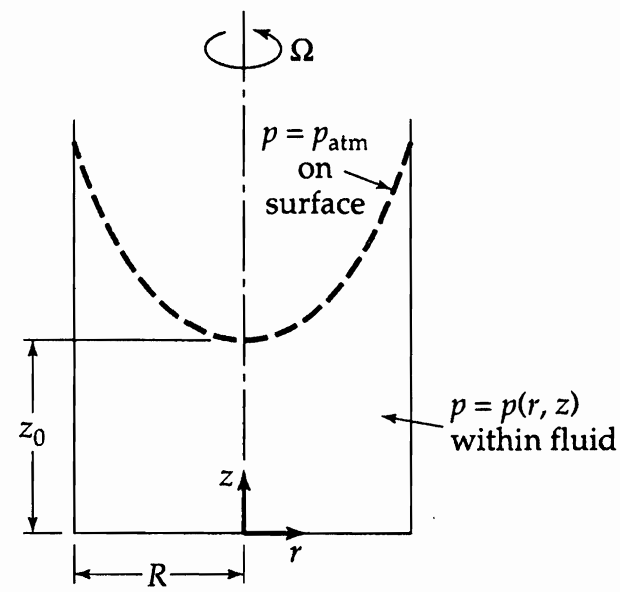
Use of the Equations of Change
To Solve Steady State Problems

Example 1: Shape of the surface of a rotating fluid

A liquid of constant density and viscosity is in a cylindrical container of radius R as shown in Fig. 3.6-6. The container is caused to rotate about its own axis at an angular velocity Ω. The cylinder axis is vertical, so that $g_r = 0$, $g_\theta = 0$, and $g_z = -g$, in which g is the magnitude of the gravitational acceleration. Find the shape of the free surface of the liquid when steady state has been established.



Solution:

The following assumptions are applicable

1. steady state
2. Laminar Incompressible flow
3. Newtonian fluid
4. Axisymmetric flow $\frac{\partial}{\partial \theta} = 0$.
5. Single velocity v_θ and $v_r = v_z = 0$.

Continuity eq. $\vec{\nabla}\cdot\vec{v} = 0 = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$ ②

--- A.7-2 (A)

since $v_r = v_z = 0 \Rightarrow \frac{\partial v_\theta}{\partial \theta} = 0$

Navier Stokes Eq. $v_r = v_z = 0$, $\frac{\partial}{\partial \theta} = 0$, z-direction neutral $\frac{\partial}{\partial z} = 0$. $g_\theta = g_r = 0$,

B.6-4 $-\rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}$ ------ ①

B.6-5 $0 = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)\right]$ -- ②

B.6-6 $0 = \frac{\partial p}{\partial z} + \rho g_z$ ---- ③

solving ②: $\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = C_1$

$r v_\theta = \frac{1}{2}C_1 r^2 + C_2$.

$v_\theta = C_1' r + \frac{C_2}{r}$.

Boundary Conditions:

$r = 0$ $v_\theta = 0$. $\Rightarrow C_2 = 0$.

$r = R$ $v_\theta = \Omega R \Rightarrow C_1 = \Omega$

$\Rightarrow \boxed{v_\theta = \Omega r}$

Solving (3) : $\frac{\partial p}{\partial z} = \rho g \quad (g_z = -g)$ (3)

integrating $p = \rho g z + f_1(r)$ ---- (4)

$\uparrow$

the const. of integration could be $f(r)$

solving (1) : $\frac{\partial p}{\partial r} = \rho \Omega^2 r$.

integrating $p = \frac{1}{2} \rho \Omega^2 r^2 + f_2(z)$ ---- (5)

by comparing the results in (4) and (5)

we can select $f_1 = \frac{1}{2} \rho \Omega^2 r^2 + C$

$f_2 = \rho g z + C$

where C is a constant satisfying the equations (1) and (3).

$\Rightarrow p = -\rho g z + \frac{1}{2} \Omega^2 r^2 + C$

boundary condition :

$r = 0 \quad \& \quad z = z_0 \quad p = P_{atm}$.

$\Rightarrow C = \rho g z_0 + P_{atm}$.

