Example 1: Potential Flow around a Cylinder

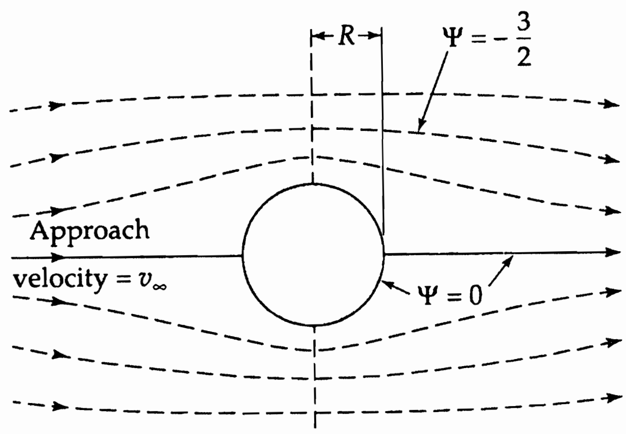
(a) Show that the complex potential

$$w(z) = -v_\infty R\left(\frac{z}{R} + \frac{R}{z}\right) \tag{4.3-16}$$

describes the potential flow around a circular cylinder of radius R, when the approach velocity is v_∞ in the positive x direction.

(b) Find the components of the velocity vector.

(c) Find the pressure distribution on the cylinder surface, when the modified pressure far from the cylinder is $\mathcal{P}_\infty$.



Solution:

(a) First we find the stream function from the complex potential: $w(z) = \phi(x,y) + i\psi(x,y)$

$$w(z) = -V_\infty R\left(\frac{x+iy}{R} + \frac{R}{x+iy} \times \frac{x-iy}{x-iy}\right)$$

conjugate of

$$= -V_\infty R\left(\frac{x+iy}{R} + \frac{(x-iy)R}{x^2+y^2}\right)$$

$$= -V_\infty R\left(\left[\frac{x}{R} + \frac{xR}{x^2+y^2}\right] + i\left[\frac{y}{R} - \frac{yR}{x^2+y^2}\right]\right)$$

$$= -V_\infty x\left(1 + \frac{R^2}{x^2+y^2}\right) - iV_\infty y\left(1 - \frac{R^2}{x^2+y^2}\right)$$

$$\Rightarrow \psi(x,y) = -V_\infty y\left(1 - \frac{R^2}{x^2+y^2}\right)$$

let $\Psi = \frac{\psi}{V_\infty R}$, $X = \frac{x}{R}$ & $Y = \frac{y}{R}$.

$$\Rightarrow \Psi(X,Y) = -Y\left(1 - \frac{1}{X^2+Y^2}\right)$$

(b) $\frac{dW}{dz} = -v_x + i v_y$

$$\frac{dW}{dz} = -V_\infty R\left(\frac{1}{R} - \frac{R}{z^2}\right)$$

$$= -V_\infty\left(1 - \frac{R^2}{z^2}\right) = -V_\infty\left(1 - \frac{R^2}{(x+iy)^2}\right)$$

recall $z = x+iy = r\cos(\theta) + ir\sin(\theta) = re^{i\theta}$.

$$\Rightarrow \frac{dW}{dz} = -V_\infty\left(1 - \frac{R^2}{r^2 e^{2i\theta}} \times \frac{e^{-2i\theta}}{e^{-2i\theta}}\right)$$

$$= -V_\infty\left(1 - \frac{R^2}{r^2} e^{-2i\theta}\right)$$

$$= -V_\infty\left[1 - \frac{R^2}{r^2}\left[\cos(2\theta) + i\sin(2\theta)\right]\right]$$

$\Rightarrow v_x = V_\infty\left(1 - \frac{R^2}{r^2}\cos(2\theta)\right)$, $v_y = -V_\infty \frac{R^2}{r^2}\sin(2\theta)$

also $v_x = -\partial\psi/\partial y$ $\quad v_y = \partial\psi/\partial x$

(c) At the cylinder surface $r = R$.

$$v^2 = v_x^2 + v_y^2 = v_\infty^2 \left[(1 - \cos 2\theta)^2 + (\sin(2\theta))^2 \right].$$

$=1$

$$= v_\infty^2 \left[1 + (\cos(2\theta))^2 - 2\cos(2\theta) + (\sin(2\theta))^2 \right]$$

$$= v_\infty^2 \left[2 - 2\cos(2\theta) \right]$$

recall, $\cos(a+b) = \cos a \cos b - \sin a \sin b$.

if $a = b$ $\Rightarrow \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

$$= (1 - \sin^2\theta) - \sin^2\theta.$$

$$= 1 - 2\sin^2\theta$$

$$\Rightarrow v^2 = v_\infty^2 \left[2 - 2(1 - 2\sin^2\theta) \right]$$

$$= \left[2 - 2 + 4\sin^2(\theta) \right) = 4\sin^2\theta.$$

$$\Rightarrow v^2 = 4 v_\infty^2 \sin^2(\theta)$$

recall $\frac{1}{2}\rho(v_x^2 + v_y^2) + P = \text{constant}$.

