HW #2

Q1 $f = (a \oplus b)c$

(i) $\frac{\partial f}{\partial a} = f_a \oplus f_{\bar{a}}$

$f_a = \bar{b}c$ $\quad f_{\bar{a}} = bc$

$\Rightarrow \frac{\partial f}{\partial a} = \bar{b}c \oplus bc = \bar{b}c + bc = c$

$C_a(f) = f_a \cdot f_{\bar{a}} = \bar{b}c \cdot bc = 0$

$S_a(f) = f_a + f_{\bar{a}} = \bar{b}c + bc = c$

(ii) Orthonormal basis: $\phi_1 = a$, $\phi_2 = \bar{a}b$, $\phi_3 = \bar{a}\bar{b}$

$f \cdot \phi \le f_\phi \le f + \bar{\phi}$

f_{ϕ_1}: $(a \oplus b)c \cdot a \le f_{\phi_1} \le (a \oplus b)c + \bar{a}$

$(a\bar{b} + \bar{a}b)c \cdot a \le f_{\phi_1} \le a\bar{b}c + \bar{a}bc + \bar{a}$

$a\bar{b}c \le f_{\phi_1} \le a\bar{b}c + \bar{a}$

$a\bar{b}c \le f_{\phi_1} \le \bar{b}c + \bar{a}$

Any function contained within this bound is a valid cofactor. We will choose the cofactor obtained by substituting $a=1$ in f i.e. $f_{\phi_1} = \bar{b}c$.

$f_{\phi_2} = f_{a=0, b=1} = c$

$f_{\phi_3} = f_{a=0, b=0} = 0$

$f = a\,[\bar{b}c] + \bar{a}b\,[c] + \bar{a}\bar{b}\,[0]$

(iii) Orthonormal basis: $\phi_1 = a + b$, $\phi_2 = \bar{a}\bar{b}$

$(a \oplus b)c \cdot (a+b) \subseteq f_{\phi_1} \subseteq (a \oplus b)c + \bar{a}\bar{b}$

$(\bar{a}(bc) + a(\bar{b}c))(\bar{a}(b) + a(1)) \subseteq f_{\phi_1} \subseteq a\bar{b}c + \bar{a}bc + \bar{a}\bar{b}$

$\bar{a}(bc) + a(\bar{b}c) \subseteq f_{\phi_1} \subseteq \bar{a}\bar{b} + \bar{b}c + \bar{a}c$

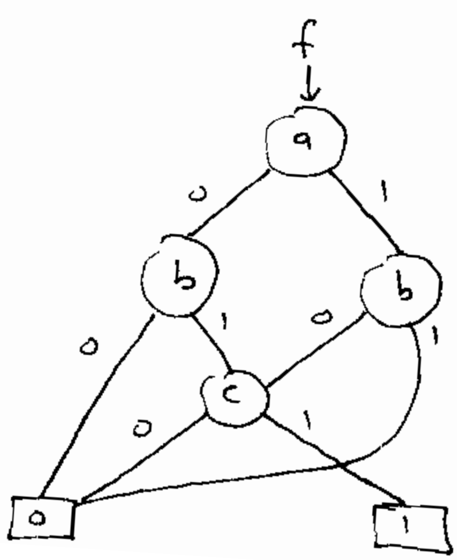
Let $f_{\phi_1} = \bar{a}bc + a\bar{b}c = (a \oplus b)c$

$f_{\phi_2} = f_{a=0, b=0} = 0$

$\Rightarrow f = (a+b)(a \oplus b)c + \bar{a}\bar{b}(0)$

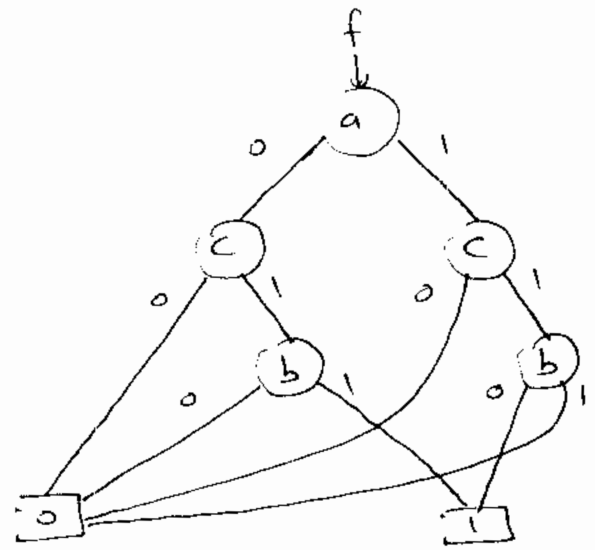
(iv) ROBDD with variable order $\{a, b, c\}$