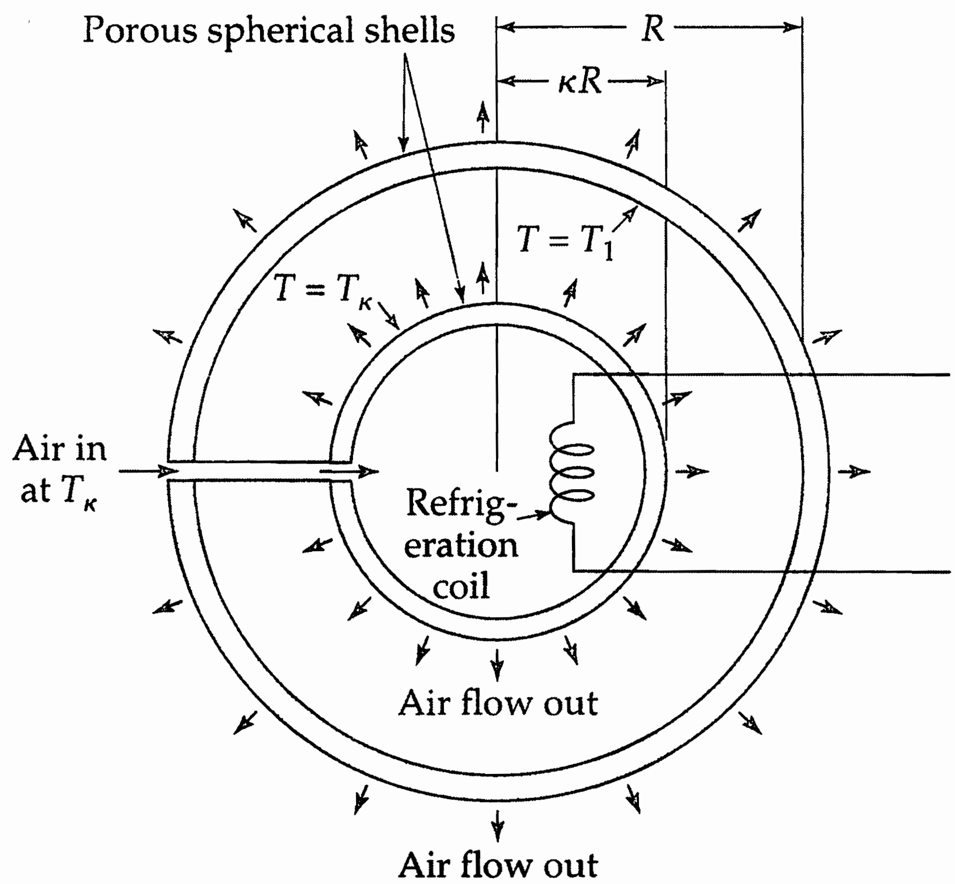
$$\Rightarrow \quad P - P_{atm} = -\rho g (z - z_0) + \frac{1}{2} \rho \Omega^2 r^2 .$$

The above equation provides the pressure at all points in the liquid. The shape of the fluid surface can be obtained by setting $P = P_{atm}$:

$$z - z_0 = \frac{\Omega^2}{2g} r^2$$

Example 2: Transpiration Cooling

A system with two concentric porous spherical shells of radii κR and R is shown in Fig. 11.4-1. The inner surface of the outer shell is at temperature T_1, and the outer surface of the inner shell is at a lower temperature T_κ. Dry air at T_κ is blown outward radially from the inner shell into the intervening space and then through the outer shell. Develop an expression for the required rate of heat removal from the inner sphere as a function of the mass rate of flow of the gas. Assume steady laminar flow and low gas velocity.



Solution:

The following assumptions are applicable

1. steady state, laminar and incompressible flow

2- Newtonian fluid.

3- Single velocity $V_r = f(r)$, $V_\theta = V_\phi = 0$.

4- Constant gas thermal conductivity and no viscous dissipation.

$\underline{\text{Continuity Eq}}$ $\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = 0$ ⑥

$$\Rightarrow \quad r^2 v_r = \text{Constant}$$

mass flow rate $w_r = 4\pi r^2 \rho v_r$

$\underline{\text{Navier Stokes Eq.}}$

$$\rho v_r \frac{dv_r}{dr} = -\frac{dP}{dr} + \mu \cancel{\left(\frac{1}{r^2}\frac{d^2}{dr^2}(r^2 v_r)\right)} \quad (B.6-7)$$

