text{ Or}$$
$$p_0 = 1 - \lambda E[\tau]$$

The quantity $\lambda E[\tau]$ is defined as the utilization for a single server. Usually, it is given the symbol ρ

$$\rho = \lambda E[\tau]$$

For a c-server system, we define the utilization (the fraction of busy servers) to be

$$\rho = \lambda E[\tau] / c$$

The M/M/1 Queue

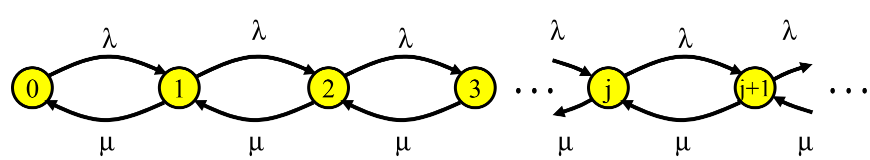
- **Consider a single server system where customers arrive according to a Poisson process of rate λ**
  - **➔ inter-arrival times are iid exponential r.v. with mean $1/\lambda$**
- **Assume the service times are iid exponential r.v. with mean $1/\mu$**
- **Assume the inter-arrival times and service times are independent**
- **Assume the system can accommodate unlimited number of customers**

The M/M/1 Queue – cont'd

- **What is the steady state pmf of N(t), the number of customers in the system?**
- **What is the PDF of T, the total customer delay in the system?**

The M/M/1 Queue – cont'd

- **Consider the transition rate diagram for M/M/1 system**



- **Note:**
  - **System state – number of customers in systems**
  - **λ is rate of customer arrivals**
  - **μ is rate of customer departure**

The M/M/1 Queue – Distribution of Number of Customers

- Writing the global balance equations for this Markov chain and solving for Prob[N(t) = j], yields (refer to previous example)

$$p_j = \text{Prob}[N(t) = j] = (1-\rho)\rho^j$$

for $\rho = \lambda/\mu < 1$

Note that for $\rho = 1$ ➔ arrival rate λ = service rate μ

The M/M/1 Queue – Expected Number of Customers

- The mean number of customer is given by

$$E[N] = \sum_j j \,\text{Prob}[N(t) = j]$$

$$= \rho / (1-\rho)$$

The M/M/1 Queue – Mean Customer Delay

- The mean total customer delay in the system is found using Little's formula

$$E[T] = E[N]/\lambda = (\rho/\lambda) / (1-\rho) = 1/(\mu - \lambda)$$

The M/M/1 Queue – Mean Queueing Time

- The mean waiting time in queue is given by

$$E[W] = E[T] - E[\tau] = \rho / (1-\rho) \quad E[\tau]$$

The M/M/1 Queue – Mean Number in Queue

- **Again we employ Little's formula:**

$$E[Nq] = \lambda E[W]$$

$$= \rho^2 / (1-\rho)$$

Remember:

server utilization $\rho = \lambda/\mu = 1-p_0$

All previous quantities E[N], E[T], E[W], and E[Nq] $\rightarrow \infty$ **as** $\rho \rightarrow 1$

Scaling Effect for M/M/1 Queues

- **Consider a queue of arrival rate λ whose service rate is μ**
  - $\rho = \lambda/\mu$,
  - **The expected delay E[T] is given by**

    $$E[T] = (1/\mu) / (1-\rho)$$
- **If the arrival rate increases by a factor of K, then we either**
  1. **Have K queueing systems, each with a server of rate μ**
  2. **Have one queueing system with a server of rate Kμ**
- **Which of the two options will perform better?**

Scaling Effect for M/M/1 Queues – cont'd

- **Case 1: K queueing systems**
  - **Identical systems**
  - **E[T] is the same for all – E[T] = $(1/\mu) / (1-\rho)$**

- **Case 2: 1 queueing system with server of rate $K\mu$**
  - **ρ for this system = $(K\lambda) / (K\mu) = \lambda/\mu$ – same as the original system**
  - **E[T'] = $(1/(K\mu)) / (1-\rho) = (1/K)$ E[T]**

- **Therefore, the second option will provide a less total delay figure – significant delay performance improvement!**

Arriving Customer's Distribution

- **Let Na be the number of customers found in the system by a customer arrival**

