Example: Laminar Flow along a Flat Plat (Exact Solution)

Obtain an exact solution of the velocity and stress near the wall of a semi-infinite flat plate with an approach velocity v_∞ .

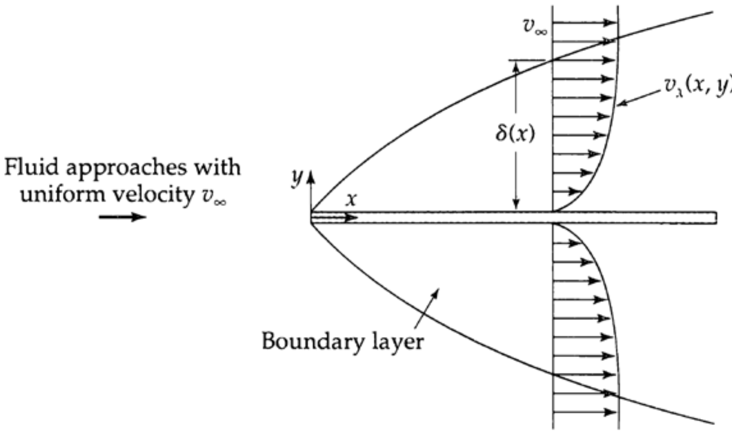

Fig. 4.4-2. Boundary-layer development near a flat plate of negligible thickness.

Solution:

For this problem the potential flow solution $v_e = v_\infty$ and $dv_e/dx = 0$. Therefore, the following simplified equations for the boundary layer flow over a flat plate:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

subject to the following boundary conditions:

$$\begin{aligned} y = 0: &\qquad v_x = 0 \\ y = 0: &\qquad v_y = 0 \\ y = \infty: &\qquad v_x = v_\infty \end{aligned}$$

The above PDE's can be reduced to a BVP using the similarity transformation method by introducing a new space variable, ζ, that combines x and y as follows:

$$\zeta \equiv y\sqrt{\frac{v_\infty}{\nu\, x}}$$

Also by introducing the stream function, ψ, such that:

$$v_x = -\frac{\partial \psi}{\partial y},\ v_y = \frac{\partial \psi}{\partial x}$$

which automatically satisfy the continuity equation.

Introducing the following separation of variables of the form:

$$\psi = g(x)\, f(\zeta).$$

one can show that:

$$v_x = -\frac{\partial \psi}{\partial y} = -\sqrt{\frac{v_\infty}{\nu\, x}}\, g(x) \frac{df}{d\zeta}$$
$$v_y = \frac{\partial \psi}{\partial x} = \frac{dg}{dx}\, f(\zeta) - \frac{1}{2x} \zeta\, g(x) \frac{df}{d\zeta}$$

Introducing $g(x) = \sqrt{\nu\, x\, v_\infty}$, one can show that the system of PDE's reduce to the following boundary value problem (BVP):

$$f \frac{d^2 f}{d\zeta^2} + 2 \frac{d^3 f}{d\zeta^3} = 0$$

Subject to the following boundary conditions:

$$\zeta = 0: \qquad f = 0$$
$$\zeta = 0: \qquad \frac{df}{d\zeta} = 0$$
$$\zeta = \infty: \qquad \frac{df}{d\zeta} = 1$$

Numerical Solution, the Shooting Method:

Introducing: $y_1 = f,\ y_2 = \frac{df}{d\zeta},\ y_3 = \frac{d^2 f}{d\zeta^2}$ one can show that the BVP can be represented by the following system initial value problems IVP's:

$$\frac{dy_1}{d\zeta} = y_2$$
$$\frac{dy_2}{d\zeta} = y_3$$
$$\frac{dy_3}{d\zeta} = -\frac{1}{2} y_1\, y_3$$

Subject to the following initial conditions:

$$\zeta = 0: \qquad y_1 = 0$$
$$\zeta = 0: \qquad y_2 = 0$$
$$\zeta = 0: \qquad y_3 = \text{unknown}$$

Moreover, by utilizing the boundary condition: $y = \infty:\ v_x = v_\infty$, we can show that:

$$\zeta = \infty: \qquad y_2 = 1$$

which can be utilized to evaluate the unknown initial condition for y_3.

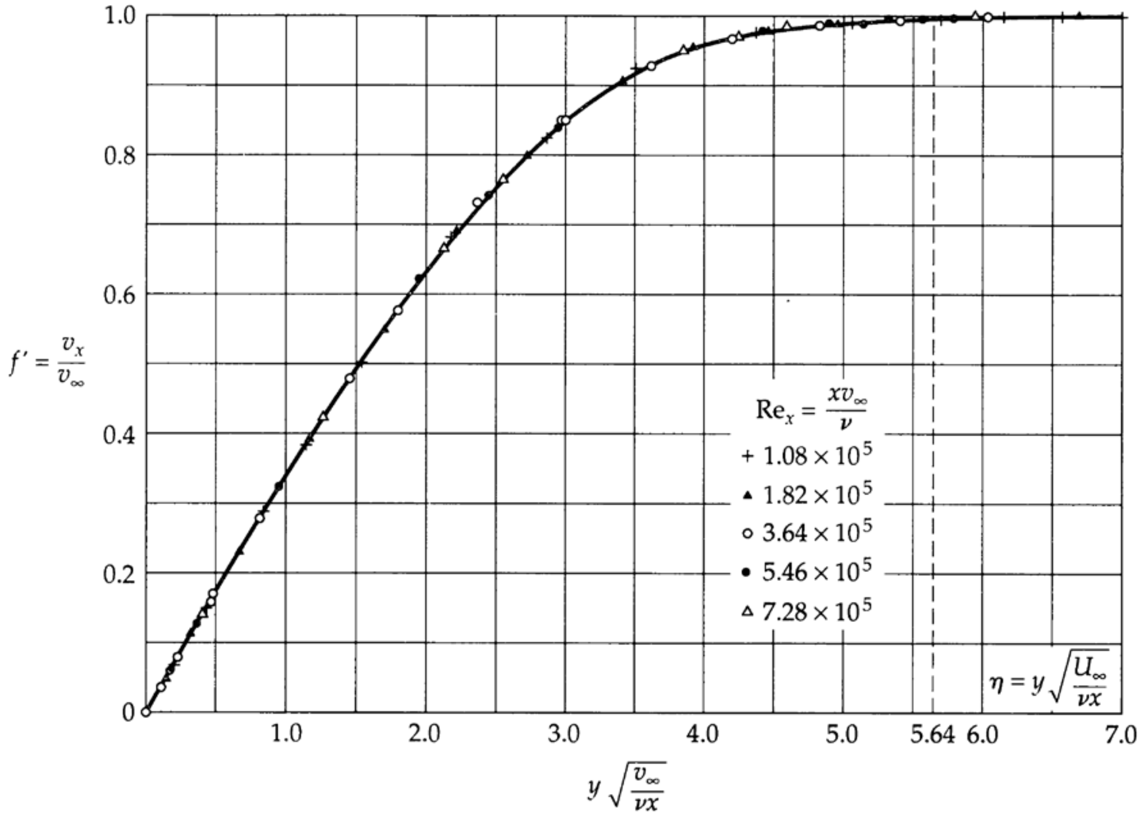

Fig. 4.4-3. Predicted and observed velocity profiles for tangential laminar flow along a flat plate. The solid line represents the solution of Eqs. 4.4-20 to 24, obtained by Blasius [see H. Schlichting, *Boundary-Layer Theory*, McGraw-Hill, New York, 7th edition (1979), p. 137].