0

$$\boxed{v_r = \frac{w_r}{\rho 4\pi r^2}} \quad \Rightarrow \quad \frac{dv_r}{dr} = -\frac{2w_r}{\rho 4\pi r^3}$$

$$\Rightarrow \frac{dP}{dr} = \rho \frac{w_r}{\rho 4\pi r^2}\frac{2w_r}{\rho 4\pi r^3}$$

integrating $\quad dP = \frac{w_r^2}{\pi^2 \rho} \int_r^R \frac{1}{r^5}\,dr$

$$P\Big|_r^R = \frac{w_r^2}{8\pi^2\rho}\left[-\frac{1}{4}\frac{1}{r^4}\right]_r^R$$

$$P(R) - P(r) = \frac{w_r^2}{32\pi^2\rho}\left[\frac{1}{r^4} - \frac{1}{R^4}\right]$$

$$\boxed{P(r) - P(R) = \frac{w_r^2}{32\pi^2\rho R^4}\left[1 - \left(\frac{R}{r}\right)^4\right]}$$

Energy Equation:

$$\rho \hat{C}_p \, v_r \frac{dT}{dr} = k\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)\right] \qquad (B.9-3)$$

re-arrange:

$$\frac{dT}{dr} = \frac{4\pi k}{w_r \hat{C}_p} \cdot \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right)$$

Let $u = r^2 \frac{dT}{dr}$.

$$\Rightarrow \frac{u}{r^2} = \frac{4\pi k}{w_r \hat{C}_p} \frac{du}{dr}.$$

$$\rightarrow \ln(u) = -\frac{w_r \hat{C}_p}{4\pi k} \frac{1}{r} + c_1$$

R_0.

$$\Rightarrow u = e^{-\frac{R_0}{r}} + c_1' = r^2 \frac{dT}{dr}.$$

$$\frac{dT}{dr} = \frac{1}{r^2} e^{-\frac{R_0}{r}} + \frac{c_1'}{r^2}.$$

$$T = \int \frac{1}{r^2} e^{-\frac{R_0}{r}} dr + \frac{c_1''}{r} + c_2$$

$$T = +\frac{1}{R_0} e^{-\frac{R_0}{r}} + \frac{c_1''}{r} + c_2.$$

$r = \varkappa R \qquad T = T_\varkappa$

$r = R \qquad T = T_1$

$\vdots$

$$\frac{T - T_1}{T_\varkappa - T_1} = \frac{e^{-\frac{R_0}{r}} - e^{-\frac{R_0}{R}}}{e^{-R_0/\varkappa R} - e^{-R_0/R}}.$$

Rate of heat transfer from inner sphere

$$Q = -4\pi(\varkappa R)^2 \, q_r\big|_{r=\varkappa r}.$$

$$= -4\pi \varkappa^2 R^2 \left(-k \frac{dT}{dr}\Big|_{r=\varkappa R}\right).$$

$\vdots$

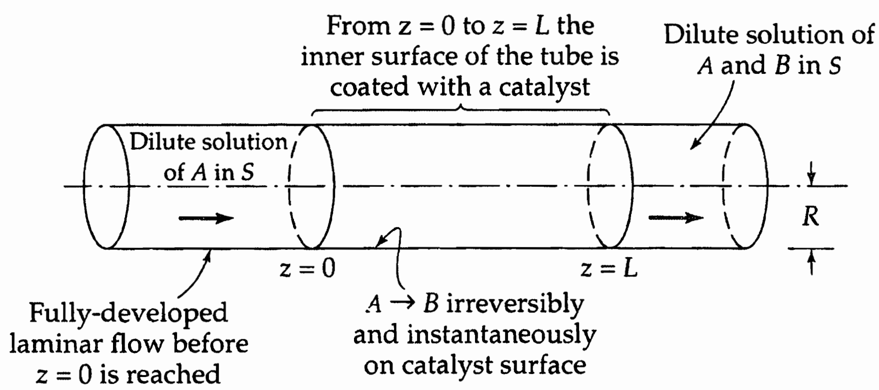
$$= \frac{4\pi R_0 k (T_1 - T_\varkappa)}{\exp\left[\frac{R_0}{\varkappa R}(1-\varkappa)\right] - 1}.$$

in the limit $\omega_r \to 0 \Rightarrow R_0 \to 0$.

$$\Rightarrow Q_0 = \frac{4\pi \varkappa R k (T_1 - T_\varkappa)}{1 - \varkappa}. \qquad \text{L-Hopital Rule.}$$

Example 3: Concentration Profile in a Tubular Reactor

A catalytic tubular reactor is shown in Fig. 19.4-2. A dilute solution of solute A in a solvent S is in fully developed, laminar flow in the region $z < 0$. When it encounters the catalytic wall in the region $0 \le z \le L$, solute A is instantaneously and irreversibly rearranged to an isomer B. Write the diffusion equation appropriate for this problem, and find the solution for short distances into the reactor. Assume that the flow is isothermal and neglect the presence of B.



