King Fahd University of Petroleum & Minerals Computer Engineering Dept

COE 200 – Fundamentals of Computer Engineering

Term 022

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Machine Representation of Numbers

- Computers store numbers in special digital electronic devices called REGISTERS
- REGISTERS consist of a fixed number of storage elements
- Each storage element can store one BIT of data (either 1 or 0)
- A register has a FINITE number of bits
  - Register size (n) is the number of bits in this register
  - N is typically a power of 2 (e.g. 8, 16, 32, 64, etc.)
  - A register of size n can represent 2^n distinct values
  - Numbers stored in a register can be either signed or unsigned

N-bit Register

- N-storage elements

MSB LSB

bit n-1	bit n-2	…	bit 2	bit 1	bit 0

- Each storage element capable of holding ONE bit (either 1 or –0
- n-bits can represent 2^n distinct values
  - For example if unsigned integer numbers are to be represented, we can represent all numbers from 0 to 2^n-1 (recall the number ranges for n-bits)
  - If we use it to represent signed numbers, still it can hold 2n different numbers – we will learn about the ranges of these numbers in the coming slides

N-bit Register – cont'd

- Using a 4-bit register, $(13)_{10}$ or $(D)_H$ is represented as follows:

1	1	0	1

- Using an 8-bit register, $(13)_{10}$ or $(D)_H$ is represented as follows:

0	0	0	0	1	1	0	1

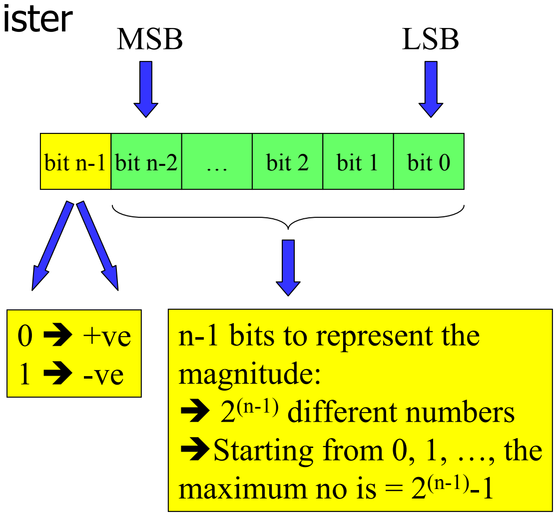
- Note that ZEROS are used to pad the binary representation of 13 in the 8-bit register
- We are still using UNSIGNED NUMBERS

Signed Number Representation

- To report a "signed" number, you need to specify its:
  - Magnitude (or absolute value), and
  - Sign (positive or negative)
- There are to major techniques to represent signed numbers
  1. Signed Magnitude Representation
  2. Complement Method

Signed Magnitude Representation

- N-bit register



Signed Magnitude Representation – Example 1:

- Show how +6, -6, +13, and –13 are represented using a 4-bit register

- Solution: Using a 4-bit register, the leftmost bit is reserved for the sign, which leaves 3 bits only to represent the magnitude

  ➔ The largest magnitude that can be represented = $2^{(4-1)} - 1 = 7 < 13$

  Hence, the numbers +13 and –13 can NOT be represented using the 4-bit register

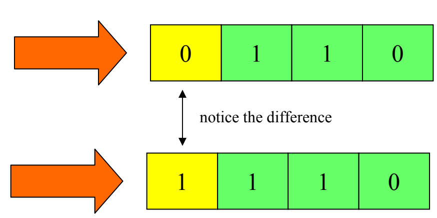
Signed Magnitude Representation – Example 1: cont'd

- Solution (cont'd):

  However both –6 and +6 can be represented as follows:

Singed-Magnitude representation of +6

Singed-Magnitude representation of -6



Signed Magnitude Representation – Example 2:

- Show how +6, -6, +13, and –13 are represented using an 8-bit register

- <u>Solution</u>: Using an 8-bit register, the leftmost bit is reserved for the sign, which leaves 7 bits only to represent the magnitude

  ➔ The largest magnitude that can be represented = $2^{(8-1)} - 1 = 127$

  Hence, the numbers can be represented using the 8-bit register

Signed Magnitude Representation – Example 2: cont'd

- <u>Solution (cont'd)</u>:

  Since 6 and 13 are equal to : 110 and 1101 respectively, the required representations are