$f_{\bar{a}} = bc$



$f_a = \bar{b}c$

(v) ROBDD with variable order $\{a, c, b\}$



Q2 $f = (a \oplus b)\, c \quad , \quad g = abc + \bar{a}\bar{b}\bar{c}$

(i) $f \cdot g \; , \; f + g \; , \; f \oplus g$

To simplify computation, we expand both functions using the orthonormal basis

$\phi_1 = \bar{a}\bar{b} \quad \phi_2 = \bar{a}b \; , \; \phi_3 = a\bar{b} \; , \; \phi_4 = ab$

$$f = \bar{a}\bar{b}\,(0) + \bar{a}b\,(c) + a\bar{b}\,(c) + ab\,(0)$$

$$g = \bar{a}\bar{b}\,(\bar{c}) + \bar{a}b\,(0) + a\bar{b}\,(0) + ab\,(c)$$

$$\Rightarrow f \cdot g = \bar{a}\bar{b}\,(0) + \bar{a}b\,(0) + a\bar{b}\,(0) + ab\,(0) = 0$$

$$f + g = \bar{a}\bar{b}\,(\bar{c}) + \bar{a}b(c) + a\bar{b}\,(c) + ab(c)$$

$$= \bar{a}\bar{b}\bar{c} + ac + bc$$

$$f \oplus g = \bar{a}\bar{b}\,(\bar{c}) + \bar{a}b(c) + a\bar{b}(c) + ab\,(c)$$

$$= \bar{a}\bar{b}\bar{c} + ac + bc$$

(ii)

$* \; f \cdot g = ite(f, g, 0) = ite((a \oplus b)c, \; abc + \bar{a}\bar{b}\bar{c}, 0)$

we assume the order $\{a, b, c\}$

- $x = a$

$t = ITE(f_a, g_a, 0) = ITE(\bar{b}c, bc, 0)$

$e = ITE(f_{\bar{a}}, g_{\bar{a}}, 0) = ITE(bc, \bar{b}\bar{c}, 0)$

- $ITE(\bar{b}c, bc, 0)$

$x = b$

$t = ITE(0, c, 0) \Rightarrow$ trivial case $= 0$.

Let us assume that the identifier for 0 is 1. So, $t = 1$

$e = ITE(c, 0, 0) \Rightarrow$ trivial case $= 0$

$\Rightarrow e = 1$.

Since $t = e$, the identifier 1 will be returned for $ITE(\bar{b}c, bc, 0)$

- $ITE(bc, \bar{b}\bar{c}, 0)$

$x = b$

$t = ITE(c, 0, 0) \Rightarrow$ trivial case $= 0$

$\Rightarrow t = 1$

$e = ITE(0, \bar{c}, 0) \Rightarrow$ trivial case $= 0$

$\Rightarrow e = 1$

since $t = e$, the identifier 1 will be returned for $ITE(bc, \bar{b}\bar{c}, 0)$.

Since $t = e = 1$, the identifier 1 will be returned for $ITE((a \oplus b)c,\ abc + \bar{a}\bar{b}\bar{c},\ 0)$ and the constructed diagram will be

$$ITE((a \oplus b)c,\ abc + \bar{a}\bar{b}\bar{c},\ 0)$$
$$\downarrow$$
$$\boxed{0}$$

* $f + g = ITE(f, 1, g)$

$= ITE((a \oplus b)c,\ 1,\ abc + \bar{a}\bar{b}\bar{c})$

- $x = a$

$t = ITE(\bar{b}c,\ 1,\ bc)$

$e = ITE(bc,\ 1,\ \bar{b}\bar{c})$

• $ITE(\bar{b}c,\ 1,\ bc)$

$x = b$

$t = ITE(0, 1, c) \Rightarrow$ trivial case $= c$

Let us assume the identifier for 0 is 1, for 1 is 2, and for c is 3.

So, $t = 3$

$e = ITE(c, 1, 0) \Rightarrow$ trivial case $= c$

So, $e = 3$.

Since $t = e$, the identifier 3 will be returned for $ITE(\bar{b}c,\ 1,\ bc)$.

- ITE (bc, 1, $\bar{b}\bar{c}$)

  $x = b$

  t = ITE (c, 1, 0) $\Rightarrow$ trivial case $= c$

  So, $t = 3$

  e = ITE (0, 1, $\bar{c}$) $\Rightarrow$ trivial case $= \bar{c}$

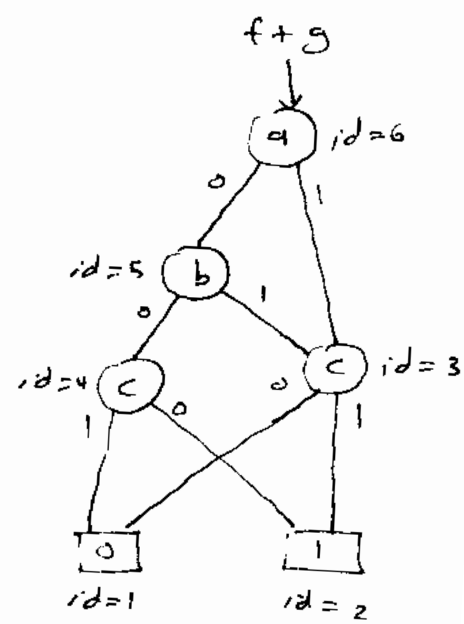
  Let us assume that the identifier for $\bar{c}$ is 4. So, $e = 4$

  Since $t \neq e$, a new identifier will be added in the table for ITE (bc, 1, $\bar{b}\bar{c}$), Say 5.