Solution :

The following assumptions are applicable

1. steady state
2. laminar flow.
3. const. ρ D_{AS}.
4. const. c
5. negligible concentration $C_B \cong 0$

Recall, const. ρD_{AB}

$$\rho\left(\frac{\partial w_A}{\partial t} + \vec{v}\cdot\vec{\nabla} w_A\right) = \rho D_{AS}\,\nabla^2 w_A + r_A \quad (19.1-16)$$

$\div$ by molecular weight of A. ρ = const.

$$\Rightarrow \frac{\partial c_A}{\partial t} + \vec{v}\cdot\vec{\nabla} c_A = D_{AS}\,\nabla^2 c_A + R_A.$$

simplifying:

$$v_z \frac{\partial c_A}{\partial z} = D_{AS}\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial c_A}{\partial r} + \frac{\partial^2 c_A}{\partial z^2}\right].$$

$$v_z = v_{z\,max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

$D_{AS}\dfrac{\partial^2 c_A}{\partial z^2} \simeq 0.$ negligible axial diffusion in z-direction convection is dominating.

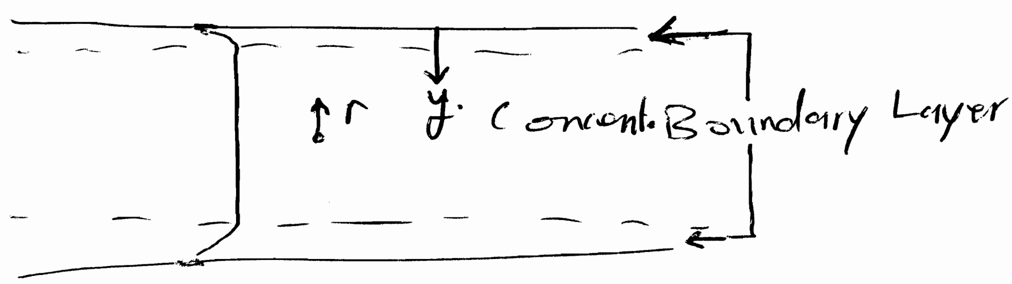
$$\Rightarrow v_{z\,max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial c_A}{\partial z} = D_{AS}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c_A}{\partial r}\right)$$

$z = 0 \qquad c_A = c_{A0}.$

$r = 0 \qquad \dfrac{\partial c_A}{\partial r} = 0.$

$r = R \qquad c_A = 0.$ all A is consumed at the surface.

This problem can be simplified further by assuming that the concentration of A changes in the vicinity of the tube wall. The concentration is more or less constant far a way from the wall.



we can change the variable : $y = R - r$

$$\Rightarrow \quad V_{z\,max}\left[1-\left(\frac{R-y}{R}\right)^2\right]\frac{\partial C_A}{\partial z} = D_{AS}\cdot\frac{\partial^2 C_A}{\partial y^2}$$

curvature is eleminated.

$\because \left(\frac{y}{R}\right)^2 \approx 0.$

$$2V_{z\,max}\,\frac{y}{R}\,\frac{\partial C_A}{\partial z} = D_{AS}\,\frac{\partial^2 C_A}{\partial y^2}.$$

$z = 0 \qquad C_A = C_{A0}$

$y = 0 \qquad C_A = 0$

$y = \infty \qquad C_A = C_{A0}.$

The above PDE with infinite domain $y \to \infty$ can be solve using the method of similarity transformation.

