Number Systems

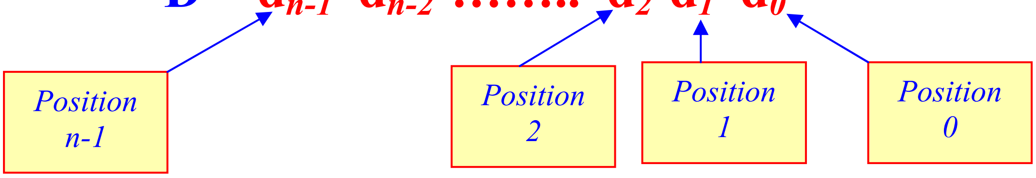
Introduction & Objectives:

- Before the inception of *digital* computers, the only number system that was in common use is the *decimal* number system (النظام العشري) which has a total of 10 digits (0 to 9).
- As discussed in the previous lesson, signals in *digital* computers may represent a digit in some number system. It was also found that the binary number system is more reliable to use compared to the more familiar decimal system
- In this lesson, you will learn:
  - What is meant by a weighted number system.
  - Basic features of weighted number systems.
  - Commonly used number systems, e.g. decimal, binary, octal and hexadecimal.
  - Important properties of these systems.

Weighted Number Systems:

- A number **D** consists of *n* digits with each digit has a particular *position.*

$$\mathbf{D} = \mathbf{d}_{n-1}\ \ \mathbf{d}_{n-2}\ \ldots\ldots\ \mathbf{d}_{2}\ \mathbf{d}_{1}\ \ \mathbf{d}_{0}$$



- Every digit *position* is associated with a *fixed weight*.
- If the weight associated with the $i^{th.}$ position is w_i, then the value of **D** is given by:

$$\mathbf{D} = \mathbf{d}_{n-1}\ \mathbf{w}_{n-1} + \mathbf{d}_{n-2}\ \mathbf{w}_{n-2} + \ldots + \mathbf{d}_{2}\ \mathbf{w}_{2} + \mathbf{d}_{1}\ \mathbf{w}_{1} + \mathbf{d}_{0}\ \mathbf{w}_{0}$$

Example of Weighted Number Systems:

- The Decimal number system (النظام العشري) is a weighted system.
- For Integer decimal numbers, the weight of the rightmost digit (*at position 0*) is **1**, the weight of *position 1* digit is **10**, that of *position 2* digit is **100**, *position 3* is **1000**, etc.

Thus,

$w_0 = 1$, $w_1 = 10$, $w_2 = 100$, $w_3 = 1000$, **etc.**

Example **Show** how the value of the decimal number **9375** is estimated

First Position Index ←

First Position Index (0) → 0

Position	3	2	1	0
Number	**9**	**3**	**7**	**5**
Weight	**1000**	**100**	**10**	**1**
Value	**9 x 1000**	**3x100**	**7x10**	**5x1**
Value	**9000 +**	**300 +**	**70 +**	**5**

The Radix (Base)

1. For *digit position i*, most weighted number systems use weights (**w_i**) that are ***powers of some constant valu***e called the **radix (r)** or the **base** such that $w_i = r^i$.
2. A number system of radix ***r***, typically has a set of ***r*** allowed digits ∈ {0,1, …,(r-1)} → *See the next example*
3. The leftmost digit has the highest weight → **Most Significant Digit (MSD)** → *See the next example*
4. The rightmost digit has the lowest weight → **Least Significant Digit (LSD)** → *See the next example*

Example **Decimal Number System**

1. **Radix (Base) = *Ten***
2. **Since $w_i = r^i$., then**
   - $w_0 = 10^0 = 1$,
   - $w_1 = 10^1 = 10$,
   - $w_2 = 10^2 = 100$,
   - $w_3 = 10^3 = 1000$, **etc.**
3. **Number of Allowed Digits is Ten →** {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

Thus: MSD LSD

$$9375 = 5\text{x}10^0 + 7\text{x}10^1 + 3\text{x}10^2 + 9\text{x}10^3$$

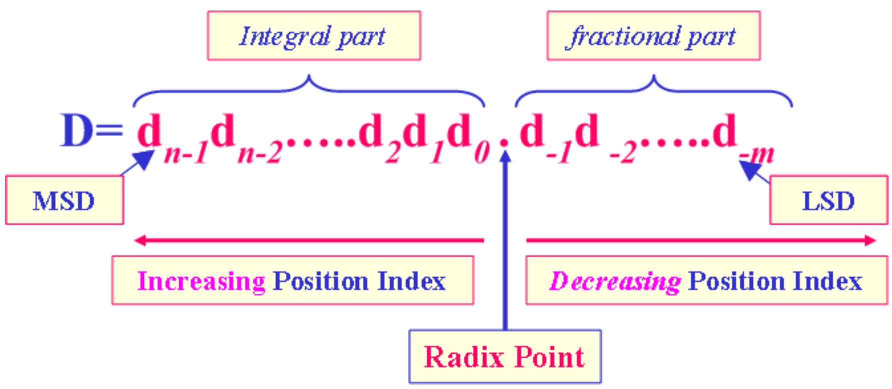
$$= 5\text{x}1 + 7\text{x}10 + 3\text{x}100 + 9\text{x}\ 1000$$

