King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 342 – Data and Computer Communications

Term 031

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Lecture Contents

1. Fourier Analysis
    a. Fourier Series Expansion
    b. Fourier Transform
    c. Ideal Low/band/high pass filters

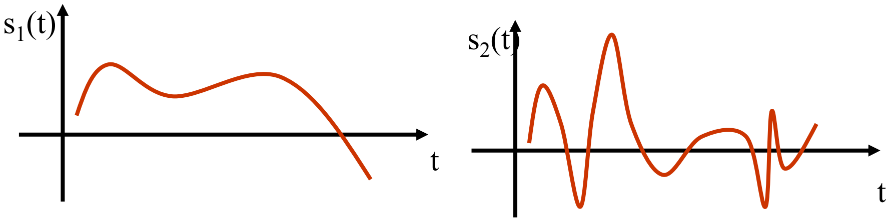
Signals

- A signal is a function representing information
  - Voice signal – microphone output
  - Video signal – camera output
  - Etc.

- Types of Signals
  - Analog – continuous-value continuous-time
  - Discrete – discrete-value continuous-time
    - Digital – predetermined discrete levels – much easier to reproduce at receiver with no errors
    - Binary – only two predetermined levels: e.g. 0 and 1

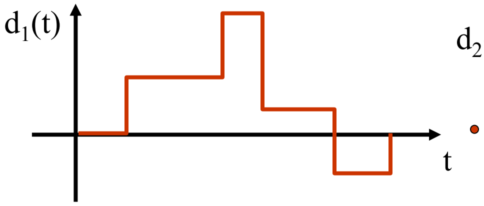
Example of Continuous-Value Continuous-time signal

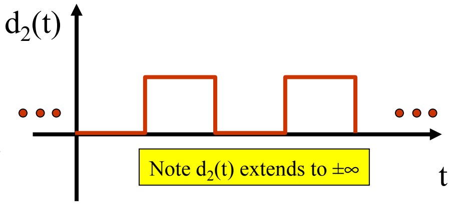
- $s_1(t)$ and $s_2(t)$ are two example of analog signals



Example of Discrete-Value Continuous-time signal

- **$d_1(t)$ and $d_2(t)$ are two example of discrete signals**
  - **$d_1(t)$ – takes more than two levels**
  - **$d_2(t)$ – takes only two levels - binary**



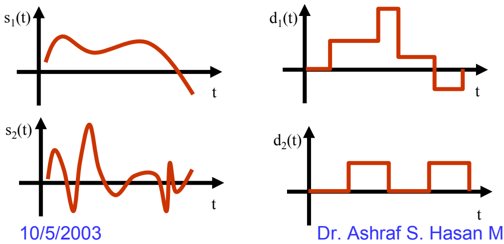


Note $d_2(t)$ extends to ±∞

Applies to BOTH analog and digital signals

Time Domain Representation

- **<u>Time domain representation</u> – we plot value (voltage, current, electric field intensity, etc.) versus time**
  - **Can infer rate of change (speed or frequency) information – e.g. $s_2(t)$ seems faster than $s_1(t)$**
    - **Using calculus terms: *rate of change* for $s_2(t)$ > rate of change for $s_1(2)$**



Applies to BOTH analog and digital signals

Frequency - Bandwidth

- $S_2(t)$ faster than $s_1(t)$ ➔
  - $S_2(t)$ contains higher frequencies than those contained in s1(t)
- $S_1(t)$ and $s_2(t)$ contain more than one frequency
  - Minimum frequency = f_{min}
  - Maximum frequency = f_{max}
- Bandwidth = Range of frequencies contained in signal

$$= f_{max} - f_{min}$$

Applies to BOTH analog and digital signals

Frequency – Bandwidth (2)

- For our example signals, assume:
  - S1(t): fmin = 10 Hz, fmax = 500 Hz
  - S2(t): fmin = 5 Hz, fmax = 1000 Hz
- This means:
  - BW for $s_1(t)$ = 500 – 10 = 490 Hz
  - BW for $s_2(t)$ = 1000 – 5 = 995 Hz
- Note that: because $s_2(t)$ is "faster than" $s_1(t)$ it should contain frequencies higher than those in $s_1(t)$
  - E.g. $s_2(t)$ contains frequencies (500,100] which do not exist in $s_1(t)$

Applies to BOTH analog and digital signals

Frequency – Bandwidth (3)

- Consider the discrete signals $d_1(t)$ and $d_2(t)$
- The function plots have points of infinite slope
  - rate of change = ∞ ➔ frequency = ∞
- Therefore for signals that look like $d_1(t)$ and $d_2(t)$, fmax = ∞
- Furthermore, BW = ∞
- Example:
  - $d_2(t)$ contains frequencies from some minimum fmin Hz to fmax = ∞ Hz

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Example of Signal BW

- Consider the human speech

- Typically fmin ~ 100Hz
- fmax ~ 3500 Hz

- BW of the human speech signal = 3100 Hz

Bandwidth for Systems

- For a system to respond (amplify, process, Tx, Rx, etc.) for a particular signal with all its details, the system should have an equal or greater bandwidth compared to that of the signal
- Example:
  - The system required to process $s_2(t)$ should have a greater bandwidth than the system required to process $s_1(t)$

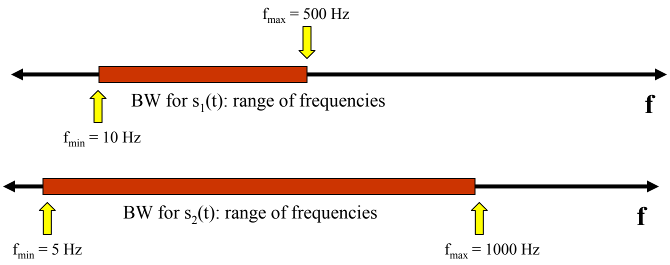
Bandwidth for Systems (2)

- Example 2: consider the human ear system
  - Responds to a range of frequencies only
  - fmin = 20 Hz fmax = 20,000 Hz → BW = 19,980 Hz
  - It does not respond to sounds with frequencies outside this range
- Example 3: consider the copper wire
  - It passes (electric) signals only between a certain fmin and a certain fmax
  - The higher the quality of the wire – the wider the BW
- More on Systems BW later!

Applies to BOTH analog and digital signals

Frequency Representation

- **How to represent signals and indicate their frequency content?**
- **The X-axis: frequency (in Hertz or Hz)**
- **What is the Y-axis then?** **– the answer will be postponed!**



Applies to BOTH analog and digital signals

Periodic Signals

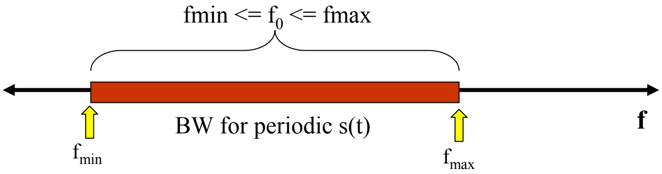
- **A periodic signal repeats itself every T seconds**
  - **Period ➔ T seconds**
- **In calculus terms:**
  - **S(t) is periodic if s(t) = s(t+T) for any** $-\infty < t < \infty$
- **For previous examples: $s_1(t)$, $s_2(t)$, and $d_1(t)$ are not periodic – however, $d_2(t)$ is periodic**

Applies to BOTH analog and digital signals

Periodic Signals (2)