- **Prob[Na = k] ← is the arriving customer distribution**

- **(Refer to handout for proof) –**

$$\text{Prob}[Na = k] = \text{Prob}[N(t) = k] = (1-\rho)\rho^k$$

where Prob[N(t) = k] is the customer distribution at any time!! –

- **This is valid only for a POISSON ARRIVAL!**

Delay Distribution for M/M/1

- We have shown before the mean delay, E[T] = $(1/\mu) / (1-\rho)$
  - But what is the distribution for T?
- An arriving customer see's k customers ahead
  - Has to wait for k iid exp r.v. service times, each with mean $1/\mu$
  - Then, our arriving customer will go to service for an exp r.v. service time of mean $1/\mu$

Delay Distribution for M/M/1 – cont'd

- Therefore, total delay, T, is the sum of k+1 iid exponential r.v. each with mean $1/\mu$
- The conditional (Na = k) distribution of T is given by the Gamma PDF (refer to Probability Theory slides)

$$f_T(x / N_a = k) = \frac{(\mu x)^k}{k!} \mu e^{-\mu x} \qquad x > 0$$

Delay Distribution for M/M/1 – cont'd

- The PDF of T can be found be de-conditioning on Na -

$$f_T(x) = \sum_{k=0}^{\infty} f_T(x / N_a = k) \Pr[N_a = k]$$

$$= \sum_{k=0}^{\infty} \frac{(\mu x)^k}{k!} \mu e^{-\mu x} (1-\rho)\rho^k$$

$$= (\mu - \lambda) e^{-(\mu-\lambda)x} \qquad x > 0$$

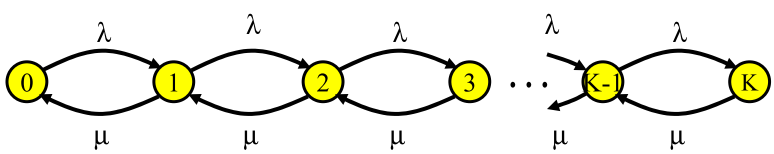
Therefore, the total delay, T, is a random variable *exponentially distributed* with mean $1/(\mu-\lambda)$

M/M/1/K – Finite Capacity Queue

- Consider an M/M/1 with finite capacity K < ∞
- For this queue – there can be at most K customers in the system
  - 1 being served
  - K-1 waiting
- A customer arriving while the system has K customers is BLOCKED (does not wait)!

M/M/1/K – Finite Capacity Queue – cont'd

- **Transition rate diagram for this queueing system is given by:**
  - **N(t) - A continuous-time Markov chain which takes on the values from the set {0, 1, …, K}**



M/M/1/K – Finite Capacity Queue – cont'd

- **The global balance equations:**

$$\lambda \quad p_0 = \mu p_1$$
$$(\lambda + \mu)p_j = \lambda p_{j-1} + \mu p_{j+1} \qquad \text{for } j=1, 2, \ldots, K-1$$
$$\mu \quad p_K = \lambda p_{K-1}$$

➔ **Prob[N(t) = j] = p_j** $\qquad$ **$j=0,1, \ldots, K; \rho<1$**

$$= (1-\rho)\rho^j/(1-\rho^{K+1})$$

When $\rho = 1$, $p_j = 1/(K+1)$ (all states are equiprobable)

M/M/1/K – Mean Number of Customers

- **Mean number of customers, E[N] is given by:**

$$E[N] = \sum_{j=0}^{K} j \Pr[N(t) = j]$$

$$= \begin{cases} \dfrac{\rho}{1-\rho} - \dfrac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} & \rho < 1 \\ K/2 & \rho = 1 \end{cases}$$

M/M/1/K – Blocking Rate

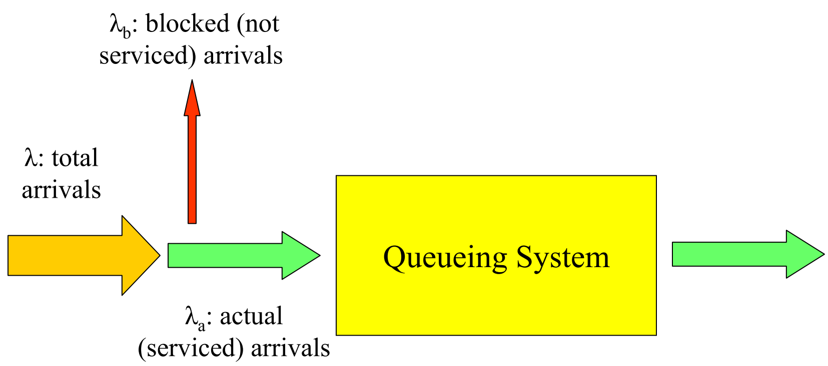
- **A customer arriving while the system is in state K is BLOCKED (does not wait)!**
- **Therefore, rate of blocking, λ_b is given by**