Position	3	2	1	0
Weight	1000 = 10^3	100 = 10^2	10 = 10^1	1 = 10^0

The Radix Point

Consider a number system of radix r,

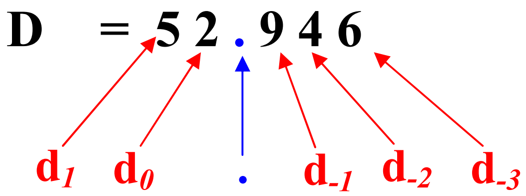
- A number D of *n* *integral* digits and *m* *fractional* digits is represented as shown



- Digits to the left of the radix point (*integral digits*) have positive position indices, while digits to the right of the radix point (*fractional digits*) have negative position indices
- Position *indices* of digits to the *left* of the ***radix point*** (*the integral part of **D***) start with a **0** and are incremented **as we move lefts ($d_{n-1}d_{n-2}\ldots d_2 d_1 d_0$.)**
- Position *indices* of digits to the *right* of the ***radix point*** (*the fractional part of **D***) are *negative* starting with **–1** and are decremented **as we move rights ($d_{-1}d_{-2}\ldots d_{-m}$).**
- The *weight* associated with digit position *i* is given by $w_i = r^i$, *where i is the position index*
  - $\forall i = -m, -m+1, \ldots, -2, -1, 0, 1, \ldots\ldots, n-1$
- The Value of **D** is Computed as :

$$D = \sum_{i=-m}^{n-1} d_i r^i$$

Example Show how the value of the following decimal number is estimated



Number	5	2	.	9	4	6
Position	1	0	.	-1	-2	-3
Weight	10^1 = 10	10^0 = 1	.	10^{-1} = 0.1	10^{-2} = 0.01	10^{-3} = 0.001
Value	5 x 10	2 x 1	.	9 x 0.1	2 x 0.01	6 x 0.001
Value	50 + 2 + 0.9 +0.02 +0.006					

$$D = 5x10^1 + 2x10^0 + 9x10^{-1} + 4x10^{-2} + 6x10^{-3}$$

Notation

- **Let $(D)_r$** denotes a number **D** expressed in a number system of radix **r.**

Note: *In this notation, r will be expressed in decimal*

Example:

- $(29)_{10}$ Represents a decimal value of 29. The radix "10" here means ten.
- $(100)_{16}$ is a Hexadecimal number since r = "16" here means sixteen. This number is equivalent to a decimal value of 16^2.
- $(100)_2$ is a Binary number (radix =2, i.e. two) which is equivalent to a decimal value of $2^2 = 4$.

Important Number Systems

The Decimal System

- **r = 10 (*ten* → *Radix is not a Power of 2*)**
  - **Ten** Possible Digits {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

The Binary System

- **r = 2**
- **Two** Allowed Digits {**0**, **1**}
- A **B**inary Dig**it** is referred to as **Bit**
- The leftmost bit has the highest weight → **Most Significant Bit (MSB)**
- The rightmost bit has the lowest weight → **Least Significant Bit (LSB)**

Examples

Find the decimal value of the two Binary numbers $(101)_2$ *and* $(1.101)_2$

MSB LSB

- $(1\ 0\ 1)_2 = 1x2^0 + 0x2^1 + 1x2^2$
- $= 1x1 + 0x2 + 1x\ 4$
- $= (5)_{10}$

MSB LSB

- $(1\ .\ 1\ 0\ 1)_2 = 1x2^0 + 1x2^{-1} + 0x2^{-2} + 1x2^{-3}$
- $= 1 + 0.5 + 0.25 + 0.125$
- $= (1\ .\ 8\ 7\ 5)_{10}$

Octal System:

- **r** = **8** (***Eight** = 2^3*)
  - **Eight** Allowed Digits {0, 1, 2, 3, 4, 5, 6, 7}