Singed-Magnitude representation of +6	0	0	0	0	0	1	1	0
Singed-Magnitude representation of -6	1	0	0	0	0	1	1	0
Singed-Magnitude representation of +13	0	0	0	0	1	1	0	1
Singed-Magnitude representation of -13	1	0	0	0	1	1	0	1

Things We Learned About Signed-Magnitude Representation

- For an n-bit register
  - Leftmost bit is reserved for the sign (0 for +ve and 1 for –ve)
  - Remaining n-1 bits represent the magnitude
  - $2^{(n-1)}$ different numbers: – minimum is zero and maximum is $2^{(n-1)}-1$
- Two representations for zero: +0 and –0
- Range of numbers: from $-\{2^{(n-1)}-1\}$ to $+\{2^{(n-1)}-1\}$ ➔ symmetric range

Complement Representation

- +ve numbers (+N) are represented exactly the same way as in signed-magnitude representation
- -ve numbers (-N) are represented by the *complement* of N or N'

How is the complement of N or N' defined?

$N' = M - N$ where M is some constant

Properties of the Complement Representation

- The complement of the complement of N is equal to N:

Proof: (N')' = M – (M – N) = -(-N) = N

Same as with –ve numbers definition!

- The complement method representation of signed numbers simplifies implementation of arithmetic operations like subtraction:

e.g.: A – B can be replaced by A + (-B) or A + B' using the complement method

Therefore to perform subtraction using computers we complement and add the subtrahend

How to Choose M?

- Consider the following number:

$$X = X_{n-1} \ldots X_2 X_1 X_0 \,.\, X_{-1} X_{-2} \ldots X_{-(m-1)} X_{-m}$$

(n integral digits – m fractional digits)

- Using the base-r number system, there can be two types of the complement representation
  - Radix Complement (R's Complement)

  ➔ $M = r^n$

  - Diminished Radix Complement (R-1's Complement):

  ➔ $M = r^n - r^{-m}$

  $= r^n - \text{ulp}$

Recall that $r^n = 1_n 0_{n-1} \ldots 0_1 0_0$
= 1 followed by n zeros
Recall that $r^{-m} = 0\ldots00.00..01$
= unit in the least position

How to Choose M? – cont'd

- Note that:
  - $M = r^n - r^{-m}$ is the LARGEST unsigned number that can be represented
  - From the definitions of M, Rs complement of N is equal to R-1's complement of N plus ulp

Summary of Complement Method

- R's Complement:

Number System	R's Complement	Complement of X
Decimal	10's Complement	$X'_{10} = 10^n - X$
Binary	2's Complement	$X'_2 = 2^n - X$
Octal	8's Complement	$X'_8 = 8^n - X$
Hexadecimal	16's Complement	$X'_{16} = 16^n - X$

Summary of Complement Method – cont'd

- R-1's Complement:

Number System	R-1's Complement	Complement of X
Decimal	9's Complement	$X'_9 = (10^n - 10^{-m}) - X$ $= 99...99.99...99 - X$
Binary	1's Complement	$X'_1 = (2^n - 2^{-m}) - X$ $= 11...11.11...11 - X$
Octal	7's Complement	$X'_7 = (8^n - 8^{-m}) - X$ $= 77...77.77...77 - X$
Hexadecimal	15's Complement	$X'_{15} = (16^n - 16^{-m}) - X$ $= FF...FF.FF...FF - X$

Example 1a:

- Find the 9's and 10's complement of 2357?
- Solution:

$X = 2357 \rightarrow n = 4$

$X'_9 = (10^4 - \text{ulp}) - X$
$= (10000 - 1) - 2357$
$= 9999 - 2357$
$= 7642$

$X'_{10} = 10^4 - X$
$= 10000 - 2357$
$= 7643$

Note that: $X + X'_9 = 2357 + 7642 = 9999 = M$
While $X + X'_{10} = 2357 + 7643 = 1\ 0000 = M$

Or alternatively,

$X'_{10} = X'_9 + \text{ulp} = 7642 + 1 = 7643$

Example 1b:

- Find the 9's and 10's complement of 2895.786?
- Solution:

$X = 2895.786 \rightarrow n = 4, m = 3$

$X'_9 = (10^4 - ulp) - X$

$= (10000 - 0.001) - 2895.786$

$= 9999.999 - 2895.786$

$= 7104.213$

$X'_{10} = 10^4 - X$

$= 10000 - 2895.786$

$= 7104.214$

Note that: $X + X'_9 = 2895.786 + 7104.213$
$= 9999.999 = M$
While $X + X'_{10} = 2895.786 + 7104.214$
$= 1\ 0000.000 = M$

Or alternatively,

$X'_{10} = X'_9 + ulp = 7642 + 1 = 7104.214$

Example 2a:

- Find the 1's and 2's complement of 110101010?
- Solution:

$X = 110101010 \rightarrow n = 9$

$X'_1 = (2^9 - ulp) - X$

$= (1000000000 - 1) - 110101010$

$= 111111111 - 110101010$

$= 001010101$

$X'_2 = 2^9 - X$

$= 1000000000 - 110101010$

$= 001010110$

Note that: $X + X'_1 = 110101010 + 001010101$
$= 111111111 = M$
While $X + X'_2 = 110101010 + 001010110$
$= 1\ 000000000 = M$

Or alternatively,

$X'_2 = X'_1 + ulp = 001010101 + 1 = 001010110$

Example 2b:

- Find the 1's and 2's complement of 1010.001?
- Solution:

$X = 1010.001 \rightarrow n = 4, m = 3$

$X'_1 = (2^4 - ulp) - X$

$= (10000 - 0.001) - 1010.001$

$= 1111.111 - 1010.001$

$= 0101.110$

$X'_2 = 2^4 - X$

$= 10000 - 1010.001$

$= 0101.111$

Note that: $X + X'_1 = 1010.001 + 0101.110 = 1111.111 = M$
While $X + X'_2 = 1010.001 + 0101.110 = 1\ 0000.000 = M$

Or alternatively,

$X'_2 = X'_1 + ulp = 0101.110 + 0.001 = 0101.111$

Notes On 1's and 2's Complements Computation:

- 1's complement can be obtained by bitwise complementing the bits of X

Examples (from previous slide)

$X = 1010.001 \rightarrow X'_1 = 0101.110$

- 2's complement of X can be obtained by:
  1. Adding ulp to its 1's complement, or

upl is added

$X = 1010.001 \rightarrow X'_1 = 0101.110 \rightarrow X'_2 = 0101.111$

  2. Scanning X from right to left, copy all digits including first 1, complement all remaining digits

$X = \underbrace{1010.00}_{\text{You complement subsequent bits}}1 \rightarrow X'_2 = \underbrace{0101.11}_{}1$

1st one you keep

Example 3a:

- Find the 7's and the 8's complement of the following octal number 6770?
- Solution:

$X = 6770 \rightarrow n = 4$

$X'_7 = (8^4 – ulp) – X$

$= (10000 – 1) – 6770$

$= 7777 – 6770$

$= 1007$

$X'_8 = 8^4 – X$

$= 10000 – 6770$

$= 1010$

Or alternatively,

$X'_8 = X'_7 + ulp = 1007 + 1 = 1010$

Example 3b:

- Find the 7's and the 8's complement of the following octal number 541.736?
- Solution:

$X = 541.736 \rightarrow n = 3, m = 3$

$X'_7 = (8^3 – ulp) – X$

$= (10000 – 0.001) – 541.736$

$= 777.777 – 541.736$

$= 236.041$

$X'_8 = 8^3 – X$

$= 10000 – 541.736$

$= 236.042$

Or alternatively,

$X'_8 = X'_7 + ulp = 236.041 + 0.001 = 236.042$

Example 4a:

- Find the 15's and the 16's complement of the following Hex number 3FA9?
- Solution:

$X = 3FA9 \rightarrow n = 4$

$X'_{15} = (16^4 - ulp) - X$

$= (10000 - 1) - 3FA9$

$= FFFF - 3FA9$

$= C056$

$X'_{16} = 16^4 - X$

$= 10000 - 3FA9$

$= C057$

Or alternatively,

$X'_{16} = X'_{15} + ulp = C056 + 1 = C057$

Example 4b:

- Find the 15's and the 16's complement of the following Hex number 9B1.C70?
- Solution:

$X = 9B1.C70 \rightarrow n = 3, m = 3$

$X'_{15} = (16^3 - ulp) - X$

$= (1000 - 0.001) - 9B1.C70$

$= FFF.FFF - 9B1.C70$

$= 64E.38F$

$X'_{16} = 16^3 - X$

$= 1000 - 9B1.C70$

$= 64E.390$

Or alternatively,

$X'_{16} = X'_{15} + ulp = 64E.38F + 0.001 = 64E.390$

Complement Representation – Example 5:

- Show how +53 and -53 are represented in 8-bit registers using signed-magnitude, 1's complement and 2's complement?
- Solution:

Note that 53 = 32 + 16 + 4 + 1,

Therefore using 8-bit signed-magnitude:

  - +53 → **0**0110101 -53 → **1**0110101

- To find the representation in complement method:

Complement Representation – Example 5: cont'd

- Solution: cont'd

To find the representation in complement method.
(53)10 = $(00110101)_2$ when written in 8-bit binary

1's complement → 11001010 (inverting every bit)

2's complement → 11001011 (adding ulp to X'_1)

Complement Representation – Example 5: cont'd

- Solution: cont'd

Putting all the results together in a table

	+53	-53
Signed-Magnitude	00110101	10110101
1's Complement	00110101	11001010
2's Complement	00110101	11001011

Note:

- +53 representation is the same for all methods
- For +53, the leftmost bit is 0 (+ve number)
- For -53, the leftmost bit is 1 (-ve number)

Example 6:

- For the shown 4-bit representations, indicate the corresponding decimal value in the shown representation

Example 6: cont'd

- Signed-Magnitude and 1's complement are symmetrical representations with TWO representations for ZERO
- Range from signed-magnitude and 1's complement is from -7 to +7
- 2's complement representation is not symmetrical
- Range for 2's complement is from -8 to +7 – with one representation for ZERO

	Unsigned	Signed-Magnitude	1's Complement	2's Complement
0000	0	0	0	0
0001	1	1	1	1
0010	2	2	2	2
0011	3	3	3	3
0100	4	4	4	4
0101	5	5	5	5
0110	6	6	6	6
0111	7	7	7	7
1000	8	-0	-7	-8
1001	9	-1	-6	-7
1010	10	-2	-5	-6
1011	11	-3	-4	-5
1100	12	-4	-3	-4
1101	13	-5	-2	-3
1110	14	-6	-1	-2
1111	15	-7	-0	-1

Summary

- The following table summarizes the properties and ranges for the studied signed number representations

	Signed-Magnitude	1's Complement	2's Complement
Symmetric	Y	Y	N
No of Zeros	2	2	1
Largest	$2^{(n-1)}-1$	$2^{(n-1)}-1$	$2^{(n-1)}-1$
Smallest	$-\{2^{(n-1)}-1\}$	$-\{2^{(n-1)}-1\}$	$-2^{(n-1)}$

Exercise

- Find the binary representation in signed magnitude, 1's complement, and 2's complement for the following decimal numbers: +13, -13, +39, +1, -1, +73, and -73. For all numbers, show the required representation for 6-bit and 8-bit registers

10's Complement

- For n = 1 and 2

X'_{10} (n=1)	X'_{10} using+/- in decimal
0	0
1	1
2	2
3	3
4	4
5	-5
6	-4
7	-3
8	-2
9	-1

X'_{10} (n=2)	X'_{10} using+/- in decimal
00	0
01	1
02	2
..	..
09	9
10	10
11	11
12	12
...	..
49	49
50	-50
51	-49
52	-48
...	...
98	-2
99	-1

8's Complement

- For n = 1 and 2

X'_8 (n=1)	X'_8 using+/- in decimal
0	0
1	1
2	2
3	3
4	-4
5	-3
6	-2
7	-1

X'_8 (n=2)	X'_8 using+/- in decimal
00	0
01	1
02	2
..	..
07	7
10	8
11	9
12	10
…	..
36	30
37	31
40	-32
41	-31
…	…
70	-8
71	-7
…	…
76	-2
77	-1

16's Complement

- For n = 1 and 2

X'_{16} (n=1)	X'_{16} using+/- in decimal
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	-8
9	-7
A	-6
B	-5
C	-4
D	-3
E	-2
F	-1

X'_{16} (n=2)	X'_{16} using+/- in decimal
00	0
01	1
…	…
0E	14
0F	15
10	16
11	17
…	…
1F	31
20	32
21	33
…	…
7E	126
7F	127
80	-128
81	-127
…	…
F0	-16
F1	-15
…	…
FD	-3
FE	-2
FF	-1