$$\lambda_b = \lambda p_K$$

- **The actual arrival rate into the system is λ_a given**

$$\lambda_a = \lambda - \lambda_b$$
$$= \lambda(1 - p_K)$$

M/M/1/K – Blocking Rate – cont'd



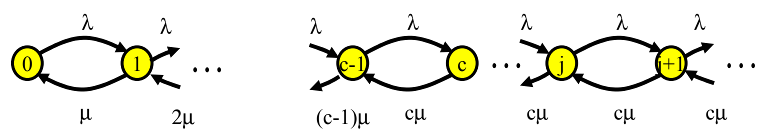
M/M/1/K – Mean Delay

- **The mean total delay E[T] is given by**

$$E[T] = E[N] / \lambda_a$$

Multi-Server Systems: M/M/c

- **The transition rate diagram for a multi-server M/M/c queue is as follows:**
  - **Departure rate = kμ when k servers are busy**



Multi-Server Systems: M/M/c – cont'd

- **When k servers are busy, the time until the next departure is given by:**

$$X = \min(\tau_1, \tau_2, \ldots, \tau_k)$$

where τ_i are iid exponential r.v. with mean $1/\mu$

The CDF for X is given by (refer to definition)

$$\begin{aligned} Prob[X > t] &= Prob[\min(\tau_1, \tau_2, \ldots, \tau_k) > t] \\ &= Prob[\tau_1>t, \tau_2>t, \ldots, \tau_k>t] \\ &= Prob[\tau_1>t]\, Prob[\tau_2>t] \ldots Prob[\tau_k>t] \\ &= e^{-\mu t} e^{-\mu t} \ldots e^{-\mu t} \\ &= e^{-k\mu t} \end{aligned}$$

Therefore, the time till the next departure (X) is an exponentially distributed r.v. with mean $1/(k\mu)$

Multi-Server Systems: M/M/c – cont'd

- **Writing the global balance equations:**

$\lambda \quad p_0 = \mu p_1$

$j\mu \quad p_j = \lambda p_{j-1} \quad$ **for** $\quad j=1, 2, \ldots, c$

$c\mu \quad p_j = \lambda p_{j-1} \quad$ **for** $\quad j= c, c+1, \ldots$

➔

$p_j = a^j/j!\ p_0 \qquad$ (for $j=1, 2, \ldots, c$) and

$p_j = \rho^{j-c}/c!\ a^c\ p_0$ (for $j=c, c+1, \ldots$)

where $a = \lambda/\mu$ **and** $\rho = a/c$

Multi-Server Systems: M/M/c – cont'd

- **To find** p_0**, we resort to the fact that** $\sum p_j = 1$

➔

$$p_0 = \left\{ \sum_{j=0}^{c-1} \frac{a^j}{j!} + \frac{a^c}{c!} \frac{1}{1-\rho} \right\}^{-1}$$

The probability that an arriving customer has to wait

$$\text{Prob}[W > 0] = \text{Prob}[N \geq c]$$

$$= p_c + p_{c+1} + p_{c+2} + \ldots$$

$$= p_c/(1-\rho)$$

Erlang-C formula

Multi-Server Systems: M/M/c – cont'd

- **The mean number of customers in queue (waiting):**

$$E[N_q] = \sum_{j=c}^{\infty} (j-c)\Pr[N(t)=j]$$

$$= \sum_{j=c}^{\infty} (j-c)\rho^{j-c} p_c$$

$$= \frac{\rho}{(1-\rho)^2} p_c$$

$$= \frac{\rho}{1-\rho} \Pr[W>0]$$

Multi-Server Systems: M/M/c – cont'd

- **The mean waiting time in queue:**

$$E[W] = E[N_q]/\lambda$$

- **The mean total delay in system:**

$$E[T] = E[W] + E[\tau]$$
$$= E[W] + 1/\mu$$

- **The mean number of customers in system:**

$$E[N] = \lambda E[T]$$
$$= E[N_q] + a$$

Why?

