CH 11 The Energy Equation

Consider a stationary control volume through which a fluid is moving. The general energy balance equation:

(1) {Rate of Accumulation. KE + Int. E} = (2) {Rate of Energy Addition by Convection. K.E. + Int. E} + (3) {Rate of Energy Addition by Molecular Transf. conduction}

+ (4) {Rate of Work done on system by molecular Mechanisims (i.e. stress + Pressure)} + (5) {Rate of work done on system by external forces (gravity)}.

before we proceed we introduce the Combined Energy Flux Vector, $\vec{e}$:

$$\vec{e}\left(\frac{J}{m^2\, s}\right) = \underbrace{\tfrac{1}{2}\rho v^2 \vec{v}}_{KE} + \underbrace{\rho \hat{U} \vec{v}}_{\text{Int. E.}} + \underbrace{p\vec{v}}_{\text{work due to Pressure}} + \underbrace{\underline{\underline{\tau}}\cdot\vec{v}}_{\text{work due to stress}} + \underbrace{\vec{q}}_{\text{conduction}}$$

note: $v^2 = \vec{v}\cdot\vec{v} = v_x^2 + v_y^2 + v_z^2$

add all terms :

$$\Delta x \Delta y \Delta z \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right) = \left(e_x|_x - e_x|_{x+\Delta x}\right)\Delta y \Delta z + \left(e_y|_y - e_y|_{y+\Delta y}\right)\Delta x \Delta z$$

$$+ \left(e_z|_z - e_z|_{z+\Delta z}\right)\Delta x \Delta y$$

$$+ \rho \, \Delta x \Delta y \Delta z \left(\vec{V} \cdot \vec{g}\right)$$

$\div$ by $\Delta x \Delta y \Delta z$ and $\Delta x \Delta y \Delta z \to 0$

$\therefore$

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right)}_{1} = \underbrace{-\vec{\nabla} \cdot \vec{e}}_{2+3+4} + \underbrace{\rho\left(\vec{g} \cdot \vec{V}\right)}_{5} \quad \left(\frac{J}{m^3 \, s}\right)$$

recall definition of $\vec{e}$ (energy flux vector)

$\therefore$

$$\boxed{\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right) = \; & -\vec{\nabla} \cdot \left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right)\vec{V} && \text{---- Convection} \\ & -\vec{\nabla} \cdot \vec{q} && \text{---- Conduction} \\ & -\vec{\nabla} \cdot p\vec{V} && \text{---- Work due to pressure} \\ & -\vec{\nabla} \cdot \left[\underline{\underline{\tau}} \cdot \vec{V}\right] && \text{---- Work due to Friction} \\ & +\rho\left(\vec{V} \cdot \vec{g}\right) && \text{---- work by external forces} \end{aligned}}$$

..... Eq. 1

Recall the equation of motion:

$$\frac{\partial}{\partial t}\rho\vec{v} = -\vec{\nabla}\cdot\rho vv - \vec{\nabla}P - \vec{\nabla}\cdot\underline{\underline{\tau}} + \rho g \qquad \left(\frac{N}{m^3}\right)$$

take dot product with $\vec{v}$

$$\vdots$$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\vec{\nabla}\cdot\frac{1}{2}\rho v^2\vec{v} - \vec{\nabla}\cdot P\vec{v} - P\left(-\vec{\nabla}\cdot\vec{v}\right) - \vec{\nabla}\cdot\left(\underline{\underline{\tau}}\cdot v\right) - \left(-\tau:\vec{\nabla}\vec{v}\right) + \rho\left(\vec{v}\cdot\vec{g}\right)} \qquad \text{Eq. } 2$$

$$\left(\frac{J}{m^3\ s}\right)$$

Eq. 1 Combines both mechanical energy and internal energy

Eq. 2 Mechanical Energy equation

Eq. 1 - Eq. 2:

$$\boxed{\frac{\partial}{\partial t}\left(\rho\hat{U}\right) = -\vec{\nabla}\cdot\rho\hat{U}\vec{v} - \vec{\nabla}\cdot\vec{q} - P\left(\vec{\nabla}\cdot\vec{v}\right) - \underline{\underline{\tau}}:\underline{\underline{\nabla v}}}$$

Eq. 3

Equation 3 Internal Energy equation and it is useful to obtain energy equation in terms of temperature

Rearranging by utilizing substancial derivative (4)

⋮

$$\rho \frac{D\hat{U}}{Dt} = -\vec{\nabla}\cdot\vec{q} - p(\vec{\nabla}\cdot\vec{v}) - \underline{\underline{\tau}} : \underline{\underline{\nabla v}}$$

Enthalpy $\quad \hat{H} \doteq \hat{U} + p\hat{V}$

specific volume$\left(\frac{1}{\rho}\right)$

⋮

$$\rho \frac{D\hat{H}}{Dt} = -(\vec{\nabla}\cdot\vec{q}) - (\underline{\underline{\tau}} : \underline{\underline{\nabla v}}) + \frac{DP}{Dt}$$

recall from thermodynamics :

$$d\hat{H} = C_p\, dT + \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p\right] dp$$

$$\doteq C_p\, dT + \left[\frac{1}{\rho} + \left(\frac{\partial \ln\rho}{\partial \ln T}\right)_p\right] dp$$

$=1$ for ideal gas.

$$\Rightarrow \boxed{\rho C_p \frac{DT}{Dt} = -\vec{\nabla}\cdot\vec{q} - (\underline{\underline{\tau}} : \underline{\underline{\nabla v}}) - \left(\frac{\partial \ln\rho}{\partial \ln T}\right)_p \frac{DP}{Dt}}\ .$$

The Energy Equation.

Special Forms of Energy Equation:

(1) Constant density fluids

$$\rho C_p \frac{DT}{Dt} = -\nabla \cdot \vec{q} - \underline{\underline{\tau}} : \nabla \underline{V}$$

(2) Newtonial Fluids:

$$\underline{\underline{\tau}} : \nabla \underline{V} = \mu \Phi_v + \kappa \psi \quad , \quad \psi = (\nabla \cdot v)^2 .$$

see Appendix. B.7

$$\rho C_p \frac{DT}{Dt} = -\nabla \cdot \vec{q} - (\mu \Phi_v + \kappa \psi) - \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p \frac{DP}{Dt} .$$

(3) Materials following Fouriers law of conduction

$$q = -k \vec{\nabla} T$$

if k is const.

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T - (\mu \Phi_v + \kappa \psi) - \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p \frac{DP}{Dt} .$$

(4) Const. ρ, Newtonian, Fourier's law & Incompressible flow

$$\rho C_p \frac{DT}{Dt} = k \nabla^2 T - \mu \Phi_v .$$