- A periodic signal has a FUNDAMENTAL FREQUENCY – f_0
  - $f_0 = 1 / T$ – where T is the period

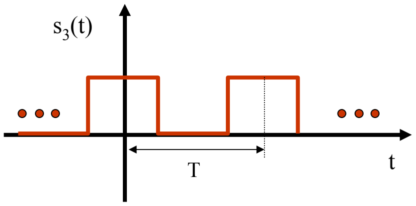
- A periodic signal may also has frequencies other than the fundamental frequency f_0

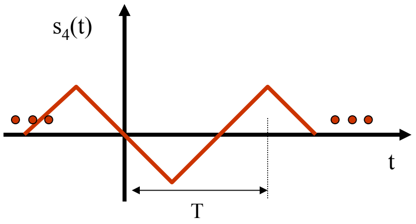


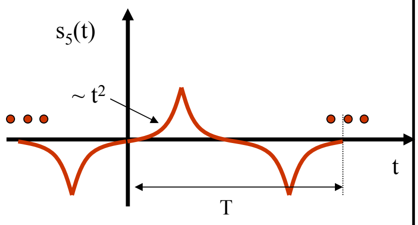
Applies to BOTH analog and digital signals

Periodic Signals (3)

- Examples of other periodic signals:







Applies to BOTH analog and digital signals

Energy/Power of Signals

- Energy for any signal is defined as

$$E_s = \int \left| s(t) \right|^2 dt$$

where the integral is carried over ALL range of t

- In other words, Es is the area under the absolute squared of the signal
- The unit of energy is Joules

Applies to BOTH analog and digital signals

Energy/Power of Signals (2)

- Note that for periodic signal E_s is equal to infinity since it is defined on (-∞, ∞)
  - However power is FINITE for these type of signals
- Power is defined as the average of the absolute squared of the signal, i.e.

$$P_s = \frac{1}{T} \int_0^T \left| s(t) \right|^2 dt$$

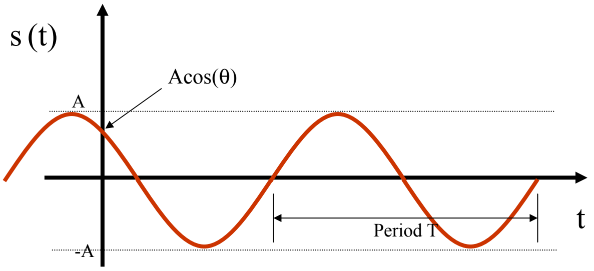
Note the integral can be performed on [0,T], [-T/2, T/2], or any continuous interval of length T

- The unit of power is Joules/sec or Watt

A VERY Analog Signal

- A function of the form

$$s(t) = A\ \cos(2\pi ft + \theta)$$

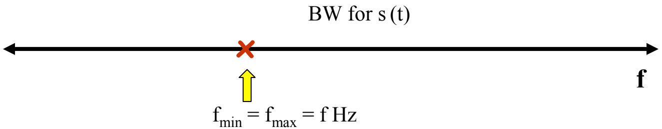


Characteristics of COSINE

- Completely specified by:
  - Amplitude – A
  - Phase - θ
  - Frequency – f
- $s(t = 0) = A \cos(\theta)$
- Periodic signal – repeats itself every T seconds
  - $T = 1 / f$
- Time to review your trigonometry !!
  - E.g. $\sin(x) = \cos(x-\pi/2)$

Characteristics of COSINE (2)

- **Energy for this signal, E_s = infinity**
- **Power for this signal, $P_g = A^2/2$**
  - **Note P_g is dependent only on the amplitude A**
  - **Exercise: Verify the above results using the power formula**
- **It contains ONLY ONE frequency f**
  - **The "purist" form of analog signals**
- **Frequency representation:**



Characteristics of COSINE (3)

- **Very Useful Properties (f = 1/T)**

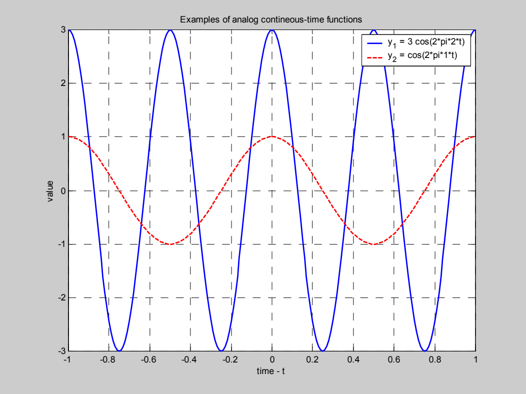
$$\int_0^T \cos(2\pi f t+\theta)dt = 0 \qquad \frac{1}{T}\int_0^T \cos^2(2\pi f t+\theta)dt = 1/2$$

$$\int_0^T \cos(2\pi n f t+\theta)dt = 0 \qquad \frac{1}{T}\int_0^T \cos^2(2\pi n f t+\theta)dt = 1/2$$

$$\frac{1}{T}\int_0^T \cos(2\pi n f t)\cos(2\pi m f t)dt = \begin{cases} 0 & n \neq m \\ 1/2 & n = m \end{cases}$$

Example of Cosine Functions

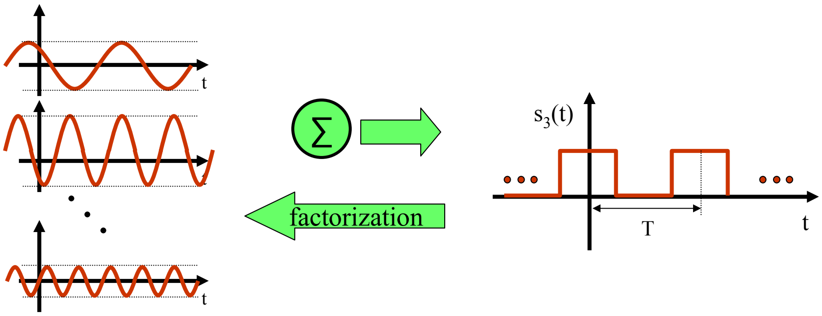
- $Y_1(t)$ – has
  - a frequency f of 2 Hz (T = ½ sec)
  - An amplitude of 3
  - $P_{Y1} = 3^2/2 = 4.5$ Watts
- $Y_2(t)$ - has
  - a frequency f of 1 Hz (T =1/1 = 1 sec)
  - An amplitude of 1
  - $P_{Y2} = 1^2/2 = 0.5$ Watts



ONLY FOR PERIODIC SIGNALS

Fourier Series Expansion

- Can we use the basic cosine functions to represent periodic signals?
- YES – Fourier Series Expansion



Fourier Series Expansion (2)

- **For a periodic signal s(t) can be represented as a sum of sinusoidal signals as in**

$$s(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(2\pi n f_0 t) + B_n \sin(2\pi n f_0 t) \right]$$

where the coefficients are computed using:

$$A_0 = \frac{2}{T} \int_0^T s(t) dt$$

f_0 is the fundamental frequency of s(t) and is equal to 1/T

$$A_n = \frac{2}{T} \int_0^T s(t) \cos(2\pi n f_0 t) dt \qquad B_n = \frac{2}{T} \int_0^T s(t) \sin(2\pi n f_0 t) dt$$

Fourier Series Expansion (3)

- **Another form for the series:**

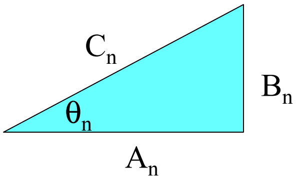
$$s(t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(2\pi n f_0 t + \theta_n)$$

where the coefficients are computed using:

$$C_0 = A_0$$

$$C_n = \sqrt{A_n^{\ 2} + B_n^{\ 2}}$$

$$\theta_n = \tan^{-1}\left(\frac{B_n}{A_n} \right)$$



Notes on Fourier Series Expansion

- The representation (the sum of sinusoids) is completely identical and equivalent to the original specification of s(t)

- It is applies to any periodic signal – analog or digital!