Example 2:

- A company has a system with four private telephone lines connecting two of its sites. Suppose that requests for these lines arrive according to a Poisson process at rate of one call every 2 minutes, and suppose that call durations are exponentially distributed with mean 4 minutes. When all lines are busy, the system delays (i.e. queues) call requests until a line becomes available. Find the probability of having to wait for a line.

Example 2: cont'd

- **Solution:**
  $\lambda = ½, 1/\mu = 4, c = 4 \Rightarrow a = \lambda/\mu = 2$
  $\Rightarrow \rho = a/c = ½$

$p_0 = \{1+2+2^2/2!+2^3/3!+2^4/4!\ (1/(1-\rho))\}^{-1}$
$= 3/23$

$p_c = a^c/c!\ p0$
$= 2^4/4!\ X\ 3/23$

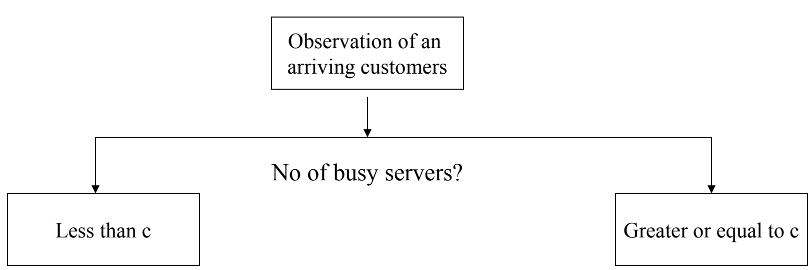
$\text{Prob}[W > 0] = p_c/(1-^r)$
$= 2^4/4!\ X\ 3/23\ X\ 1/(1-1/2)$
$= 4/23$
≈ 0.17

Waiting Time Distribution for M/M/c

- An arriving customer to the system, either
  - Does not wait, if number of busy servers is less than c
  - Does wait if number of busy servers is c

- If there are k > 0 customers waiting (as observed by an arriving customer), the total waiting time for the arriving customer = the sum of: remaining service time of the earliest job to finish + service time for these k customers
  - i.e. $W = \tau + \tau_1 + \tau_2 + \dots + \tau_k$, where τ's ~ iid exponentially distributed r.v. with mean $E[\tau] = 1/\mu$

Waiting Time Distribution for M/M/c – cont'd



Prob = $p_0 + p_1 + \dots + p_{c-1}$

Prob = $p_c + p_{c+1} + \dots$

Customer does not wait, OR
W = 0

Customer waits:
If there are k waiting, then T = sum of k+1 iid exp r.v with parameter μ
Remember k=0, 1, 2, …

Waiting Time Distribution for M/M/c – cont'd

- **We have seen before that (given there are k ahead), the distribution of W follows the gamma distribution with parameter cμ. I.e.**

$$f_W(x/N=c+k)=\frac{(c\mu x)^k}{k!}c\mu e^{-c\mu x} \qquad x>0, k=0,1,2,\ldots$$

Waiting Time Distribution for M/M/c – cont'd

- **We can find the overall pdf of W given N >= c (i.e. summing over all ks) as follows:**

$$f_W(x/W>0)=\sum_{k=0}^{\infty} f_W(x/N=c+k)\Pr[N=c+k] \qquad x>0$$

- **Equivalently, we can write:**

$$F_W(x/W>0)=\sum_{k=0}^{\infty} F_W(x/N=c+k)\Pr[N=c+k] \qquad x>0$$

Waiting Time Distribution for M/M/c – cont'd

- **But (refer to handout for proof)**

  Pro[N = c + k/N ≥c] = (1-ρ)ρ^k   k=0,1,2 …

- **Substituting in previous formula for $F_W(x/W>0)$ and simplifying, yields**

$$F_W\left(x/W>0\right)=1-e^{-c(1-\rho)x} \qquad x>0$$

This is all assuming the customer will have to wait!!