$$\Rightarrow \frac{1}{2}\rho v^2 + P = \text{constant}.$$

$$\Rightarrow \frac{1}{2}\rho\, 4 v_\infty^2 \sin^2(\theta) + P = \frac{1}{2}\rho v_\infty^2 + P_\infty.$$

$$\Rightarrow P - P_\infty = \frac{1}{2}\rho v_\infty^2 (1 - 4\sin^2\theta)$$

Example 3: Flow Near a Corner

Figure 4.3-3 shows the potential flow in the neighborhood of two walls that meet at a corner at O. The flow in the neighborhood of this corner can be described by the complex potential

$$w(z) = -cz^{\alpha} \tag{4.3-38}$$

in which c is a constant. We can now consider two situations: (i) an "interior corner flow," with $\alpha > 1$; and (ii) an "exterior corner flow," with $\alpha < 1$.

(a) Find the velocity components.

(b) Obtain the tangential velocity at both parts of the wall.

(c) Describe how to get the streamlines.

(d) How can this result be applied to the flow around a wedge?

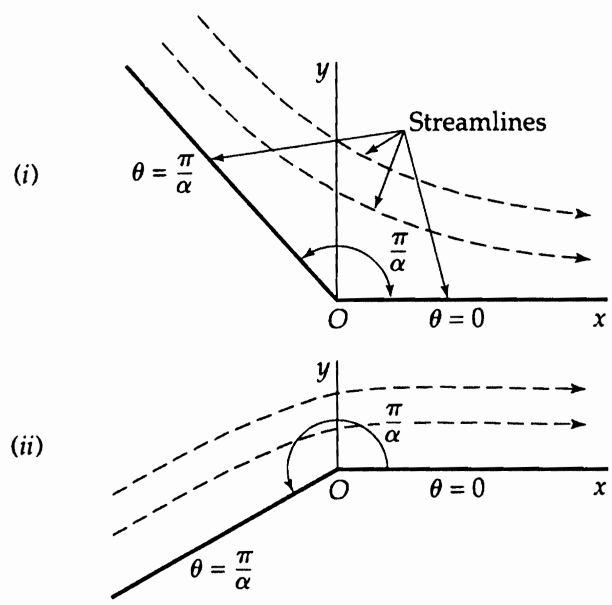


Fig. 4.3-3. Potential flow near a corner. On the left portion of the wall, $v_r = -cr^{\alpha-1}$, and on the right, $v_r = +cr^{\alpha-1}$. (i) Interior-corner flow, with $\alpha > 1$; and (ii) exterior-corner flow, with $\alpha < 1$.

Solution:

(a) recall $\frac{dw}{dz} = -v_x + i v_y$ $\qquad z = x + iy = re^{i\theta}$.

$$\frac{dw}{dz} = -c\alpha z^{\alpha-1}$$

$$= -c\alpha r^{\alpha-1} e^{i(\alpha-1)\theta}$$

$$\Rightarrow \quad v_x = +c\alpha r^{\alpha-1} \cos((\alpha-1)\theta)$$

$$v_y = -c\alpha r^{\alpha-1} \sin((\alpha-1)\theta)$$

(b) The tangential velocity V_r:

at $\theta = 0 \Rightarrow V_r = V_x = c\alpha r^{\alpha-1} = c\alpha x^{\alpha-1}$

$\theta = \frac{\pi}{\alpha} \Rightarrow V_r = V_x \cos(\theta) + V_y \sin\theta$

$$= c\alpha r^{\alpha-1}\cos((\alpha-1)\theta)\cos\theta - c\alpha r^{\alpha-1}\sin((\alpha-1)\theta)\sin\theta$$

$$= c\alpha r^{\alpha-1}\left[\cos((\alpha-1)\theta)\cos(\theta) - \sin((\alpha-1)\theta)\sin(\theta)\right]$$

$$\boxed{\begin{aligned}\sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b)\\ \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b)\end{aligned}}$$

$$= c\alpha r^{\alpha-1}\cos(\alpha\theta)$$

$$= c\alpha r^{\alpha-1}\cos\left(\alpha\frac{\pi}{\alpha}\right) \to -1$$

$$= -c\alpha r^{\alpha-1}$$

(c) recall, $w = \phi + i\psi$

$$= -cr^{\alpha}e^{i\alpha\theta}$$

$$= -cr^{\alpha}\left[\cos(\alpha\theta) + i\sin(\alpha\theta)\right]$$

$\Rightarrow \psi = -cr^{\alpha}\sin(\alpha\theta)$

$\phi = -cr^{\alpha}\cos(\alpha\theta)$