King Fahd University of Petroleum & Minerals
Computer Engineering Dept

COE 200 – Fundamentals of Computer Engineering

Term 022

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Number Systems – Base r

- General number in base r is written as:

$$A_{n-1}\,A_{n-2}\,\ldots\,A_2\,A_1\,A_0\,.\,A_{-1}\,A_{-2}\,\ldots\,A_{-(m-1)}\,A_{-m}$$

Integer Part (n digits)

Radix Point

Fraction Part (m digits)

- Note that All A_i (digits) are less than r:
  - i.e. Allowed digits are 0, 1, 2, …, r – 1 ONLY
- A_{n-1} is the MOST SIGNIFACT Digit (MSD) of the number
- A_{-m} is the LEAST SIGNIFICANT Digit (LSD) of the number

A_{n-1} is the MSD of the integer part
A_0 is the LSD of the integer part
A_{-1} is the MSD of the fraction part
A_{-m} is the LSD of the fraction part

Number Systems – Base r

- The (base r) number

$$A_{n-1}\, A_{n-2}\, \ldots\, A_2\, A_1\, A_0\, .\, A_{-1}\, A_{-2}\, \ldots\, A_{-(m-1)}\, A_{-m}$$

FORM or SHAPE OF NUMBER

is equal to

$$A_{n-1} X r^{n-1} + A_{n-2} X r^{n-2} + \ldots A_2 X r^2 + A_1 X r^1 + A_0 X r^0 + A_{-1} X r^{-1} + A_{-2} X r^{-2} + \ldots A_{-(m-1)} X r^{-(m-1)} + A_{-m} X r^{-m}$$

VALUE OF NUMBER

Example – Decimal or Base 10

- For decimal system (base 10), the number

$$(724.5)_{10}$$

is equal to

$$7X10^2 + 2X10^1 + 4X10^0 + 5X10^{-1}$$
$$= 7 X 100 + 2 X 10 + 4 X 1 + 5 X 0.1$$
$$= 700 + 20 + 4 + 0.5$$
$$= 724.5$$

It is all powers of 10:
…
$10^3 = 1000$,
$10^2 = 100$,
$10^1 = 10$,
$10^0 = 1$,
$10^{-1} = 0.1$,
$10^{-2} = 0.01$,
…

Example –Base 5

- Base 5 ➔ r = 5
- Allowed digits are: 0, 1, 2, 3, and 4 ONLY
- The number

$$(312.4)_5$$

is equal to

$3X5^2 + 1X5^1 + 2X5^0 + 4X5^{-1}$
= 3 X 25 + 1 X 5 + 2 X 1 + 4 X 0.2
= 75 + 5 + 2 + 0.8
= $(82.8)_{10}$

Therefore $(312.4)_5 = (82.8)_{10}$

It is all powers of 5:
…
$5^3 = 125$,
$5^2 = 25$,
$5^1 = 5$,
$5^0 = 1$
$5^{-1} = 0.2$
$5^{-2} = 0.04$,
…

A Third Example –Base 2

- Base 2 ➔ r = 2
  - This is referred to as the **BINARY SYSTEM**
- Allowed digits are: 0 and 1 ONLY
- The number

$$(110101.11)_2$$

is equal to

$1X2^5 + 1X2^4 + 0X2^3 + 1X2^2 + 0X2^1 + 1X2^0 + 1X2^{-1} + 1X2^{-2}$
= 1 X 32 + 1 X 16 + 1 X 4 + 1 X 2 + 1 X 0.5 + 1 X 0.25
= 32 + 16 + 4 + 1 + 0.5 + 0.25
= $(53.75)_{10}$

Therefore $(110101.11)_2 = (53.75)_{10}$

It is all powers of 5:
…
$2^4 = 16$
$2^3 = 8$,
$2^2 = 4$,
$2^1 = 2$,
$2^0 = 1$
$2^{-1} = 0.5$
$2^{-2} = 0.25$,
…

Decimal to Binary Conversion of Integer Numbers

- Conversion from base 2 to base 10 (for real numbers) – See previous slide
- To convert a decimal *integer* to binary ➔ decompose into powers of 2
  - Example: $(37)_{10} = (?)_2$

    37 has ONE 32 ➔ remainder is 5
    5 has ZERO 16 ➔ remainder is 5
    5 has ZERO 8 ➔ remainder is 5
    5 has ONE 4 ➔ remainder is 1
    1 has ZERO 2 ➔ remainder is 1
    1 has ONE 1 ➔ remainder is 0

Therefore $(37)_{10} = (100101)_2$

Decimal to Binary Conversion of Integer Numbers– cont'd

- Or we can use the following (see table):
- You stop when the division result is ZERO
- Note the order of the resulting digits