First we derive the similarity variable that combines both y and z in one parameter that changes the PDE to ODE 12

Scaling of the PDE:

$$2\, V_{z\,max}\, \frac{y}{R}\, \frac{\cancel{C_A}}{z} = D_{AS}\, \frac{\cancel{C_A}}{y^2}$$

$$\Rightarrow \quad \eta = \frac{y}{\left(\frac{R\, D_{AS}}{2\, V_{z\,max}}\, z\right)^{1/3}}$$

similarity variable.

$$\frac{\partial C_A}{\partial y} = \frac{1}{\left(\frac{R\, D_{AS}}{2 V_{z\,max}}\, z\right)^{1/3}} = \frac{\partial C_A}{\partial \eta}\frac{d\eta}{dy}, \qquad \frac{\partial^2 C_A}{\partial y^2} = \frac{\frac{d^2 C_A}{d\eta^2}}{\left(\frac{R\, D_{AS}}{2\, V_{z\,max}}\, z\right)^{2/3}}$$

$$\frac{\partial C_A}{\partial z} = \frac{d C_A}{d\eta}\frac{\partial \eta}{\partial z} = -\frac{1}{3}\, z^{-4/3}\, \frac{y}{\left(\frac{R\, D_{AS}}{2\, V_{z\,max}}\right)^{1/3}}\, \frac{dC_A}{d\eta}$$

$$= -\frac{1}{3}\,\frac{1}{z}\,\eta\, \frac{dC_A}{d\eta}$$

substitute in PDE:

$$-\frac{1}{3}\, V_{z\,max}\, \frac{y}{R}\, \frac{1}{z}\, \eta\, \frac{dC_A}{d\eta} = D_{AS}\, \frac{1}{\left(\frac{R\, D_{AS}}{2\, V_{z\,max}}\, z\right)^{2/3}}\, \frac{d^2 C_A}{d\eta^2}.$$

$$\frac{d^2 C_A}{d\eta^2} + \frac{1}{3}\, 2\, \frac{V_{z\,max}\, y}{R\;\; z\, D_{AS}} \cdot \left(\frac{R\, D_{AS}}{2\, V_{z\,max}} \cdot z\right)^{2/3} \eta\, \frac{dC_A}{d\eta} = 0.$$

$$\frac{d^2 C_A}{d\eta^2} + \frac{1}{3}\left(\frac{z^{1/3} V_{z\,max}^{1/3}\, y}{R^{1/3} z^{1/3} D_{AB}^{1/3}}\right)^2 \frac{dC_A}{d\eta} = 0.$$

$\downarrow$

η.

$$\frac{d^2 C_A}{d\eta^2} + \frac{1}{3}\eta^2 \frac{dC_A}{d\eta} = 0.$$

$\eta = 0 \qquad C_A = 0$

$\eta = \infty \qquad C_A = C_{A0} \qquad \leftarrow$ this represents BC 1 and 3 combined.

So similarity transformation converted PDE to ODE and combined 3 BC's into two BC's.

Formular C.1-9 can be useful.

$$\frac{d^2 y}{dx^2} + 3x^2 \frac{dy}{dx} = 0$$

$$y = c_1 \int_0^x e^{-\bar{x}^3}\, d\bar{x} + c_2$$

let $x = \dfrac{\eta}{(9)^{1/3}}$

$$\frac{d^2 C_A}{d\eta^2} = \frac{d^2 C_A}{dx^2}\left(\frac{dx}{d\eta}\right)^2 = \frac{1}{(9)^{2/3}}\frac{d^2 C_A}{dx^2}, \quad \frac{dC_A}{d\eta} = \frac{1}{(9)^{1/3}}\frac{dC_A}{dx}.$$

$$\frac{1}{(9)^{2/3}}\frac{d^2C_A}{dx^2} + \frac{1}{3}(9)^{2/3}x^2\frac{1}{(9)^{1/3}}\frac{dC_A}{dx} = 0.$$

$$\frac{d^2C_A}{dx^2} + 3x^2\frac{dC_A}{dx} = 0.$$

$$\Rightarrow \quad C_A = C_1\int_0^x e^{-\bar{x}^3}d\bar{x} + C_2.$$

BC1 $\quad x = 0 \quad C_A = 0. \quad \Rightarrow \boxed{C_2 = 0}$

BC2 $\quad x = \infty \quad C_A = C_{A0}.$

$$\Rightarrow \quad C_{A0} = C_1\int_0^\infty e^{-\bar{x}^3}d\bar{x}$$

$$\Rightarrow \quad \frac{C_A}{C_{A0}} = \frac{\int_0^x e^{-\bar{x}^3}d\bar{x}}{\int_0^\infty e^{-\bar{x}^3}d\bar{x}} = \frac{\int_0^x e^{-\bar{x}^3}d\bar{x}}{\Gamma\left(\frac{4}{3}\right)}$$

see C.4