Examples

Find the decimal value of the two Octal numbers $(375)_8$ *and* $(2.746)_8$

MSD LSD

$$(375)_8 = 5x8^0 + 7x8^1 + 3x8^2$$
$$= 5x1 + 7x8 + 3x64$$
$$= (253)_{10}$$

MSD LSD

$$(2.746)_8 = 2x8^0 + 7x8^{-1} + 4x8^{-2} + 6x8^{-3}$$
$$= (2.94921875)_{10}$$

Hexadecimal System:

- **r** = **16** (***Sixteen** = 2^4*)
- **Sixteen** Allowed Digits {*0-to-9 and A, B, C, D, E, F*}
  - *Where:* ***A = ten,*** ***B = Eleven,*** ***C = Twelve,*** ***D = Thirteen,*** ***E = Fourteen &*** ***F = Fifteen***.

- **Q:** Why is the digit following **9** assigned the character **A** and not "**10**"?
- **A:** What we need is a *single* digit whose value is *ten*, but "**10**" is actually *two digits* not *one*.
  - Thus, in Hexadecimal system the 2-digit number $\mathbf{(10)_{16}}$ actually represents a value of *sixteen* not *ten* {$\mathbf{(10)_{16} = 0x16^0 + 1x16^1 = (16)_{10}}$}.

Examples

Find the decimal value of the two Hexadecimal numbers $(9E1)_{16}$ *and* $(3B.C)_{16}$

MSD LSD

$$(9E1)_{16} = 1x16^0 + Ex16^1 + 9x16^2$$
$$= 1x1 + 14x16 + 9x256$$
$$= (2529)_{10}$$

MSD LSD

$$(3B.C)_{16} = Cx16^{-1} + Bx16^0 + 3x16^1$$
$$= 12x16^{-1} + 11x16^0 + 3x16$$
$$= (59.75)_{10}$$

Important Properties

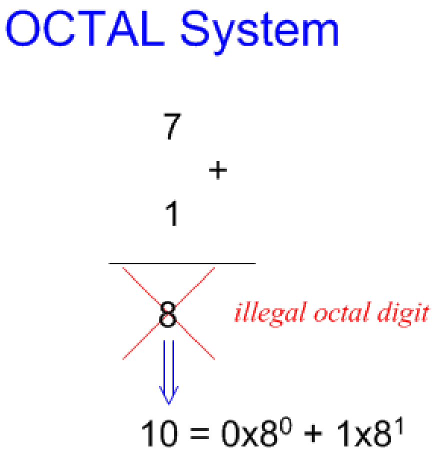
1. The number of possible digits in any number system with radix *r* equals *r*. (*Give examples in decimal, binary, octal and hexadecimal*)
2. The smallest digit is 0 and the largest possible digit has a value = (*r-1*)
3. The Largest value that can be expressed in *n integral* digits is $(\mathbf{r^n-1})$ → *Prove (Hint add 1 to the LSD position of the largest number)*
4. The Largest value that can be expressed in *m fractional* digits is $(\mathbf{1-r^{-m}})$ → *Prove* (*Hint add 1 to the LSD position of the largest number*)
5. The Largest value that can be expressed in *n integral* digits and *m fractional* digits is $(r^n - r^{-m})$ → *Prove* (*Hint- add results of properties 3 &4 above*)
6. Total number of values (patterns) representable in *n* digits is r^n

Clarification (a)

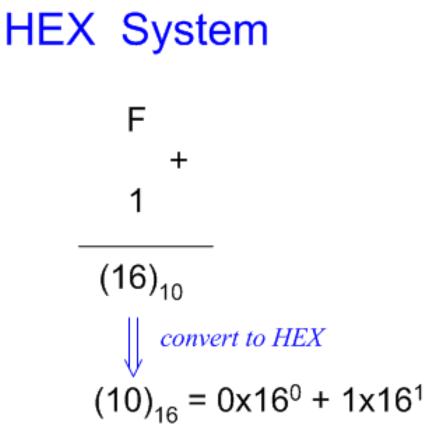
Q. What is the result of adding 1 to the largest digit of some number system??

A.

- For the decimal number system, $(1)_{10} + (9)_{10} = (10)_{10}$
- For the octal number system, $(1)_8 + (7)_8 = (10)_8 = (8)_{10}$



- For the hex number system, $(1)_{16} + (F)_{16} = (10)_{16} = (16)_{10}$



- For the binary number system, $(1)_2 + (1)_2 = (10)_2 = (2)_{10}$

Conclusion. Adding 1 to the largest digit in any number system always has a result of (10) in that number system.