No	No/2	Remainder	
37	18	1	LSD
18	9	0	
9	4	1	
4	2	0	
2	1	0	
1	0	1	MSD

- Therefore $(37)_{10} = (100101)_2$
- To check:

$1X2^5+1X2+1 = 32+4+1 = 37$

In general: to convert a decimal integer to its equivalent in base r we use the above procedure but dividing by r

A Very Useful Table

- To represent decimal numbers from 0 till 15 (16 numbers) we need FOUR binary digits $B_3B_2B_1B_0$
- In general to represent N numbers, we need $\lceil \log_2 N \rceil$ bits
- Note how
  - B_0 flipped or COMPLEMENTED at every increment
  - B_1 flipped or COMPLEMENTED every 2 steps
  - B_2 flipped or COMPLEMENTED every 4 steps
  - B_3 flipped or COMPLEMENTED every 8 steps

Decimal	Binary	Decimal	Binary
0	0000	8	1000
1	0001	9	1001
2	0010	10	1010
3	0011	11	1011
4	0100	12	1100
5	0101	13	1101
6	0110	14	1110
7	0111	15	1111

A Very Useful Table – cont'd

- Note that zeros to the left of the number do not add to its value
- When we need DIGITS beyond 9, we will use the alphabets as shown in Table
  - Example: base 16 system has 16 digits; these are: 0, , 1, 2, 3, …, 8, 9, A, B, C, D, E, F
  - This is referred to as HEXADECIMAL or HEX number system

Decimal	Binary	Decimal	Binary
0	0000	8	1000
1	0001	9	1001
2	0010	10 → A	1010
3	0011	11 → B	1011
4	0100	12 → C	1100
5	0101	13 → D	1101
6	0110	14 → E	1110
7	0111	15 → F	1111

Decimal to Binary Conversion of Fractions

- Example: $(0.234375)_{10} = (?)_2$
- Solution: We use the following procedure
- **<u>Note:</u>**
  - The binary digits are the integer part of the multiplication process
  - The process stops when the number is 0
- There are situations where the process DOES NOT end – See next slide
- Therefore $(0.234375)_{10} = (0.001111)_2$
- To check: $(0.001111)_2 = 1X2^{-2} +1X2^{-3}+1X2^{-4}+ 1X2^{-5}+1X2^{-5} 1X2^{-6} = (0.234375)_{10}$

No	NoX2	Integer Part	
0.234375	0.46875	0	← MSD
0.46875	0.9375	0	
0.9375	1.875	1	
0.875	1.75	1	
0.75	1.5	1	
0.5	1.0	1	← LSD
0			

In general: to convert a decimal fraction to its equivalent in base r we use the above procedure but multiplying by r

Decimal to Binary Conversion of Fractions – cont'd

- Example: $(0.513)_{10} = (?)_2$
- Solution: As in previous slide

No	NoX2	Integer Part
0.513	1.026	1
0.026	0.052	0
0.052	0.104	0
0.104	0.208	0
0.208	0.416	0
0.416	0.832	0
0.832	1.664	1
0.664	1.328	1
0.328	0.656	0
…		

Therefore $(0.513)_{10} = (0.100000110 \ldots)_2$

If we chose to round to 1 significant figure ➔ $(0.1)_2$

Or to 7 significant figures ➔ $(0.1000001)_2$

Etc.

Octal Number System

- Base r = 8
- Allowed digits are = 0, 1, 2, …, 6, 7

- Example: the number $(127.4)_8$ has the decimal value $1X8^2 + 2X8^1 + 7X8^0 + 4X8^{-1}$

= 1 X 64 + 2 X 8 + 7 + 0.5

= $(87.5)_{10}$

<u>It is all powers of 8</u>:
…
$8^4 = 4096$
$8^3 = 512$,
$8^2 = 64$,
$8^1 = 8$,
$8^0 = 1$
$8^{-1} = 0.125$
$8^{-2} = 0.015625$,
…

Conversion between Octal and Binary

- **<u>Example:</u>** $(127)_8 = (?)_2$
- **<u>Solution:</u>** we can find the decimal equivalent (see previous slide) and then convert from decimal to binary

$(127)_8 = (87)_{10} \rightarrow (?)_2$

From long division

$(127)_8 = (87)_{10} = (1010111)_2$

To check:

$1X2^6+1X2^4+1X2^2+1X2^1+1X2^0$

= 64 + 16 + 4 + 2 + 1

= 87

No	No/2	Remainder
87	43	1
43	21	1
21	10	1
10	5	0
5	2	1
2	1	0
1	0	1

Conversion between Octal and Binary– cont'd

- **NOTE:** $(127)_8 = (1010111)_2$
- Lets group the binary digits in groups of 3 starting from the LSD

$$(1010111)_2 \rightarrow (001 \quad 010 \quad 111)_2$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$1 \quad 2 \quad 7$$