Waiting Time Distribution for M/M/c – cont'd

- **The general expression for the CDF (waiting and not waiting):**

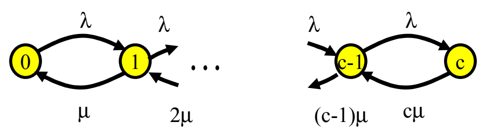
$$F_W\left(x\right)=\Pr[W=0]\times 1+F_W\left(x/W>0\right)\Pr[W>0]$$

$$=1-\Pr[W>0]e^{-c\mu(1-\rho)x} \qquad x>0$$

$$=1-\frac{p_c}{1-\rho}e^{-c\mu(1-\rho)x} \qquad x>0$$

Multi-Server Systems: M/M/c/c

- **The transition rate diagram for a multi-server with no waiting room (M/M/c/c) queue is as follows:**
  - **Departure rate = kμ when k servers are busy**



PMF for Number of Customers for M/M/c/c

- **Writing the global balance equations, one can show:**

  $p_j = a^j / j! \, p_0$ (for j=0, 1, …, c)

where a = λ/μ (the offered load)

- **To find p_0, we resort to the fact that $\sum p_j = 1$**

$$p_0 = \left\{ \sum_{j=0}^{c} \frac{a^j}{j!} \right\}^{-1}$$

Erlang-B Formula

- **Erlang-B formula is defined as the probability that all servers are busy:**

$$\Pr[N=c]=p_c$$

$$=\frac{a_c / j!}{1+a+a^2/2!+\ldots+a^c/c!}$$

Expected Number of customers in M/M/c/c

- **The actual arrival rate *into* the system:**

$$\lambda_a=\lambda(1-p_c)$$

- **Average total delay figure:**

$$E[T]=E[\tau]$$

Why?

- **Average number of customers:**

$$E[N]=\lambda_a E[\tau]$$

M/G/1 Queues

- **Poisson arrival process (i.e. exponential r.v. interarrival times)**
- **Service time: general distribution $f_\tau(x)$**
  - **For M/M/1, $f_\tau(x) = \mu e^{-\mu x}$ for $x > 0$**
- **The state of the M/G/1 system at time t is specified by**
  1. **N(t)**
  2. **The remaining (residual) service time of the customer being served**

The Residual Service Time

- **Mean residual time (see example and derivation in handout) is given by**

$$E[R] = \frac{E[\tau^2]}{2E[\tau]}$$

Mean Waiting Time in M/G/1

- **The waiting time of a customer is the sum of the residual service time R′ of the customer (if any) found in service and the Nq(t)= k-1 service time of the customers (if any) found in queue**

$$E[W] = E[R'] + E[Nq]\,E[\tau]$$
$$= E[R'] + \lambda E[W]E[\tau]$$
$$= E[R'] + \rho\,E[W]$$

Mean Waiting Time in M/G/1 – cont'd

- **But residual service time R′ (as observed by an arriving customers) is either**
  - **0 is the server is free**
  - **R if the server is busy**

- **Therefore, mean of R′ is given by**

$$E[R'] = 0 \times Pro[N(t)=0] + E[R](1-Pro[N(t)=0])$$
$$= E[\tau^2]/(2E[\tau]) \times \rho$$
$$= \lambda E[\tau^2]/2$$

Mean Waiting Time in M/G/1 – cont'd

- **Substituting back, yields**

$$E[W] = \frac{\lambda E[\tau^2]}{2(1-\rho)}$$

$$= \frac{\lambda(\delta^2_\tau + E[\tau]^2)}{2(1-\rho)}$$

$$= \frac{\rho\,(1 + C_\tau^2)}{2(1-\rho)} E[\tau]$$

Remember:
- $E[\tau^2] = \delta^2_\tau + E[\tau]^2$
- $C^2_\tau = \delta^2\tau / E[\tau]^2$

Pollaczek-Khinchin (P-K) Mean Value Formula

Mean Delay in M/G/1 – cont'd

- **The mean waiting time, E[T] is found by adding mean service time to E[W]:**

$$E[T] = E[\tau] + E[W]$$

$$= E[\tau] + \frac{\rho\,(1 + C_\tau^2)}{2(1-\rho)} E[\tau]$$

Example 3:

- **Problem: Compare E[W] for M/M/1 and M/D/1 systems.**

- **Answer:**

M/M/1: service time, τ, is exponential r.v. with parameter μ

➔ $E[\tau] = 1/\mu$, $E[\tau^2] = 2/\mu^2$, $\delta^2_\tau = 1/\mu^2$, $C^2_\tau = 1$

