Standard & Canonical Forms

CHAPTER OBJECTIVES

- **Learn** ***Binary Logic*** **and** ***BOOLEAN Algebra*****Learn How to Map a** ***Boolean Expression*** **into Logic** ***Circuit Implementation*** **Learn How To Manipulate** ***Boolean Expressions*** **and** ***Simplify Them*****Lesson Ojectives**Learn how to derive a Boolean expression of a function defined by its truth table. The derived expressions may be in one of two possible standard forms: *The Sum of Min-terms* or the *Product of Max-Terms*.

2. Learn how to map these expressions into logic circuit implementations (2-Level Implementations).

MinTerms

- Consider a system of 3 input signals (variables) x, y, & z.
- A term which ANDs all input variables, either in the true or complement form, is called a minterm.
- Thus, the considered 3-input system has 8 minterms, namely:

$$\bar{x}\bar{y}\bar{z}, \bar{x}\bar{y}z, \bar{x}y\bar{z}, \bar{x}yz, x\bar{y}\bar{z}, x\bar{y}z, xy\bar{z} \ \& \ xyz$$

- Each minterm equals 1 at exactly one particular input combination and is equal to 0 at all other combinations
- Thus, for example, $\bar{x}\,\bar{y}\,\bar{z}$ is always equal to 0 except for the input combination ***xyz*** = **000**, where it is equal to 1.
- Accordingly, the minterm $\bar{x}\,\bar{y}\,\bar{z}$ is referred to as m_0.
- In general, minterms are designated m_i, where i corresponds the input combination at which this minterm is equal to 1.

- For the 3-input system under consideration, the number of possible input combinations is 2^3, or 8. This means that the system has a total of 8 minterms as follows:
  - $m_0 = \bar{x}\,\bar{y}\,\bar{z} = 1$ IFF xyz = **000**, *otherwise it equals 0*
  - $m_1 = \bar{x}\,\bar{y}z = 1$ IFF xyz = **001**, *otherwise it equals 0*
  - $m_2 = \bar{x}y\bar{z} = 1$ IFF xyz = **010**, *otherwise it equals 0*
  - $m_3 = \bar{x}yz = 1$ IFF xyz = **011**, *otherwise it equals 0*
  - $m_4 = x\bar{y}\,\bar{z} = 1$ IFF xyz = **100**, *otherwise it equals 0*
  - $m_5 = x\bar{y}z = 1$ IFF xyz = **101**, *otherwise it equals 0*
  - $m_6 = xy\bar{z} = 1$ IFF xyz = **110**, *otherwise it equals 0*
  - $m_7 = xyz = 1$ IFF xyz = **111**, *otherwise it equals 0*

In general,

- For n-input variables, the number of minterms = the total number of possible input combinations = 2^n.
- A minterm = 0 at all input combinations except one where the minterm = 1.

MaxTerms

- Consider a circuit of 3 input signals (variables) x, y, & z.
- A term which ORs all input variables, either in the true or complement form, is called a Maxterm.
- With 3-input variables, the system under consideration has a total of 8 Maxterms, namely:

$$(x+y+z),(x+y+\bar{z}),(x+\bar{y}+z),(x+\bar{y}+\bar{z}),\ (\bar{x}+y+z),(\bar{x}+y+\bar{z}),\ (\bar{x}+\bar{y}+z)\,\&\,(\bar{x}+\bar{y}+\bar{z})$$

- Each Maxterm equals 0 at exactly one of the 8 possible input combinations and is equal to 1 at all other combinations.
- For example, $(x+y+z)$ equals 1 at all input combinations except for the combination xyz = **000**, where it is equal to 0.
- Accordingly, the Maxterm $(x+y+z)$ is referred to as M_0.
- In general, Maxterms are designated M_i, where i corresponds to the input combination at which this Maxterm is equal to 0.

- For the 3-input system, the number of possible input combinations is 2^3, or 8. This means that the system has a total of 8 Maxterms as follows:
  - $M_0 = (x+y+z) = 0$ IFF $xyz = \mathbf{000}$, *otherwise it equals 1*
  - $M_1 = (x+y+\overline{z}) = 0$ IFF $xyz = \mathbf{001}$, *otherwise it equals 1*
  - $M_2 = (x+\overline{y}+z) = 0$ IFF $xyz = \mathbf{010}$, *otherwise it equals 1*
  - $M_3 = (x+\overline{y}+\overline{z}) = 0$ IFF $xyz = \mathbf{011}$, *otherwise it equals 1*
  - $M_4 = (\overline{x}+y+z) = 0$ IFF $xyz = \mathbf{100}$, *otherwise it equals 1*
  - $M_5 = (\overline{x}+y+\overline{z}) = 0$ IFF $xyz = \mathbf{101}$, *otherwise it equals 1*
  - $M_6 = (\overline{x}+\overline{y}+z) = 0$ IFF $xyz = \mathbf{110}$, *otherwise it equals 1*
  - $M_7 = (\overline{x}+\overline{y}+\overline{z}) = 0$ IFF $xyz = \mathbf{111}$, *otherwise it equals 1*

In general,

- For n-input variables, the number of Maxterms = the total number of possible input combinations = 2^n.
- A Maxterm = 1 at all input combinations except one where the Maxterm = 0.