- That is the decimal equivalent of the first group 111 ➔ 7
  of the second group 010 ➔ 2
  of the third group 001 ➔ 1
- Hence, to convert from Octal to Binary one can perform direct translation of the Octal digits into binary digits: ONE Octal digit ←→ THREE Binary digits

Conversion between Octal and Binary – cont'd

- To convert from Binary to Octal, Binary digits are grouped into groups of three digits and then translated to Octal digits

- Example: $(1011101.10)_2 = (?)_8$
- Solution:

$$(1011101.10)_2 = (001\ 011\ 101\ .\ 100)_2$$
$$= (1 \quad 3 \quad 5 \ .\ 4)_8$$
$$= (135.4)_8$$

Note:
We can add zeros to the left of the number or to the right of the number after the radix point to form the groups

Conversion From Decimal to Octal

- **Problem**: What is the octal equivalent of $(\mathbf{32.57})_{10}$?
- **Solution**:

a) We can covert (32.57)10 to binary and then to Octal or

b) We can do:

32_{10} ➔ 32/8 = 4 and remainder is 0 → 0

4/8 = 0 and remainder is 4 → 4

hence, $32_{10} = 40_8$

$(0.57)_{10}$ ➔ 0.57 X 8 = 4.56 → 4

0.56 X 8 = 4.48 → 4

0.48 X 8 = 3.84 → 3

0.84 X 8 = 6.72 → 6

…

hence, $(0.57)_{10} = (0.4436)_8$

Therefore, $(32.57)_{10} = (40.4436)_8$

What is (0.4436)8 rounded for
-Two fraction digits?
-One fraction digit?

Hexadecimal Number Systems

- Base r = 16
- Allowed digits: 0, 1, 2, …, 8, 9, A, B, C, D, E, F
- The values for the alphabetic digits are as show in Table

Hex	Value
A	10
B	11
C	12
D	13
E	14
F	15

- **Example 1:**

$(B65F)_{16} = BX16^3+6X16^2+5X16^1+FX16^0$

$= 11X4096 + 6X256 + 5X16 + 15$

$= (46687)_{10}$

- **Example 2:**

$(1B.3C)_{16} = 1X16^1+BX16^0+3X16^{-1}+CX16^{-2}$

$= 16+11+3X0.0625+ 12X0.00390625$

$= (27.234375)_{10}$

Conversion Between Hex and Binary

- **Example:** $(1B.3C)_{16} = (?)_2$
- **Solution:** we can find the decimal equivalent (see previous slide) and then convert from decimal to binary

  $(1B)_{16} = (27)_{10} \rightarrow (?)_2$

  From long division

  $(1B)_{16} = (27)_{10} = (11011)_2$

  $(0.3C)16 = (0.234375)_{10} = (0.001111)_2$

  $\rightarrow$ Therefore $(1B.3C)_{16} = (11011.\ 001111)_2$

  Verify This Result

Conversion Between Hex and Binary – cont'd

- **Note:**

$(1B.3C)_{16} = (11011.\ 001111)_2$ from previous example

Lets group the binary bits in groups of 4 starting from the radix point, adding zeros to the left of the number or to the right as needed

$\rightarrow$ (0001 1011 . 0011 1100)

↕ ↕ ↕ ↕

1 B . 3 C

- Hence, to convert from Hex to Binary one can perform direct translation of the Hex digits into binary digits: ONE Hex digit $\leftrightarrow$ FOUR Binary digits

Conversion between Hex and Binary – cont'd

- To convert from Binary to Hex, Binary digits are grouped into groups of four digits and then translated to Hex digits

- Example: $(1011101.10)_2 = (?)_{16}$
- Solution:

$$(1011101.10)_2 = (0101\ 1101\ .\ 1000)_2$$
$$= (\ 5\quad D\ .\ 8\ )_{16}$$
$$= (5D.8)_{16}$$

Note:
We can add zeros to the left of the number or to the right of the number after the radix point to form the groups

Sample Exam Problem

- **Problem**: What is the radix r if

$$((33)_r + (24)_r)\ X\ (10)_r = (1120)_r$$

- **Solution**:

$(33)_r = 3r + 3$,
$(24)_r = 2r + 4$,
$(10)_r = r$,
$(1120)_r = r^3 + r^2 + 2r$
therefore:

$$[(3r+3)+(2r+4)]\ X\ r$$
$$= r^3 + r^2 + 2r \rightarrow r^3 - 4\ r^2 - 5\ r = 0, \text{ or}$$
$$r(r - 5)(r + 1) = 0$$

This means, the radix r is equal to 5

Number Ranges - Decimal

- Consider a decimal integer number of n digits:

  $A_{n-1}A_{n-2}\ldots A_1A_0$ where $A_i \in \{0,1,2, \ldots, 9\}$

Smallest integer is $0_{n-1}0_{n-2}\ldots0_10_0 = 0$
Largest integer is $9_{n-1}9_{n-2}\ldots9_19_0 = 10^n - 1$

Example: for n equal to 3 ➔ 3 digits integer decimals; the maximum integer is 999 or $10^3 - 1$

Number Ranges – Decimal – cont'd

- Consider a decimal fraction of m digits:

  $0.A_{-1}A_{-2}\ldots A_{-(m-1)}A_{-m}$ where $A_i \in \{0,1,2, \ldots, 9\}$

Smallest non-zeros fraction is $0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m} = 10^{-m}$
Largest fraction is $0.9_{-1}9_{-2}\ldots9_{-(m-1)}9_{-m} = 1 - 10^{-m}$

Example: for m equal to 3 ➔ 3 digits decimal fraction;

The minimum fraction is 10^{-3} or 0.001
The maximum number is $1 - 10^{-3}$ or 0.999

Number Ranges – Base-r Numbers

- Consider a base-r integer of n digits:

$$A_{n-1}A_{n-2}\ldots A_1A_0 \quad \text{where } A_i \in \{0,1,2, \ldots, r-1\}$$

Smallest integer is $0_{n-1}0_{n-2}\ldots0_10_0 = 0$

Largest integer is $(r-1)_{n-1}(r-1)_{n-2}\ldots(r-1)_1(r-1)_0 = r^n – 1$

Example: for r = 5, n equal to 3 ➔ 3 digits base-5 integer;

The maximum integer is $(444)_5$ or $(5^3 – 1)_{10}$

To check:

the decimal equivalent of $(444)_5$ is $4X5^2+4X5^1+4 = (124)_{10}$ or simply $5^3 – 1 = (124)_{10}$

Number Ranges - Base-r Numbers

- Consider a base-r fraction of m digits:

$$0.A_{-1}A_{-2}\ldots A_{-(m-1)}A_{-m} \quad \text{where } A_i \in \{0,1,2, \ldots, r-1\}$$

Smallest non-zero fraction is

$$(0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m})_r = (r^{-m})_{10}$$

Largest fraction is

$$(0.(r-1)_{-1}(r-1)_{-2}\ldots(r-1)_{-(m-1)}(r-1)_{-m})_r = (1 – r^{-m})_{10}$$

Example: for r = 5 and m equal to 3 ➔ 3 digits base-5 fraction;

The maximum number is $(0.444)_5$ or $1 – 5^{-3} = 0.992$

Number Ranges - Base-r Numbers – cont'd

		Decimal (r=10)	Binary (r = 2)	Octal (r = 8)	Hex (r = 16)
Integer	Min	$0_{n-1}0_{n-2}\ldots0_10_0$ = 0	$0_{n-1}0_{n-2}\ldots0_10_0$ = 0	$0_{n-1}0_{n-2}\ldots0_10_0$ = 0	$0_{n-1}0_{n-2}\ldots0_10_0$ = 0
	Max	$9_{n-1}9_{n-2}\ldots9_19_0 = 10^n - 1$	$(1_{n-1}1_{n-2}\ldots1_11_0)_2 = (2^n - 1)_{10}$	$(8_{n-1}8_{n-2}\ldots8_18_0)_8 = (8^n - 1)_{10}$	$(F_{n-1}F_{n-2}\ldots F_1F_0)_{16} = (16^n - 1)_{10}$
fraction	Min	$0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m} = 10^{-m}$	$(0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m})_2 = (2^{-m})_{10}$	$(0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m})_8 = (8^{-m})_{10}$	$(0.0_{-1}0_{-2}\ldots0_{-(m-1)}1_{-m})_{16} = (16^{-m})_{10}$
	Max	$0.9_{-1}9_{-2}\ldots9_{-(m-1)}9_{-m} = 1 - 10^{-m}$	$(0.1_{-1}1_{-2}\ldots1_{-(m-1)}1_{-m})_2 = (1 - 2^{-m})_{10}$	$(0.7_{-1}7_{-2}\ldots7_{-(m-1)}7_{-m})_8 = (1 - 8^{-m})_{10}$	$(0.F_{-1}F_{-2}\ldots F_{-(m-1)}F_{-m})_{16} = (1 - 16^{-m})_{10}$

Exercises

- What is 8^4 equal to in octal?
  $(8^4)_{10} = (10000)_8$
- What is 2^5 equal to in binary?
  $(2^5) = (100000)_2$
- What is $16^4 - 1$ equal to in Hex?
- What is $2^3 - 2^{-2}$ equal to in Binary?
- What is $16^5 - 16^4$ equal to in Hex?
- What is $3^4 - 3^{-2}$ equal to in base-3?
- What is $2^4 - 2^{-2}$ equal to in base-3?