M/D/1: service time, τ, is constant with value $\tau = 1/\mu$

➔ $E[t] = 1/\mu$, $E[\tau^2] = 1/\mu^2$, $\delta^2_\tau = 0$, $C^2_\tau = 0$

Example 3: cont'd

- **Answer: cont'd**

Substitute in P-K mean value formula

M/M/1:

$$E[W_{M/M/1}] = \frac{\lambda E[\tau^2]}{2(1-\rho)} = \frac{\rho}{(1-\rho)} E[\tau]$$

M/D/1:

$$E[W_{M/D/1}] = \frac{\lambda E[\tau^2]}{2(1-\rho)} = \frac{\rho}{2(1-\rho)} E[\tau]$$

$$= \frac{1}{2} E[W_{M/M/1}]$$

The waiting time in an M/D/1 queue is half of that of an M/M/1 system

M/G/1 with Priority Service Discipline

- Handles K priority classes of customers
- Head-of-line priority service discipline
- Type k ={1, 2, …, K} arrive according to Poisson arrival process
- A separate queue is kept for each priority class
- Server utilization from type k customers:

$$\rho_k = \lambda_k E[\tau_k]$$

- Total server utilization

$$\rho = \rho_1 + \rho_2 + \ldots + \rho_K < 1$$

for a stable system

- Assume class 1 is the highest priority while class K is the lowest

Mean Waiting Time in M/G/1 with Priority Service Discipline

- An arriving customer of type 1 finds $N_{q1}(t)$ = k1 type 1 customers in queue
- Assuming FCFS for each queue
- The mean waiting time for type one customer:

$$E[W_1] = E[R''] + E[N_{q1}] E[\tau_1]$$

Where E[R''] is the residual time of the customer (if any) found in service

Mean Waiting Time in M/G/1 with Priority Service Discipline – cont'd

- We also know (Little's formula) that:

$$E[N_{q1}] = \lambda_1 E[W_1]$$

Substituting and solving for E[W1], yields,

$$E[W_1] = E[R''] / (1-\rho_1)$$

Mean Waiting Time in M/G/1 with Priority Service Discipline – cont'd

- Consider a type 2 customer – Because of the priority scheme one can write

$$E[W_2] = E[R''] + E[N_{q1}] E[\tau_1] + E[N_{q2}] E[\tau_2] + E[M_1] E[\tau_1]$$

Where

- $E[R'']$ is the residual time of the customer (if any) found in service
- $E[N_{q1}] E[\tau_1]$ time to service already existing class 1 customers (remember $E[N_{q1}] = \lambda_1 E[W_1]$)
- $E[N_{q2}] E[\tau_2]$ time to service already existing class 2 customers (remember $E[N_{q2}] = \lambda_2 E[W_2]$)
- $E[M_1] E[\tau_1]$ time to service class 1 customers arriving during our customer waiting time - $E[M_1] = \lambda_1 E[W_2]$

Mean Waiting Time in M/G/1 with Priority Service Discipline – cont'd

- $E[M_1]$ is given by

$$E[M_1] = \lambda_1 E[W_2]$$

- Substituting and solving for $E[W_2]$, yields,

$$E[W_2] = \frac{E[R'']}{(1 - \rho_1)(1 - \rho_1 - \rho_2)}$$

Mean Waiting Time in M/G/1 with Priority Service Discipline – cont'd

- In general we can show the mean waiting time for a customer of type k, $E[W_k]$ is given by

$$E[W_k] = \frac{E[R'']}{(1 - \rho_1 - \ldots - \rho_{k-1})(1 - \rho_1 \ldots - \rho_k)}$$

Mean Waiting Time in M/G/1 with Priority Service Discipline – cont'd

- **What is E[R"]?**
- **Remember R" is the residual service time of a customer (if any) found in service – of any type**
- **Recall that mean residual time E[R"] is computed by**

$$E[R''] = \lambda\, E[\tau^2]/2 \quad \text{(refer to slide 52)}$$

But $E[\tau^2]$ for which type of customers?