Very power tool - it reveals all frequencies contained in the original periodic signal s(t)

Notes on Fourier Series Expansion (2)

- In general, s(t) contains
  - DC term – the zero frequency term = $A_0/2$
  - A (possibly infinite) number of harmonics (or sinusoids) at multiples of the fundamental frequency, f_0
- The contribution of a harmonic with frequency nf_0 is proportional to $|A_n^2+B_n^2|$ or C_n^2
  - E.g. if $C_n^2 \sim 0$, then we say the harmonic at nf_0 (or higher does not contribute significantly towards building s(t) – more on this when we discuss total power!

Notes on Fourier Series Expansion (3)

- **A harmonic with frequency equal to nf_0 (n>0), has a period of 1/(nT)**
- **In general the series expansion of s(t) contains INFINITE number of terms (harmonics)**
- **However for less than 100% accurate representation one can ignore higher terms – terms with frequencies greater than certain n_*f_0**

Notes on Fourier Series Expansion (4)

- **Lets define the following function:**

s_e(n=k)

To be the series expansion of s(t) up to and including the n = k term

It should be noted that s_e(n=k) is periodic with period T

- **Examples:**

$$s\_e(n=0) = A_0/2$$

$$s\_e(n=1) = A_0/2 + A_1\cos(2\pi f_0 t) + B_1\sin(2\pi f_0 t)$$
$$= A_0/2 + C_1\cos(2\pi f_0 t + \theta_1)$$

Notes on Fourier Series Expansion (5)

- **Examples – cont'd:**

$$
\begin{aligned}
s\_e(n=2) &= A_0/2 + A_1\cos(2\pi f_0 t) + B_1\sin(2\pi f_0 t) \\
&\quad + A_2\cos(2\pi\times 2f_0 t) + B_2\sin(2\pi\times 2f_0 t) \\
&= A_0/2 + C_1\cos(2\pi f_0 t + \theta_1) + C_2\cos(2\pi\times 2f_0 t + \theta_2)
\end{aligned}
$$

$$\ddots$$

$$
\begin{aligned}
s\_e(n=\infty) &= \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n\cos(2\pi nf_0 t) + B_n\sin(2\pi nf_0 t)\right] \\
&= A_0/2 + \sum_{n=1}^{\infty} C_n\cos(2\pi nf_0 t + \theta_n)
\end{aligned}
$$

Notes on Fourier Series Expansion (6)

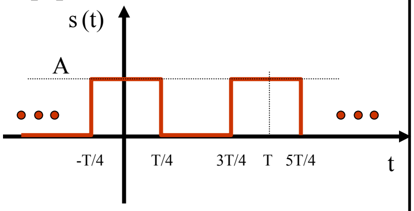
- **It is obvious that s(t) is 100% represented by s_e(n=∞)**

- **s_e(n = n* < ∞) produces a less than 100% accurate representation of the original s(t)**

- **For most practical periodic signals s_e(n=10) provides a more than enough accuracy in representing s(t)**
  - **No need for infinite number of terms**

Example 1:

- **Consider the following s(t)**



- **Over one period, the signal is defined as**

S(t) = A -T/4 < t <= T/4
= 0 T/4 < t <= 3/4

- **Finding the Series Expansion:**
  - **The DC term A_0**

$$A_0 = \frac{2}{T}\int_{-T/4}^{T/4} s(t)dt = \frac{2}{T}\times\frac{T}{2}\times A$$
$$= A$$

Example 1: cont'd

- **The term A_n:**

$$A_n = \frac{2}{T}\int_{-T/4}^{T/4} s(t)\cos(2\pi n f_0 t)dt = \frac{2A}{T}\int_{-T/4}^{T/4}\cos(2\pi n f_0 t)dt$$

$$= \frac{2A}{2\pi n f_0 T}\sin(2\pi n f_0 t)\Big|_{t=-T/4}^{t=T/4} = \frac{A}{\pi n}\times 2\times\sin(\frac{n\pi}{2})$$

$$= \begin{cases} 0 & n = 2,4,6,\ldots \\ \frac{2A}{\pi n} & n = 1,5,9,\ldots \\ -\frac{2A}{\pi n} & n = 3,7,11,\ldots \end{cases}$$

Remember
1. $f_0 = 1/T$
2. Int (cos(ax)) = 1/a sin(ax)
3. sin(nπ) = 0 for integer n
4. sin(nπ/2) = 1 for n=1,5,9, …
5. sin(nπ/2) = -1 for n=3,7,11, …

Example 1: cont'd

- **Therefore A_n is given by:**

$$= \begin{cases} 0 & n = 2,4,6,\ldots \\ (-1)^{(n-1)/2} \times \dfrac{2A}{\pi n} & n = 1,3,5,7,\ldots \end{cases}$$

Remember
$(-1)^{(n-1)/2} = 1$ for n = 1,5,9, …
= -1 for n = 3,7,11, …

Example 1: cont'd

- **The term B_n:**

$$B_n = \frac{2}{T}\int_{-T/4}^{T/4} s(t)\sin(2\pi n f_0 t)dt = \frac{2A}{T}\int_{-T/4}^{T/4}\sin(2\pi n f_0 t)dt$$

$$= \frac{-2A}{2\pi n f_0 T}\cos(2\pi n f_0 t)\Big|_{t=-T/4}^{t=T/4} = \frac{-2A}{\pi n}\times\left\{\cos(\frac{n\pi}{2}) - \cos(-\frac{n\pi}{2})\right\}$$

$$= 0$$

Remember
1. Int (cos(ax)) = -1/a sin(ax)
2. cos(x) = cos(-x)

Example 1: cont'd

- **Therefore, the overall series expansion is given by**

$$s(t) = \frac{A}{2} + \frac{2A}{\pi}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{(n-1)/2}}{n}\times\cos(2\pi n f_0 t)$$

$$s(t) = \frac{A}{2} + \frac{2A}{\pi}\times\cos(2\pi f_0 t) - \frac{2A}{3\pi}\cos(2\pi\times 3 f_0 t) + \frac{2A}{5\pi}\times\cos(2\pi\times 5 f_0 t) - \frac{2A}{7\pi}\cos(2\pi\times 7 f_0 t) + \ldots$$

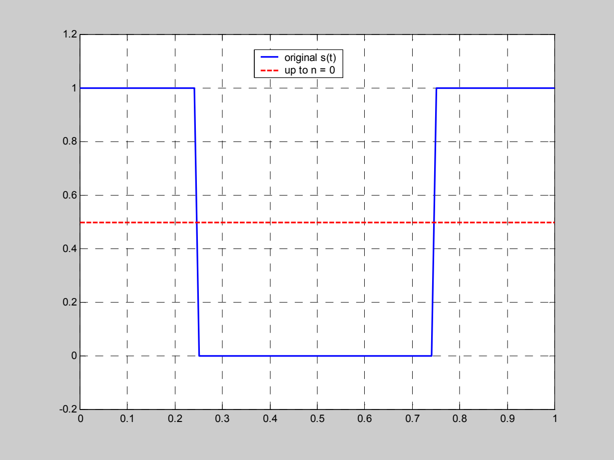
Example 1: cont'd

- **Original s(t) and the series up to and including n = 0**
- **i.e. Comparing:**