Since $t \neq e$, a new identifier will be added in the table for ITE ($(a \oplus b)c$, 1, $abc + \bar{a}\bar{b}\bar{c}$), Say 6.

The constructed unique table and the corresponding ITE DAG is shown below:

id	var	left child	right child
3	c	1	2
4	c	2	1
5	b	4	3
6	a	5	3



$* \; f \oplus g = ite(f, \bar{g}, g)$

$= ite\left((a \oplus b)c,\; a(\bar{b}+\bar{c}) + \bar{a}(b+c),\; abc + \bar{a}\bar{b}\bar{c} \right)$

- $x = a$

  $t = ITE(\bar{b}c,\; \bar{b}+\bar{c},\; bc)$

  $e = ITE(bc,\; b+c,\; \bar{b}\bar{c})$

- $ITE(\bar{b}c,\; \bar{b}+\bar{c},\; bc)$

  $x = b$

  $t = ITE(0, \bar{c}, c) \Rightarrow$ trivial case $= c$

  $\Rightarrow t = 3$

  $e = ITE(c, 1, 0) \Rightarrow$ trivial case $= c$

  $\Rightarrow e = 3$

  Since $t = e$, the identifier 3 will be returned for $ITE(\bar{b}c,\; \bar{b}+\bar{c},\; bc)$.

- $ITE(bc,\; b+c,\; \bar{b}\bar{c})$

  $x = b$

  $t = ITE(c, 1, 0) \Rightarrow$ trivial case $= c$

  $\Rightarrow t = 3$

  $e = ITE(0, c, \bar{c}) \Rightarrow$ trivial case $= \bar{c}$

  $\Rightarrow e = 4$

  Since $t \neq e$, the identifier 5 will be returned for $ITE(bc,\; b+c,\; \bar{b}\bar{c})$.

Since $t \neq e$, the identifier 6 will be returned for $ITE(f, \bar{g}, g)$. Note that it has the same table and ITE DAG as $ITE(f, 1, g)$ indicating the two functions

Q3

$$A = \begin{array}{c|cccccccc|} & c1 & c2 & c3 & c4 & c5 & c6 & c7 & c8 \\ \hline r1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ r2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ r3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ r4 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ r5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ r6 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ r7 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}$$

(i) Minimum cover using Exact_Cover procedure.

We can see that there are no essential columns.

Next, we check column dominance.

- C2 dominates C3, so we remove C3
- C1 dominates C5, so we remove C5
- C6 dominates C7, so we remove C7
- C6 dominates C8, so we remove C8
- C4 also dominates C7.

So, we remove columns C3, C5, C7, and C8 and the reduced matrix becomes

$$A = \begin{array}{|cccc|c} \overset{*}{c_1} & \overset{*}{c_2} & c_4 & \overset{*}{c_6} & \\ \hline 1 & 1 & 1 & 1 & r_1 \\ 0 & 1 & 1 & 0 & r_2 \\ 1 & 0 & 0 & 1 & r_3 \\ 0 & 0 & 1 & 1 & r_4 \\ 0 & 0 & 0 & 1 & r_5 \\ 0 & 1 & 0 & 0 & r_6 \\ 1 & 0 & 0 & 0 & r_7 \end{array}$$

We can see now that c_1, c_2, and c_6 become essential. So, we select them and remove them and all the rows covered by them. So, $x = (1,1,0,0,0,1,0,0)$. We can see that the matrix will become empty and the minimum solution returned will be $x = (1,1,0,0,0,1,0,0)$.

(ii) <u>Satisfiability Formulation:</u>

Let x_i be the variable associated with column i.

$$\Rightarrow (x_1 + x_2 + x_4 + x_6)(x_2 + x_4)(x_1 + x_6)(x_4 + x_6 + x_7 + x_8)$$
$$(x_6 + x_8)(x_2 + x_3)(x_1 + x_5) = 1$$

To find all possible minimum solutions, we need to express the function as sum-of-products and then each product corresponds to a solution. We select the products with the minimum number of literals. This corresponds to the minimum solutions.

The sum-of-product representation for this function is

$$x_1 x_2 x_6 + x_1 x_2 x_8 + x_1 x_3 x_4 x_6 + x_1 x_3 x_4 x_8$$
$$+ x_2 x_5 x_6 + x_3 x_4 x_5 x_6$$

This indicates that we have three minimum solutions: $x_1 x_2 x_6$, $x_2 x_5 x_6$, $x_1 x_2 x_8$.