- This is easy to prove since the largest digit in a number system of radix r has a value of (r-1). Adding 1 to this value the result is ***r*** *which is always equal to* $(10)_r = 0\text{x } r^0 + 1\text{x } r^1 = (r)_{10}$

Clarification (b)

Q. What is the largest value representable in 3-integral digits?

A. The largest value results when all 3 positions are filled with the largest digit in the number system.

- **For** the decimal system, it is $(999)_{10}$
- **For** the octal system, it is $(777)_8$
- **For** the hex system, it is $(FFF)_{16}$
- **For** the binary system, it is $(111)_2$

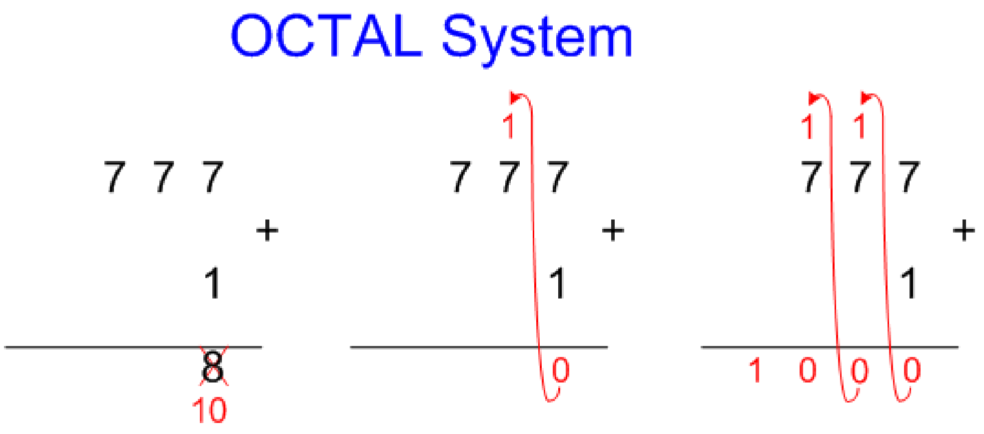
Clarification (c)

Q. What is the result of adding 1 to the largest 3-digit number?

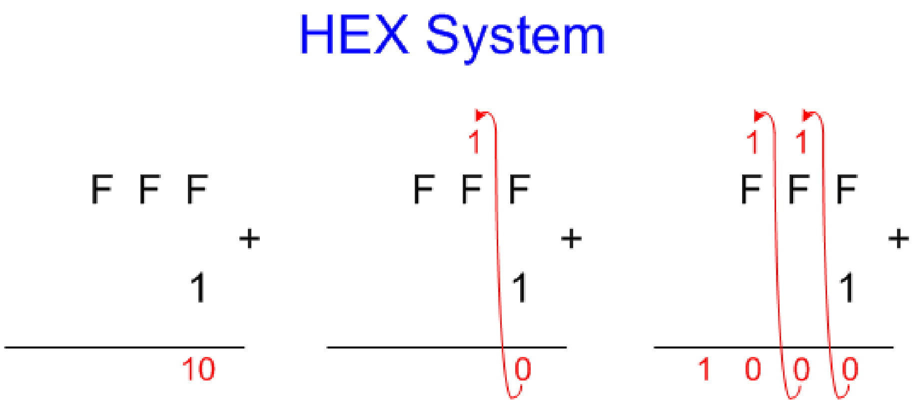
?

A.

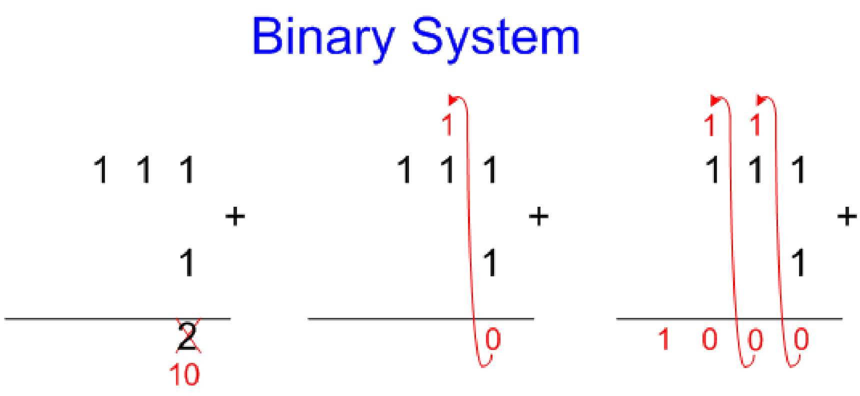
- **For** the decimal system, $(1)_{10} + (999)_{10} = (1000)_{10} = (\mathbf{10^3})_{10}$
- **For** the octal system, $(1)_8 + (777)_8 = (1000)_8 = (\mathbf{8^3})_{10}$



➢ **For** the hex system, $(1)_{16} + (FFF)_{16} = (1000)_{16} = (\mathbf{16^3})_{16}$



➢ **For** the binary system, $(1)_2 + (111)_2 = (1000)_2 = (\mathbf{2^3})_{10}$



In general, for a number system of radix **r**, adding 1 to the largest n-digit number = $\mathbf{r}^{n}$

Accordingly, the value of largest *n*-digit number = $\mathbf{r}^{n}$ -1

Conclusions.