S(t)

vs.

s_e(n=0) = A/2



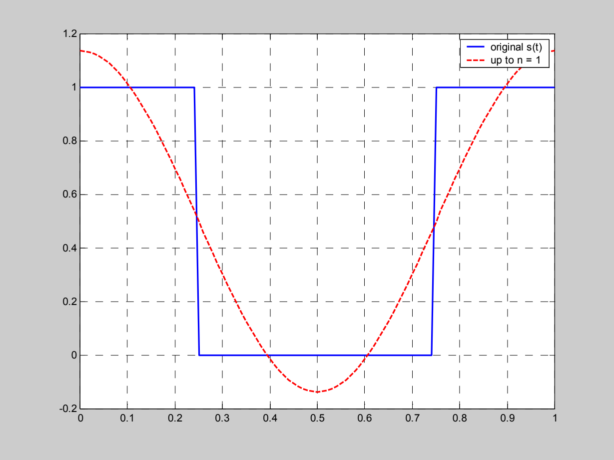
Example 1: cont’d

- **Original s(t) and the series up to and including n = 1**
- **i.e. Comparing:**

S(t)

vs.

s_e(n=1) =
$A/2 +$
$2A/\pi \cos(2\pi f_0 t)$



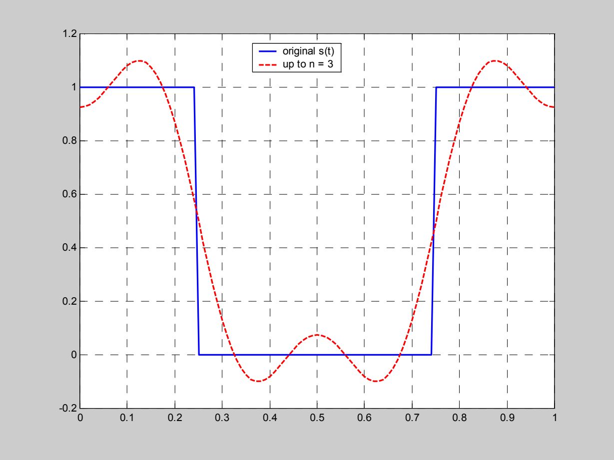
Example 1: cont’d

- **Original s(t) and the series up to and including n = 3**
- **i.e. Comparing:**

S(t)

vs.

s_e(n=3) =
$A/2 +$
$2A/\pi\cos(2\pi f_0 t) -$
$2A/(3\pi)\cos(2\pi 3 f_0 t)$



Example 1: cont'd

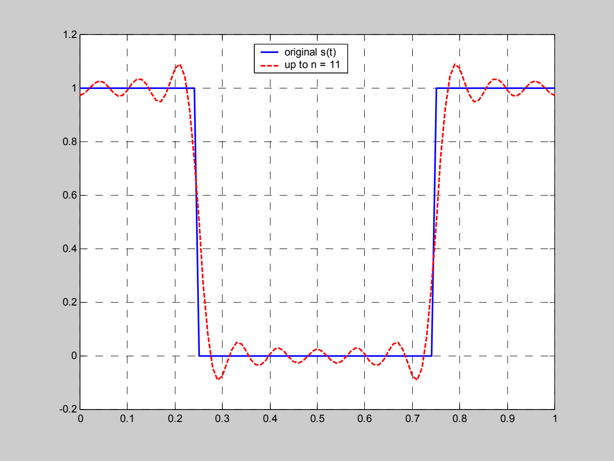
- **Original s(t) and the series up to and including n = 11**
- **i.e. Comparing:**

s(t)

vs.

s_e(n=11) =

$A/2 + 2A/\pi\cos(2\pi f_0 t)$
$- 2A/(3\pi)\cos(2\pi 3f_0 t)$
$+ 2A/(5\pi)\cos(2\pi 5f_0 t)$
$- 2A/(7\pi)\cos(2\pi 7f_0 t)$
$+2A/(11\pi)\cos(2\pi 11f_0 t)$



Example: cont'd

```
clear all

T = 1;
A = 1;
t = -1:0.01:1;
n_max = 11;

s = (A*square(2*pi/T*(t+T/4))+A)/2;

figure(1)
plot(t, s);
grid
axis([0 1 -0.2 1.2]);

s_e = A/2*ones(size(t));

for n=1:2:n_max
   s_e = s_e + (-1)^((n-1)/2) * 2*A/(n*pi) * cos(2*pi*n/T*t);
end

figure(2)
plot(t, s,'b-', t, s_e,'r--');
axis([0 1 -0.2 1.2]);
legend('original s(t)', 'up to n = 11');
grid
```

•The matlab code for plotting and evaluating the Fourier Series Expansion

•This code builds the series incrementally using the "for" loop

• Make sure you study this code!!

Notes Previous Example

- The more terms included in the series expansion → the closer the representation to the original s(t)
  - i.e. comparing s(t) with s_e(n=n*), the greater the n* the closer the representation is
- How to measure "closeness"?
- Answer: Let's use power!!

Power Calculation Using Fourier Series Expansion

- Rule: if s(t) is represented using Fourier Series expansion, then its power can be calculated using:

$$P_s = \frac{1}{T}\int_0^T |s(t)|^2 dt = \frac{A_0{}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left[A_n{}^2 + B_n{}^2\right]$$

$$= \frac{A_0{}^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} C_n{}^2$$

Power Calculation Using Fourier Series Expansion (2)

- **The previous result is based on the following two facts:**
  - **(1) For f(t) = constant**
    ➔ power of f(t) = constant^2

Proof:

power = 1/T X Int(constant^2 over one period)
= 1/T X constant^2 X T
= constant^2 Watts

Power Calculation Using Fourier Series Expansion (3)

- **The previous result is based on the following facts (continued):**
  - **(2) For f(t) = A cos(2πnf$_0$t+θ)**
    ➔ power of f(t) = A^2/2

Proof:

$$P_f = \frac{1}{T}\int_0^T |f(t)|^2 dt = \frac{A^2}{T}\int_0^T \cos^2(2\pi n f_0 t + \theta)dt$$

$$= \frac{A^2}{T}\int_0^T [\frac{1}{2} + \frac{1}{2}\cos(4\pi n f_0 t + 2\theta)]dt$$

$$= \frac{A^2}{T}[\frac{1}{2T} + 0] = \frac{A^2}{2}$$

Example 2:

- Problem: What is the power of the signal s(t) used in previous example? And find n* such that the power contained in s_e(n=n*) is 95% of that existing in s(t)?