Imprtant Result

Using De-Morgan's theorem, or truth tables, it can be easily shown that:

$$M_i = \overline{m_i} \qquad \forall i = 0,1,2,\ldots\ldots,(2^n - 1)$$

Expressing Functions as a *Sum of Minterms* and Product of Maxterms

Example: Consider the function F defined by the shown truth table

x	y	z	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Now let's rewrite the table, with few added columns.

- A column i indicating the input combination
- Four columns of minterms m_2, m_4, m_5 and m_7
- One last column **OR-ing** the above minterms ($m_2 + m_4 + m_5 + m_7$)

i	*x y z*	*F*	m_2	m_4	m_5	m_7	$m_2+m_4+m_5+m_7$
0	0 0 0	0	**0**	**0**	**0**	**0**	**0**
1	0 0 1	0	**0**	**0**	**0**	**0**	**0**
2	0 1 0	1	**1**	**0**	**0**	**0**	**1**
3	0 1 1	0	**0**	**0**	**0**	**0**	**0**
4	1 0 0	1	**0**	**1**	**0**	**0**	**1**
5	1 0 1	1	**0**	**0**	**1**	**0**	**1**
6	1 1 0	0	**0**	**0**	**0**	**0**	**0**
7	1 1 1	1	**0**	**0**	**0**	**1**	**1**

- From this table, we can clearly see that $F = m_2 + m_4 + m_5 + m_7$
- This is logical since F = 1, only at input combinations *i*= 2, 4, 5 and 7
- Thus, by ORing minterm m_2 (which has a value of 1 only at input combination *i*= 2) with minterm m_4 (which has a value of 1 only at input combination *i*= 4) with minterm m_5 (which has a value of 1 only at input combination *i*= 5) with minterm m_7 (which has a value of 1 only at input combination *i*= 7) the resulting function will equal F.
- *In general*, Any function can be expressed by *OR-ing* all minterms (m_i) corresponding to input combinations (*i*) at which the function has a value of 1.
- The resulting expression is commonly referred to as the ***SUM of minterms*** and is typically expressed as **F = ∑(2, 4, 5, 7),** where ∑ indicates OR-ing of the indicated minterms. Thus, **F = ∑(2, 4, 5, 7)** = ($m_2 + m_4 + m_5 + m_7$)

Example:

- Consider the function *F* of the previous example.
- We will, first, derive the sum of minterms expression for the complement function F\`.

i	*x y z*	*F*	F\`
0	0 0 0	0	1
1	0 0 1	0	1
2	0 1 0	1	0
3	0 1 1	0	1
4	1 0 0	1	0
5	1 0 1	1	0
6	1 1 0	0	1
7	1 1 1	1	0

The truth table of F\` shows that F\` equals 1 at *i = 0, 1, 3 and 6, then,*

F\` $= m_0 + m_1 + m_3 + m_6$, *i.e*

F\` = ∑(0, 1, 3, 6), (1)

F = ∑(2, 4, 5, 7) (2)

- Obviously, the sum of minterms expression of F\` contains *all* minterms that do not appear in the sum of minterms expression of F.

Using De-Morgan theorem on equation (2),

$$\overline{F} = \overline{(m_2 + m_4 + m_5 + m_7)} = \overline{m_2}.\overline{m_4}.\overline{m_5}.\overline{m_7} = M_2.M_4.M_5.M_7$$

This form is designated as the *Product of Maxterms* and is expressed using the ∏ symbol, which is used to designate product in regular algebra, but is used to designate AND-ing in Boolean algebra.

Thus,

F` = ∏ (2, 4, 5, 7) = *M₂. M₄. M₅. M₇* **(3)**

From equations (1) and (3) we get,

F` = Σ(0, 1, 3, 6) = ∏ (2, 4, 5, 7)

In general, *any function can be expressed both as a sum of minterms and as a product of maxterms*. Consider the derivation of F back from **F`** given in equation (3):

$$F = \overline{\overline{F}} = \overline{m_0 + m_1 + m_3 + m_6} = \overline{m_0} \cdot \overline{m_1} \cdot \overline{m_3} \cdot \overline{m_6} = M_0 \cdot M_1 \cdot M_3 \cdot M_6$$

F = Σ(2, 4, 5, 7) = ∏ (0, 1, 3, 6)