1. In any number system of radix ***r***, the result of adding 1 to the *largest n-digit* number equals r^{n}.
2. Thus, the value of the *largest n-digit* number is equal to (r^{n} *-1*)
3. Thus, *n digits* can represent r^{n} different values (digit combinations) starting from a 0 value up to the largest value of r^{n} *-1*.

Appendix A. Summary of Number Systems Properties

The following table summarizes the basic features of the Decimal, Octal, Binary, and Hexadecimal number systems as well as a number system with a general radix ***r***

	Decimal 10	**Octal 8**	**Binary 2**	**Hexadecimal 16**	**General r**
Allowed Digits	{0-9}	{0-7}	{0-1}	{0-9, A-F}	{ 0 - R} where R = (r-1)
Value of $a_{n-1} \ldots a_2\, a_1\, a_0 \bullet a_{-1} a_{-2} \ldots a_{-m}$	$a_{n-1}x10^{n-1} + a_{n-2}x10^{n-2} + \ldots + a_2x10^2 + a_1x10^1 + a_0 \times 10^0 + a_{-1} \times 10^{-1} + a_{-2} \times 10^{-2} + \ldots + a_{-m}x10^{-m}$ $a_i \in$ {0-9} i=-m,…., 0, 1, …, n-1	$a_{n-1}8^{n-1}+\ldots+ a_2 8^2+a_1 8^1+a_0 8^0+a_{-1}8^{-1}+ a_{-2}8^{-2}+\ldots+a_{-m}8^{-m}$ $a_i \in$ {0-7}	$a_{n-1}2^{n-1}+\ldots+ a_2 2^2+a_1 2^1+a_0 2^0+a_{-1}2^{-1}+ a_{-2}2^{-2}+\ldots+a_{-m}2^{-m}$ $a_i \in$ {0,1}		$a_{n-1}r^{n-1}+\ldots+ a_2 r^2 + a_1 r^1 + a_0 r^0 + a_{-1}r^{-1} + a_{-2}r^{-2} +\ldots+a_{-m}r^{-m}$ $a_i \in$ {0 - (r-1)}
Smallest n-digit number	000…..0	000…..0	000…..0	000…..0	000…..0
Largest n-digit number	999……9 = $10^n - 1$	77…..7 = $8^n - 1$	11…..1 = $2^n - 1$	FF…….F = $16^n - 1$	RR…..R = $r^n - 1$
Range of n-digit integers	0 - (10^n-1)	0- (8^n-1)	0- (2^n-1)	0- $(16^n -1)$	0 – $(r^n - 1)$
# of Possible Combinations of n-digits	10^n	8^n	2^n	16^n	r^n
Max Value of m Fractional Digits	$1-10^{-m}$	$1-8^{-m}$	$1-2^{-m}$	$1-16^{-m}$	$1-r^{-m}$

Appendix B. First 16 Binary Numbers & Their Decimal Equivalent
(*All Possible Binary Combinations in 4-Bits*)

Decimal	Bin. Equivelent	Decimal	Bin. Equivelent
0	0000	8	1000
1	0001	9	1001
2	0010	10	1010
3	0011	11	1011
4	0100	12	1100
5	0101	13	1101
6	0110	14	1110
7	0111	15	1111

Appendix C. Decimal Values of the First 10 Powers of 2

- One Kilo is defined as 1000.
- For example, one Kilogram is 1000 grams. A kilometer is 1000 meters.
- In the Binary system, the power of 2 value closest to 1000 is 2^{10} which equals 1024. This is referred to as one Kilo (or in short 1K) in binary systems.
- Thus, one Kilo (or 1K) in Binary systems is not exactly 1000 but rather equals 1024 or 2^{10}
- Thus, in binary systems 2K= 2 x 1024 = 2048, 4K=4 x 1024= 4096, and so on
- Similarly, a one Meg (one million) in binary systems is 2^{20} which equals 1,048,576.

Powers of 2	Decimal. Value
2^0	1
2^1	2
2^2	4
2^3	8
2^4	16
2^5	32
2^6	64
2^7	128
2^8	256
2^9	512
2^{10}	1024