- Solution:

Let the power of s(t) be given by P_s

$$P_s = \frac{1}{T}\int_0^T |s(t)|^2 \, dt = \frac{1}{T} \times A^2 \times \frac{T}{2} = \frac{A^2}{2} = 0.5A^2$$

Example 2: cont'd

- Now it is desired to compute the power using the Fourier Series Expansion

- What is the power in s_e(n=0) = A/2?
- Ans: we apply the power formula:

$$P_{s\_e(n=0)} = \frac{1}{T}\int_0^T |s\_e(n=0)|^2 \, dt$$

$$= \frac{1}{T} \times \frac{A^2}{4} \times T = \frac{A^2}{4} = 0.25A^2$$

Example 2: cont'd

- **What is the power in**

  s_e(n=1) = A/2 + 2A/π cos(2πf$_0$t)

- **Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:**

$$P_{s\_e(n=1)} = \frac{1}{T}\int_0^T |s\_e(n=1)|^2\,dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2}$$

$$= \left(\frac{1}{4} + \frac{2}{\pi^2}\right)A^2 = 0.4526A^2$$

Example 2: cont'd

- **What is the power in**

  s_e(n=3) = A/2 + 2A/π cos(2πf$_0$t) – 2A/(3p) cos(2π3f$_0$t)

- **Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:**

$$P_{s\_e(n=3)} = \frac{1}{T}\int_0^T |s\_e(n=3)|^2\,dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2} + \frac{2A^2}{9\pi^2}$$

$$= \left(\frac{1}{4} + \frac{2}{\pi^2} + \frac{2}{9\pi^2}\right)A^2 = 0.4752A^2$$

Example 2: cont'd

- **What is the power in**

  s_e(n=5) = A/2 + 2A/π cos(2πf$_0$t) –
  2A/(3π) cos(2π3f$_0$t) +
  2A/(5π) cos(2π5f$_0$t)

- **Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:**

$$P_{s\_e(n=5)} = \frac{1}{T}\int_0^T \left|s\_e(n=5)\right|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2} + \frac{2A^2}{9\pi^2} + \frac{2A^2}{25\pi^2}$$

$$= \left(\frac{1}{4} + \frac{2}{\pi^2} + \frac{2}{9\pi^2} + \frac{2}{25\pi^2}\right)A^2 = 0.4833A^2$$

Example 2: cont'd

- **What is the power in**

$$s\_e(n=\infty) = \frac{A}{2} + \frac{2A}{\pi}\sum_{n=1,3,5}^{\infty} \frac{(-1)^{(n-1)/2}}{n} \times \cos(2\pi n f_0 t)$$

- **Ans: we can use the result on slide Power Calculation Using Fourier Series Expansion:**

$$P_{s\_e(n=\infty)} = \frac{1}{T}\int_0^T \left|s\_e(n=\infty)\right|^2 dt = \frac{A^2}{4} + \frac{2A^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \left(\frac{1}{4} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right)A^2 = 0.5A^2$$

This the EXACT SAME power contained in s(t) –
This is expected since s(t) is 100% represented by s_e(n=∞)

Example 2: cont'd

s_e(n=k)	Expression	Power	% Power$^+$
k = 0	A/2	0.25 A^2	$(0.25A^2)/(0.5A^2)$ = 50%
k = 1	$A/2 + 2A/\pi\cos(2\pi f_0 t)$	0.4526 A^2	$(0.4526A^2)/(0.5A^2)$ = 90.5%
k = 2*	$A/2 + 2A/\pi\cos(2\pi f_0 t)$	0.4526 A^2	90.5%
k = 3	$A/2 + 2A/p\cos(2\pi f_0 t)$ – $2A/(3\pi)\cos(2\pi 3f_0 t)$	0.4752 A^2	95.0%
k = 5	$A/2 + 2A/\pi\cos(2\pi f_0 t)$ – $2A/(3\pi)\cos(2\pi 3f_0 t)$ + $2A/(5\pi)\cos(2\pi 5f_0 t)$	0.4833 A^2	96.7%

$^+$ % power = power of s_e(n=k) relative to original power in s(t) which is equal to $0.5A^2$
* For k = 2, the expression s_e(n=k) is the same as that for s_e(k=1). Why?

Example 2: cont'd

- **Therefore, s_e(n=n*) such that 95% of power is contained ➔ n* = 3**

Power Spectral Density Function

- **Fourier Series Expansion:**
  - **Specifies all the basic harmonics contained in the original function s(t)**
  - **$C_n^2/2 = (A_n^2+B_n^2)^{1/2}/2$ determines the power contribution of the nth harmonic with frequency nf_0**
- **The power Spectral Density function is a function specifying: how much power contributed at a given frequency**

Power Spectral Density Function (2)

- **Typical PSD function for periodic signals:**

Periodic *s*(*t*)

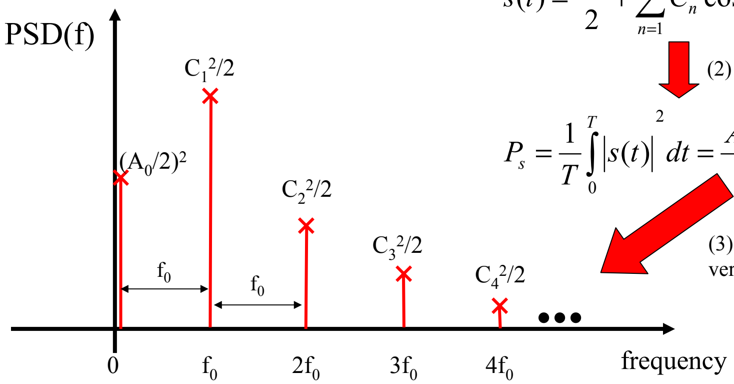
(1) Fourier Series Expansion

$$s(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} C_n \cos(2\pi n f_0 t + \theta_n)$$

(2) Power calculation

$$P_s = \frac{1}{T}\int_0^T |s(t)|^2 dt = \frac{A_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2$$

(3) Plotting power versus frequency



Power Spectral Density Function (3)

- A mathematical expression for PSD(f) can be written as

$$PSD(f) = \begin{cases} A_0{}^2/4 & f = 0 \\ C_n{}^2/2 & f = n \times f_0 \\ 0 & otherwise \end{cases}$$

- Another way (more compact) of writing PSD(f) is as follows:

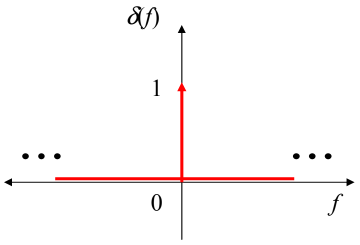
$$PSD(f) = \frac{A_0{}^2}{4} \times \delta(f) + \frac{1}{2}\sum_{n=1}^{\infty} C_n{}^2 \times \delta(f - nf_0)$$

where $\delta(t)$ is defined by

$$\delta(f) = \begin{cases} 1 & f = 0 \\ 0 & f \neq 0 \end{cases}$$

Power Spectral Density Function (4)

- $\delta(f)$ is referred to as the dirac function or unit impulse function



Note on the PSD Function

- PSD function has units of Watts/Hz
- For periodic signals ➔ PSD is a discrete function
  - Specifies the power contribution of every harmonic component $C_n^2/2$ ←→ nf_0
- The separation between the discrete components is at least f_0
  - It is exactly f_0 is all C_n's are not zeros
  - E.g. for the previous s(t) example, $C_n=0$ for even n ➔ separation = $2f_0$

Note on the PSD Function (2)

- To calculate the total power of signal ➔ Integrate PSD over all contained frequencies
  - For discrete PSD: integration = summation
- Therefore total power of s(t),

$$P_s = (A_n/2)^2 + \sum C_n^2/2 \quad \text{in Watts}$$

Example 3:

- **Find the PSD function of the periodic signal s(t) considered in Example 1.**
- **From Example 1, s(t) is given by**

$$s(t)=\frac{A}{2}+\frac{2A}{\pi}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{(n-1)/2}}{n}\times\cos(2\pi nf_0t)$$

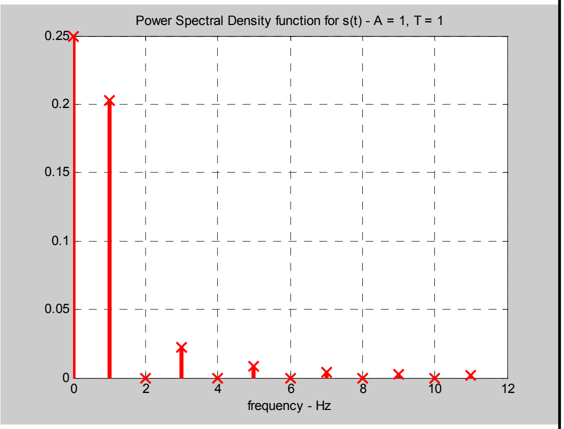
- **Using Example 2:**
  - **Power at the zero frequency = $(A/2)^2 = A^2/4$**
  - **Power at the nth harmonic (n odd) is equal to $2A^2/(n\pi)^2$**
  - **Power at the nth harmonic (n even) is zero**
  - **Therefore the PSD function is given by**

$$PSD(f)=\frac{A^2}{4}\times\delta(f)+\frac{4A^2}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\times\delta(f-nf_0)$$

Example 3: cont'd

- **The PSD is plotted as shown (A = 1, T = 1)**

$$PSD(f)=\frac{A^2}{4}\times\delta(f)+\frac{2A^2}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\times\delta(f-nf_0)$$



Example 3: cont'd

- **Matlab Code to plot PSD**

```
clear all

T = 1;
A = 1;
t = -1:0.01:1;
n_max = 11;

Frequency    = [0:1:n_max];
PwrSepctralD = zeros(size(Frequency));

% Record the DC term power at f = 0
PwrSepctralD(1) = (A/2)^2;

% Record the nth harmonic power at f = nf0
for n=1:2:n_max
   PwrSepctralD(n+1) = (2*A/(n*pi))^2 / 2;
end

figure(1)
stem(Frequency, PwrSepctralD,'rx');
title('Power Spectral Density function for s(t) - A = 1, T = 1');
xlabel('frequency - Hz');
grid
```

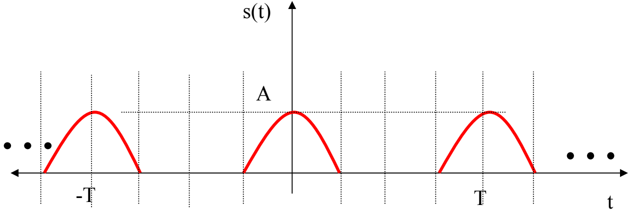
The "stem" function is typically used to plot discrete functions

---

Example 4:

This is a typical exam question

- **Problem: Consider the periodic half-wave rectified signal s(t) depicted in figure.**
  - **Write a mathematical expression for s(t)**
  - **Calculate the Fourier Series Expansion for s(t)**
  - **Calculate the total power for s(t)**
  - **Find n* such that s_e(n*) has 95% of the total power**
  - **Determine the PSD function for s(t)**
  - **Plot the PSD function for s(t)**



Example 4: cont'd

- **Answer:**

(a) To write a mathematical expression for s(t), remember that the general form of a sinusoidal function is given by

A cos(2π X Freq X t), or

A cos(2π /Period X t)

Therefore s(t) is given by

$$s(t) = A\cos(2\pi/T\ t) \quad -T/4 < t \leq T/4$$
$$= 0 \quad T/4 < t \leq 3T/4$$

Example 4: cont'd

- **Answer:**

(b) The F.S.E of s(t):

The DC term is given by

$$A_0 = \frac{2}{T}\int_{-T/4}^{T/4} s(t)dt = \frac{2A}{T}\times\int_{-T/4}^{T/4}\cos(2\pi t/T)dt$$

$$= \frac{A}{\pi}\times\sin(2\pi t/T)\Big|_{t=-T/4}^{t=T/4} = \frac{A}{\pi}\left[\sin(\pi/2)-\sin(-\pi/2)\right]$$

$$= \frac{2A}{\pi}$$

Example 4: cont'd

Remember:

Int[cos(ax)cos(bx)] = sin(ax+bx) / 2(a+b) + sin(ax-bx) / 2(a-b)

sin(a+/-b) = sin(a)cos(b) +/- cos(a)sin(b)

- **Answer:**

The An term is given by (remember 1/T = f_0)

$$A_n = \frac{2}{T}\int_{-T/4}^{T/4} s(t)\cos(2\pi n f_0 t)dt = \frac{2A}{T}\times\int_{-T/4}^{T/4}\cos(2\pi t/T)\cos(2\pi n f_0 t)dt$$

$$= \frac{2A}{T}\times\left[\frac{\sin\left(2\pi(n+1)f_0 t\right)}{4\pi(n+1)f_0} + \frac{\sin\left(2\pi(n-1)f_0 t\right)}{4\pi(n-1)f_0}\right]\Bigg|_{t=-T/4}^{t=T/4}$$

For n ≠ 1

$$= \frac{A}{\pi}\times\left[\frac{\cos(n\pi/2)}{(n+1)} + \frac{-\cos(n\pi/2)}{(n-1)}\right]$$

For n ≠ 1

This means: the n=1 should be special!

Example 4: cont'd

But

cos(nπ/2) = 0 n = odd, n≠1

= $(-1)^{(1+n/2)}$ n = even

Therefore

$$A_n = \frac{A}{\pi}\times\left[\frac{(-1)^{(1+n/2)}}{(n+1)} + \frac{(-1)(-1)^{(1+n/2)}}{(n-1)}\right]$$

For n even

$$= 0$$

For n odd, n≠1

Example 4: cont'd

The expression for An (for even n) can be further simplified to

$$A_n = \frac{A}{\pi} \times \left[\frac{(-1)^{(1+n/2)}}{(n+1)} + \frac{(-1)(-1)^{(1+n/2)}}{(n-1)} \right]$$

$$= \frac{A}{\pi} \times \left[\frac{(-1)^{(1+n/2)}(n-1) + (-1)(-1)^{(1+n/2)}(n+1)}{(n+1)(n-1)} \right]$$

$$= \frac{A}{\pi(n^2-1)} \times \left[(-1)^{(1+n/2)}(n-1) - (-1)^{(1+n/2)}(n+1) \right]$$

$$= \frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)}$$

For n even

Example 4: cont'd

An is still not completely specified – we still need to calculate it for n=1; in other words we need to calculate A1:

$$A_{n=1} = \frac{2}{T} \int_{-T/4}^{T/4} s(t)\cos(2\pi \times 1 \times f_0 t)dt = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos(2\pi t/T)\cos(2\pi f_0 t)dt$$