Mean Waiting Time in M/G/1 with Priority Service Discipline – cont'd

- **$E[\tau^2]$ – is the mean service-time squared for ANY type:**

$$E[\tau^2] = (\lambda_1/\lambda)E[\tau_1^2] + (\lambda_2/\lambda)E[\tau_2^2] + \ldots + (\lambda_K/\lambda)E[\tau_K^2]$$

where $\lambda = \lambda_1 + \lambda_2 + \ldots \lambda_K$

Mean Waiting Time in M/G/1 with Priority Service Discipline – cont'd

- **Therefore, the mean waiting time of type k customers:**

$$E[W_k] = \frac{\sum_{j=1}^{K} \lambda_j E[\tau_j^2]}{2(1-\rho_1-\cdots-\rho_{k-1})(1-\rho_1-\cdots-\rho_k)}$$

- **The mean delay for type k customer is then equal to**

$$E[T_k] = E[W_k] + E[\tau_k]$$

M/G/1 Analysis Using Embedded Markov Chain

- **Pollaczek-Khinchin (P-K) Transform Equation**

$$G_N(z) = \frac{(1-\rho)(z-1)\hat{\tau}(\lambda(1-z))}{z-\hat{\tau}(\lambda(1-z))}$$

See derivation in handout

where:

- **$G_N(z)$: moment generating function of the r.v. N(t)**
- **$\hat{\tau}(s)$ is the Laplace transform of r.v. τ**

Example 4:

- **Problem**: Use the P-K transform equation to find the steady state pmf of an M/M/1

- **Answer**:

For an M/M/1 the steady state pmf for N(t) is given by (refer to slide 13)

$$p_j = \mathrm{Prob}[N(t) = j] = (1-\rho)\rho^j$$

Example 4: cont'd

- **Answer**: cont'd

The moment generating function, $G_N(z)$, is then given by

$$G_N(z) = \sum_{j=0}^{\infty} p_j z^j = \sum_{j=0}^{\infty} (1-\rho)\rho^j z^j = \frac{(1-\rho)}{(1-\rho z)}$$

Example 4: cont'd

- **<u>Answer</u>: cont'd**

Now let's use the P-K transform and see if we get the same answer!

For M/M/1, τ is exp r.v → the pdf for τ is

$$f_\tau(t) = \mu e^{-\mu t} \qquad t > 0$$

The Laplace transform of τ is given by

$$\hat{\tau}(s) = \int_0^\infty f_\tau(t) e^{-st} dt$$

$$= \frac{\mu}{s+\mu}$$

Example 4: cont'd

- **<u>Answer</u>: cont'd**

Therefore, $\hat{\tau}(\lambda(1-z))$ is given by

$$\hat{\tau}(\lambda(1-z)) = \frac{\mu}{\lambda(1-z)+\mu}$$

We are now in a position to substitute in the P-K transform equation

Example 4: cont'd

- **Answer**: **cont'd**

$$G_N(z) = \frac{(1-\rho)(z-1)\hat{\tau}(\lambda(1-z))}{z-\hat{\tau}(\lambda(1-z))}$$

$$= \frac{(1-\rho)(z-1)(\mu/\lambda(1-z)+\mu)}{z-(\mu/\lambda(1-z)+\mu)}$$

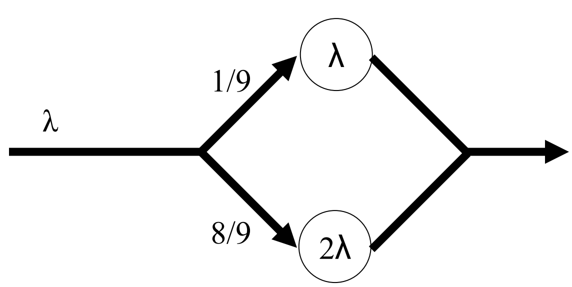
$$= \frac{(1-\rho)(z-1)\mu}{(\lambda-\lambda z+\mu)z-\mu}$$

$$= \frac{(1-\rho)}{(1-\rho z)}$$

Which the same M.G.F for N(t) derived previously!

Example 5:

- **Problem**: **M/H$_2$/1**



What is Prob[N(t) = k] =?