F` = ∏ (2, 4, 5, 7) = Σ (0, 1, 3, 6)

Conclusions:

- Any function can be expressed both as a sum of minterms (Σ m_i) and as a product of maxterms. The product of maxterms expression (∏ M_j) expression of F contains *all* maxterms M_j ($\forall j \neq i$) that do not appear in the sum of minterms expression of F.
- The sum of minterms expression of F` contains *all* minterms that do not appear in the sum of minterms expression of F.
- This is true for all complementary functions. Thus, each of the 2^n minterms will appear either in the sum of minterms expression of F or the sum of minterms expression of $\overline{F}$ but not both.
- The product of maxterms expression of F` contains *all* maxterms that do not appear in the product of maxterms expression of F.
- This is true for all complementary functions. Thus, each of the 2^n maxterms will appear either in the product of maxterms expression of F or the product of maxterms expression of $\overline{F}$ but not both.

Example:

Given that **F (a, b, c, d) = ∑(0, 1, 2, 4, 5, 7)**, derive the product of maxterms expression of **F** and the 2 standard form expressions of **F`**.

Since the system has 4 input variables (a, b, c & d) → The number of minterms and Maxterms = $2^4 = 16$

F (a, b, c, d) = ∑(0, 1, 2, 4, 5, 7)

1. List all maxterms in the Product of maxterms expression

F = ∏ (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15).

2. Cross out maxterms corresponding to input combinations of the minterms appearing in the sum of minterms expression

F = ∏ (~~0, 1, 2,~~ 3, ~~4, 5,~~ 6, ~~7,~~ 8, 9, 10, 11, 12, 13, 14, 15).

F = ∏ (3, 6, 8, 9, 10, 11, 12, 13, 14, 15).

Similarly, obtain both canonical form expressions for F`

F` = ∑ (3, 6, 8, 9, 10, 11, 12, 13, 14, 15).

F` = ∏ (0, 1, 2, 4, 5, 7)

Canonical Forms:

The sum of minterms and the product of maxterms forms of Boolean expressions are known as the canonical forms (الصيغ القانونية) of a function.

Standard Forms:

- A product term is a term with ANDed literals*. Thus, AB, A'B, A'CD are all product terms.
- A minterm is a special case of a product term where all input variables appear in the product term either in the true or complement form.
- A sum term is a term with ORed literals*. Thus, (A+B), (A'+B), (A'+C+D) are all sum terms.
- A maxterm is a special case of a sum term where all input variables, either in the true or complement form, are ORed together.
- Boolean functions can generally be expressed in the form of a Sum of Products (SOP) or in the form of a Product of Sums (POS).
- The sum of minterms form is a special case of the SOP form where all product terms are minterms.
- The product of maxterms form is a special case of the POS form where all sum terms are maxterms.
- The SOP and POS forms are Standard forms for representing Boolean functions.

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* *A Boolean variable in the true or complement forms*

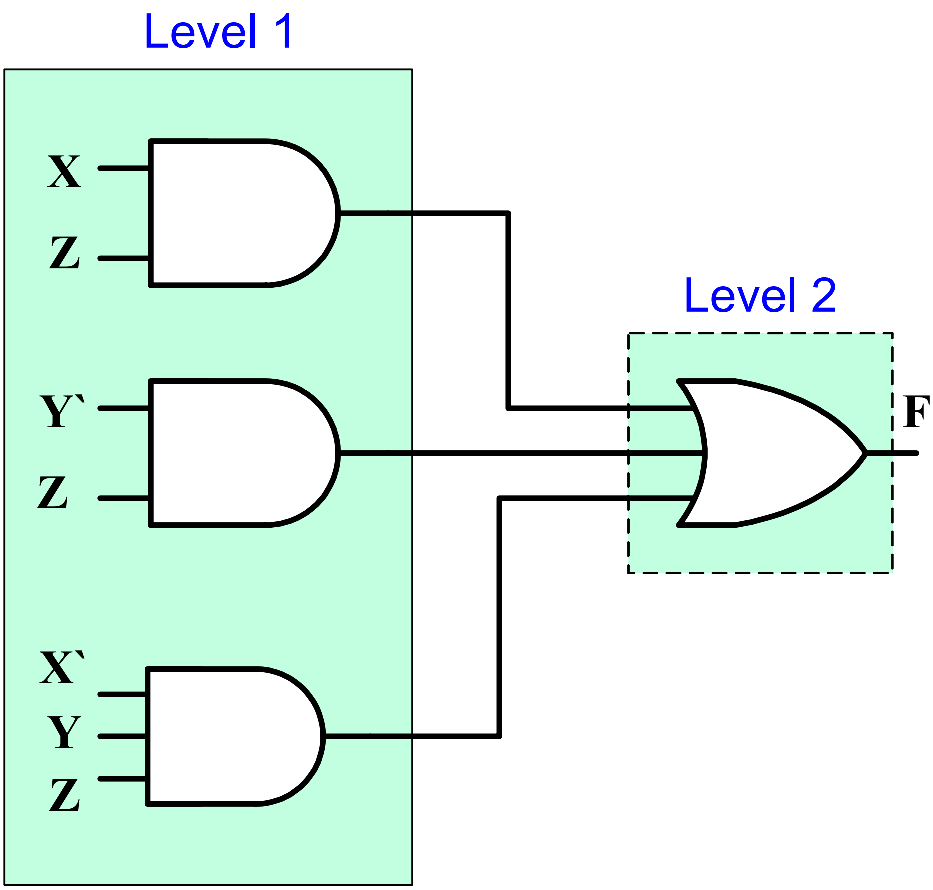
Two-Level Implementations of Standard Forms