Therefore:

$$A_1 = \frac{2A}{T} \times \int_{-T/4}^{T/4} \cos^2(2\pi f_0 t)dt$$

$$= \frac{2A}{T} \times \left[\frac{t}{2} + \frac{1}{4 \times 2\pi f_0} \sin(4\pi f_0 t) \right]_{t=-T/4}^{t=T/4} = \frac{2A}{T} \times \left[\frac{T}{4} + \frac{\sin(\pi) - \sin(-\pi)}{8\pi f_0} \right]$$

$$= \frac{A}{2}$$

Example 4: cont'd

This mean An is equal to the following:

An = 2A/π n = 0

0 n odd, n≠1

A/2 n = 1

$$\frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)} \qquad n = 2, 4, 6, \ldots$$

The above expression specifies An for ALL POSSIBLE values of n ➔ specification is complete

---

Remember:

$$\text{Int}[\sin(ax)\cos(bx)] = \frac{\cos(ax+bx)}{2(a+b)} - \frac{\cos(ax-bx)}{2(a-b)}$$

Example 4: cont'd

We still need to compute Bn:

$$B_n = \frac{2}{T}\int_{-T/4}^{T/4} s(t)\sin(2\pi n f_0 t)dt = \frac{2A}{T}\times\int_{-T/4}^{T/4}\cos(2\pi t/T)\sin(2\pi n f_0 t)dt$$

$$= \frac{2A}{T}\times\left[\frac{\cos(2\pi(n+1)f_0 t)}{4\pi(n+1)f_0} - \frac{\cos(2\pi(n-1)f_0 t)}{4\pi(n-1)f_0}\right]\Bigg|_{t=-T/4}^{t=T/4}$$

For n ≠ 1

$$= \frac{A}{2\pi}\times\left[\frac{\cos(\pi/2(n+1)) - \cos(-\pi/2(n+1))}{(n+1)} - \frac{\cos(\pi/2(n-1)) - \cos(-\pi/2(n-1))}{(n-1)}\right]$$

$$= 0$$

For n ≠ 1

This means: the n=1 should be special!

Remember:

sin(2ax) = ½ cos(ax) sin(ax)

Example 4: cont'd

Bn is still NOT completely specified – we still need to calculate it for n=1; in other words we need to calculate B1:

$$B_{n=1} = \frac{2}{T}\int_{-T/4}^{T/4} s(t)\sin(2\pi\times 1\times f_0 t)dt = \frac{2A}{T}\times\int_{-T/4}^{T/4}\cos(2\pi t/T)\sin(2\pi f_0 t)dt$$

Therefore:

$$B_1 = \frac{2A}{T}\times\int_{-T/4}^{T/4}\cos(2\pi f_0 t)\sin(2\pi f_0 t)dt = \frac{A}{T}\times\int_{-T/4}^{T/4}\sin(4\pi f_0 t)dt$$

$$= \frac{-A}{4\pi}\times\cos(4\pi f_0 t)\Big|_{t=-T/4}^{t=T/4} = \frac{-A}{4\pi}\times\left[\cos(\pi)-\cos(-\pi)\right]$$

$$=0$$

➔ This means Bn = 0 for all n

Example 4: cont'd

- **To summarize:**

An =	**2A/π**	**n = 0**
	0	**n odd, n≠1**
	A/2	**n = 1**
	$\frac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)}$	**n = 2, 4, 6, …**

And

Bn = 0 for all n

- **Having computed An and Bn we are now in a position to write the Fourier Series Expansion for s(t)**

Example 4: cont'd

- **The Fourier Series Expansion for s(t) is given by**

$$s(t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}\left[A_n\cos(2\pi nf_0t)+B_n\sin(2\pi nf_0t)\right]$$

$$=\frac{A}{\pi}+\frac{A}{2}\cos(2\pi f_0t)+\frac{2A}{\pi}\sum_{n=2,4,6}^{\infty}\frac{(-1)^{(1+n/2)}}{n^2-1}\cos(2\pi nf_0t)$$

The C_n terms (**<u>there is a typo in the textbook</u>**) are as follows:

$C_0 = A/\pi$

$C_1 = A/2$

$$C_n = \begin{cases} \dfrac{2A(-1)^{(1+n/2)}}{\pi(n^2-1)}, & n = 2, 4, 6, \ldots \\ 0 & n \text{ odd}, n\neq 1 \end{cases}$$

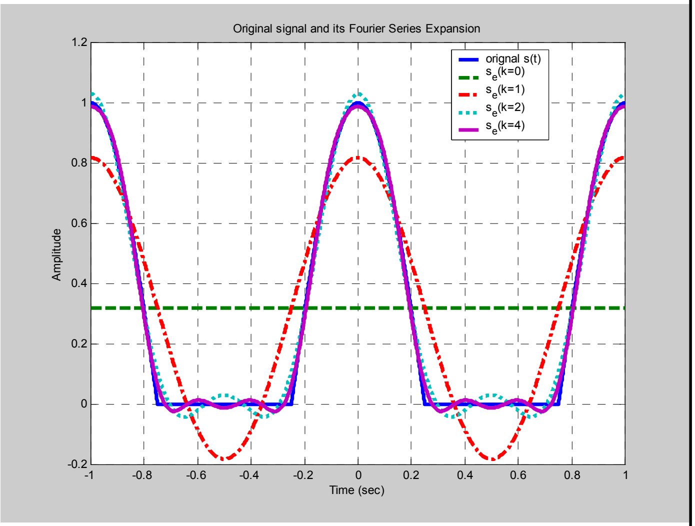
Remember:

sin(2ax) = ½ cos(ax) sin(ax)

Example 4: cont'd

Plot for s(t) and the Fourier Series Expansion for k=0, 1, 2, and 4

Note: As k increases s_e(n=k) approaches the original s(t)



Example 4: cont'd

- **The total power of s(t) is given by:**

$$P_s = \frac{1}{T}\int_{-T/4}^{3T/4} |s(t)|^2 \, dt = \frac{A^2}{T} \times \int_{-T/4}^{T/4} \cos^2(2\pi t/T)$$

$$= \frac{A^2}{T} \times \left[\frac{t}{2} + \frac{\sin(4\pi t/T)}{8\pi/T} \right]_{t=-T/4}^{t=T/4}$$

$$= \frac{A^2}{4}$$

Therefore total power of s(t) = 0.25 A^2

Example 4: cont'd

- **To find n* such that power of s_e(n=n*) = 95% of total power:**

s_e(n=k)	Expression	Power	% Power$^+$
k = 0	A/π	0.1013 A^2	$(0.1013A^2)/(0.25A^2)$ = 40.5%
k = 1	A/π + A/2 cos(2πf_0t)	0.2263 A^2	$(0.2262A^2)/(0.25A^2)$ = 90.5%
k = 2	A/π + A/2 cos(2πf_0t) + 2A/(3π) cos(2π2f_0t)	0.2488 A^2	$(0.2488A^2)/(0.25A^2)$ 99.5%

Therefore n* = 2 ➔ power of s_e(n=2) = 0.2488 A^2 which is 99.5% of total power of s(t)