Example 5: cont'd

- **Answer:**

The pdf of the service time, τ, is

$$f_\tau(t) = \frac{1}{9}\lambda e^{-\lambda t} + \frac{8}{9}2\lambda e^{-2\lambda t} \qquad t > 0$$

The mean service time, E[τ] is given by

E[τ] = (1/9)X 1/λ + (8/9)X 1/(2λ)
= 5/(9λ)

➔ ρ = λ E[τ] = 5/9

The Laplace transform is given by

$$\hat{\tau}(s) = \frac{1}{9}\frac{\lambda}{s+\lambda} + \frac{8}{9}\frac{2\lambda}{s+2\lambda}$$

and

$$= \frac{18\lambda^2 + 17\lambda s}{9(s+\lambda)(s+2\lambda)}$$

Example 5: cont'd

- **Answer:**

Substituting λ(1-z) for every s in the previous expression, and writing $G_N(z)$, yields,

$$G_N(z) = \frac{(1-\rho)(z-1)\hat{\tau}(\lambda(1-z))}{z - \hat{\tau}(\lambda(1-z))}$$

$$= \frac{(1-\rho)(35-17z)(z-1)}{9(2-z)(z-7/3)(z-5/3)}$$

$$= (1-\rho)\left\{\frac{1/3}{1-3z/7} + \frac{2/3}{1-3z/5}\right\}$$

Partial Fraction Expansion – How?

Example 5: cont'd

- **Answer**:

Therefore, $G_N(z)$ is given by

$$G_N(z) = (1-\rho)\left\{\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{3}{7}\right)^k z^k + \frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^k z^k\right\}$$

Since the coefficient of z^k is Prob[N(t) = k], then we finally have:

$$\Pr[N(t)=k] = \frac{4}{27}\left(\frac{3}{7}\right)^k + \frac{8}{27}\left(\frac{3}{5}\right)^k \qquad k = 0,1,\cdots$$

Total Delay Distribution for M/G/1 System

- **If, T is the total delay variable, then the Laplace transform of T is given by (see handout for derivation)**

$$\hat{T}(s) = \frac{(1-\rho)s\,\hat{\tau}(s)}{s-\lambda+\lambda\hat{\tau}(s)}$$

P-K transform equation

- **The pdf for T, $f_T(t)$, is obtained by inverting the above expression analytically or numerically**

Waiting Time Distribution for M/G/1 System

- **Since T = W + τ ➔ Therefore,**

$$\hat{T}(s) = \hat{W}(s)\hat{\tau}(s)$$

- **Hence, the Laplace transform of the waiting time is given by**

$$\hat{W}(s) = \frac{(1-\rho)s}{s-\lambda+\lambda\hat{\tau}(s)}$$

P-K transform equation

Example 6:

- **Problem: Verify the result obtained previously for the total delay time distribution of an M/M/1 queue using P-K transform equations for M/G/1 systems**

- **Answer: for M/M/1 the service time, τ, is exp r.v. ➔** $f_\tau(t) = \mu e^{-\mu t} \quad t > 0$

or $$\hat{\tau}(s) = \frac{\mu}{s+\mu}$$

Example 6: cont'd

- **Substituting in the P-K transform equations**

$$\hat{T}(s) = \frac{(1-\rho)s\mu}{(s+\mu)(s-\lambda)+\lambda\mu} = \frac{(1-\rho)\mu}{s-(\lambda-\mu)}$$

Inverting the above expression, yields

$$f_T(t) = \mu(1-\rho)e^{-\mu(1-\rho)t} \qquad t>0$$
$$= (\mu-\lambda)e^{-(\mu-\lambda)t} \qquad t>0$$

Example 6: cont'd

- **This means the total delay is exponentially distributed with mean1/(μ-λ) – Same result as obtained before! (refer to slide 23)**
- **The waiting time is obtained using**

$$\hat{W}(s) = \frac{(1-\rho)s}{s-\lambda+\lambda\hat{\tau}(s)} = (1-\rho)\frac{s+\mu}{s+\mu-\lambda} = (1-\rho)\left\{1+\frac{\lambda}{s+\mu-\lambda}\right\}$$

Example 6: cont'd

- **Therefore the pdf of W is given by**

$$f_W(t) = (1-\rho)\delta(t) + \lambda(1-\rho)e^{-\mu(1-\rho)t} \qquad t > 0$$

- **The $\delta(t)$ term indicates there is a ZERO waiting time with probability equal to 1-ρ – i.e. when server is free**