Sum of Products Expression (SOP):

- Any SOP expression can be implemented in 2-levels of gates.
- The first level consists of a number of AND gates which equals the number of product terms in the expression. Each AND gate implements one of the product terms in the expression.
- The second level consists of a *SINGLE* OR gate whose number of inputs equals the number of product terms in the expression.

Example Implement the following SOP function

F = XZ + Y`Z + X`YZ



Two-Level Implementation (F = XZ + Y`Z + X`YZ)

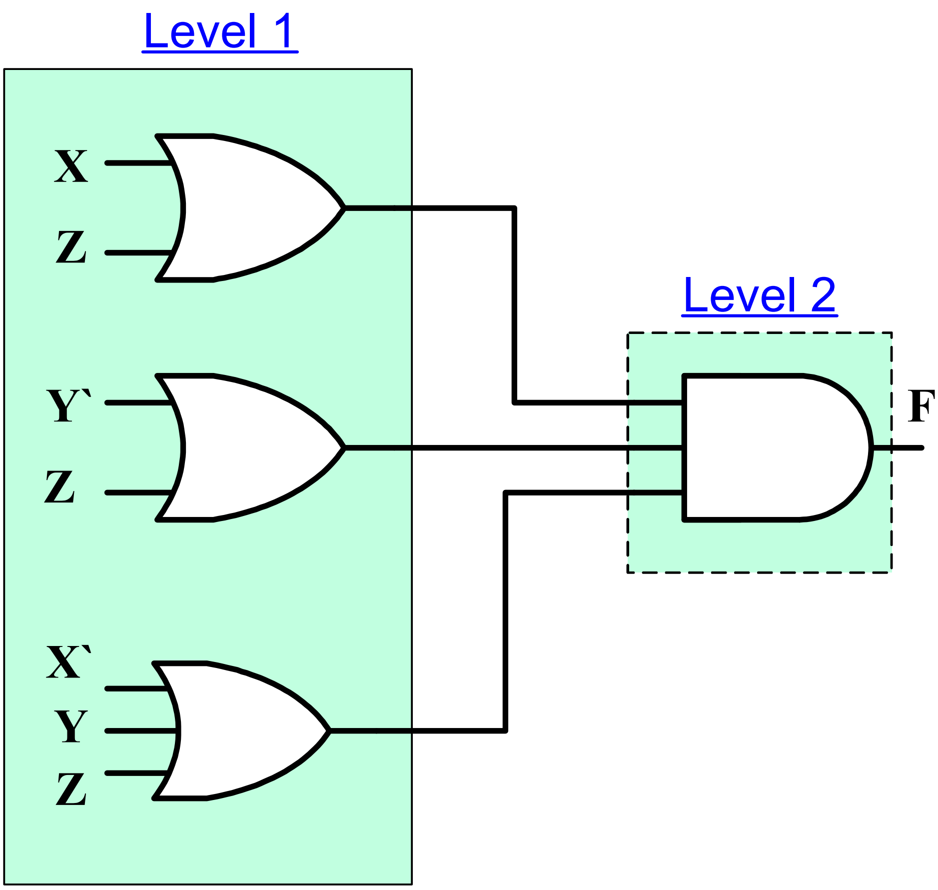
Level-1: AND-Gates ; **Level-2**: One OR-Gate

Product of Sums Expression (POS):

- Any POS expression can be implemented in 2-levels of gates
- The first level consists of a number of OR gates which equals the number of sum terms in the expression, each gate implements one of the sum terms in the expression.
- The second level consists of a *SINGLE* AND gate whose number of inputs equals the number of sum terms.

Example Implement the following SOP function

F = (X+Z)(Y`+Z)(X`+Y+Z)



Two-Level Implementation {F = (X+Z)(Y`+Z)(X`+Y+Z)}

Level-1: OR-Gates ; **Level-2**: One AND-Gate