Example 4: cont’d

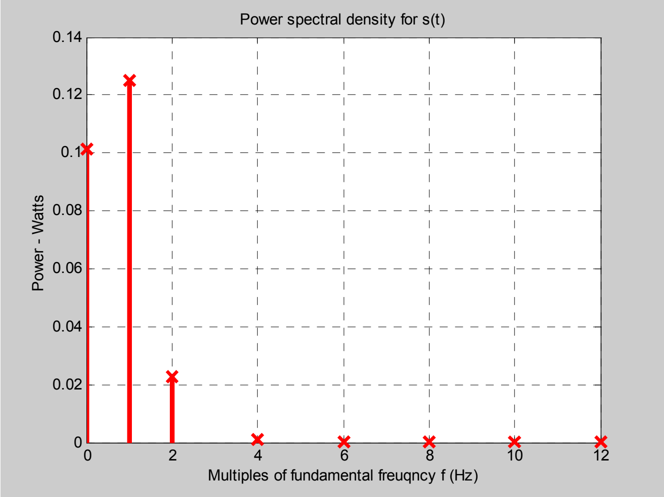
- The PSD function for s(t) is as follows:
  - Power for DC term = $(A/\pi)^2$
  - Power for harmonic at f = f_0: $(A/2)^2/2 = A^2/8$
  - Power for harmonic at f = nf_0 (n=2,4,6, …): $[2A/(\pi(n^2-1))]^2/2 = 2A^2/(\pi(n^2-1))^2$

- **Therefore PSD function equals to**

$$PSD(f) = \left(\frac{A}{\pi}\right)^2 \delta(f) + \frac{A^2}{8}\delta(f - f_0) + \frac{2A^2}{\pi^2}\sum_{n=2,4,6}^{\infty}\frac{\delta(f - nf_0)}{\left(n^2 - 1\right)^2}$$

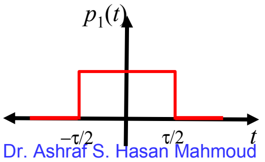
Example 4: cont’d

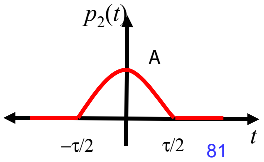
- **Plot of The PSD function for s(t)**



Fourier Transform

- Fourier Series Expansion analysis is applicable for PERIODIC signals ONLY
- There are important signals that are not periodic such as
  - Your voice waveform
  - Pulse signal p(t) – used for modulation and transmission
  - Examples: $p_1(t)$ and $p_2(t)$





Fourier Transform (2)

- How to find the frequency content of such signals?
- Use FOURIER TRANSFORM

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi jft}\,dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{2\pi jft}\,df$$

Notes on Fourier Transform

- **F.T describes a two-way transformation**

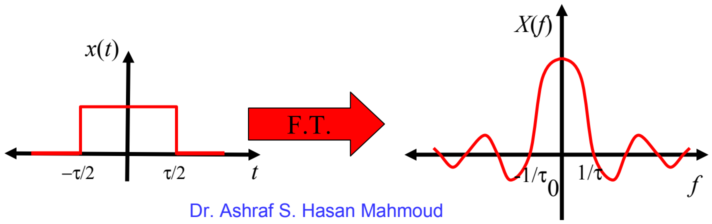
x(t) ← → X(f)

where x(t) is the time representation of the signal, while X(f) is the frequency representation of the signal

- **X(f) is defined on a continuous range of frequencies**
  - **All frequencies within the range of X(f) where X(f) is not zero contribute towards building x(t)**

Notes on Fourier Transform (2)

- **The magnitude of the contribution of a particular frequency f* in x(t) is proportional to $|X(f^*)|^2$**
- **Example: Consider the F.T. pair shown below – clearly frequencies belonging to $(-1/\tau,1/\tau)$ contribute more significantly compared to frequencies belonging to $(1/\tau,\infty)$ or $(-\infty,-1/\tau)$**

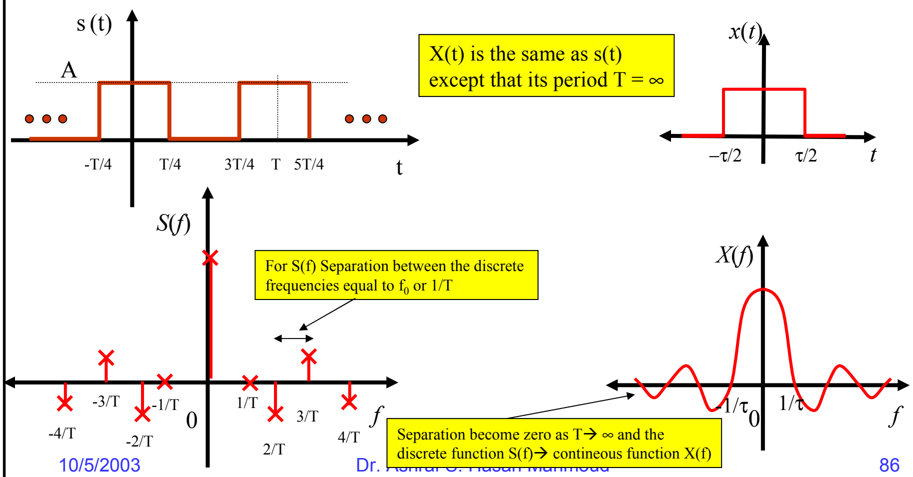


Properties of Fourier Transform

- **If x(t) is time-limited ➔ X(f) is not frequency-limited**
  - **i.e. the range of X(f) = (- ∞, ∞)**
- **If x(t) is a real-valued symmetric ➔ X(f) is real-valued**

Relation between Fourier Series Expansion and Fourier Transform

- **Consider the following two signals:**



X(t) is the same as s(t) except that its period T = ∞

For S(f) Separation between the discrete frequencies equal to f_0 or 1/T

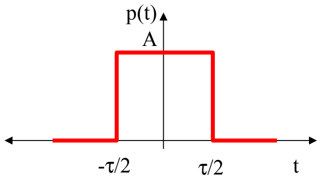
Separation become zero as T→ ∞ and the discrete function S(f)→ contineous function X(f)

Relation between Fourier Series Expansion and Fourier Transform (2)

- The separation between spectral lines for a periodic signal is 1/T
- As T → infinity and s(t) becomes non periodic → the separation between spectral lines → zero (i.e. it becomes continuous)

Example 5:

- Problem: Consider the square pulse function shown in figure:
  - Write a mathematical expression for p(t)
  - Find the Fourier transform for p(t)
  - Plot P(f)



Example 5: cont'd

- **Answer: p(t) can be expressed as**

p(t) = A |t| ≤ τ/2
= 0 otherwise

The F.T. for p(t), P(f) is given by

$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-2\pi j f t} dt$$

Example 5: cont'd

- **Which is equal to**

$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-2\pi j f t} dt = \int_{-\tau/2}^{\tau/2} A e^{-2\pi j f t} dt$$

$$= \frac{A}{-2\pi j f} \int_{-\tau/2}^{\tau/2} e^{-2\pi j f t} dt = -\frac{A}{2\pi j f} \times \left(e^{-\pi j f \tau} - e^{\pi j f \tau}\right)$$

$$= \frac{A}{\pi f} \times \frac{\left(e^{\pi j f \tau} - e^{-\pi j f \tau}\right)}{2j} = A\tau \frac{\sin(\pi f \tau)}{\pi f \tau}$$

$$= A\tau \frac{\sin(\pi f \tau)}{\pi f \tau}$$

Example 5: cont'd

- P(f) plot for A = 1 and τ = 1
- Note:
  - P(f) is define on (-∞, ∞)
  - P(f) is continuous
  - <u>P(f) = ZERO for f = n/τ</u>
  - For practical pulses – P(f) approaches zero as f → ±∞
  - Most of the energy of p(t) is contained in the period of (-1/τ, 1/τ)