Example 1: Laminar Flow of a Power Law Fluid in a Pipe

Derive the expression for the mass flow rate of a polymer liquid, described by the power law model. The fluid is flowing in a long circular tube of radius R and length L, as a result of a pressure difference, gravity, or both.

Solution:

For non-Newtonian incompressible pipe flow:

$\nabla \cdot V = 0$ / $V_r = V_\theta = 0$; $V_z = f(r)$, η not μ.

$$\tau_{rr} = \cancelto{0}{-\mu\left[2\frac{\partial v_r}{\partial r}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})} \qquad \text{(B.1-8)}^a$$

$$\tau_{\theta\theta} = \cancelto{0}{-\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})} \qquad \text{(B.1-9)}^a$$

$$\tau_{zz} = \cancelto{0}{-\mu\left[2\frac{\partial v_z}{\partial z}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})} \qquad \text{(B.1-10)}^a$$

$$\tau_{r\theta} = \tau_{\theta r} = \cancelto{0}{-\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]} \qquad \text{(B.1-11)}$$

$$\tau_{\theta z} = \tau_{z\theta} = \cancelto{0}{-\mu\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]} \qquad \text{(B.1-12)}$$

$$\tau_{zr} = \tau_{rz} = -\mu\left[\cancelto{0}{\frac{\partial v_r}{\partial z}} + \frac{\partial v_z}{\partial r}\right] \qquad \text{(B.1-13)}$$

$$\Rightarrow \quad \tau_{rz} = \eta \frac{dV_z}{dr}$$

$$\eta = m\,\dot{\gamma}^{\,n-1} \quad , \quad \dot{\gamma} = \sqrt{\tfrac{1}{2}\,\dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}}} = \underbrace{-\frac{dV_z}{dr}}_{+ve}$$

$$\Rightarrow \quad \eta = m\left(-\frac{dV_z}{dr}\right)^{n-1}$$

$$\Rightarrow \quad \tau_{rz} = -m\left(-\frac{dV_z}{dr}\right)^{n-1}\left(\frac{dV_z}{dr}\right)$$

$$0 = -\frac{dP}{dz} - \frac{1}{r}\frac{d}{dr}(r\,\tau_{rz}) \qquad (B.5-6)$$

$$\frac{dP}{dz} = \frac{P_L - P_0}{L}$$

$$\Rightarrow \frac{d}{dr}(r\,\tau_{rz}) = \left(\frac{P_0 - P_L}{L}\right) r$$

$$\tau_{rz} = \frac{P_0 - P_L}{2L}\, r + \cancel{\frac{C_1}{r}} \xrightarrow{} 0$$ τ_{rz} = finite @ $r=0$

$$\tau_{rz} = \left(\frac{P_0 - P_L}{2L}\right) r$$

$$+m\left(-\frac{dv_z}{dr}\right)^n = \left(\frac{P_0 - P_L}{2L}\right) r$$

$$\frac{dv_z}{dr} = -\left(\frac{P_0 - P_L}{2mL}\right)^{\frac{1}{n}} r^{\frac{1}{n}}$$

$$v_z = -\left(\frac{P_0 - P_L}{2mL}\right)^{\frac{1}{n}} \frac{1}{\frac{1}{n}+1}\, r^{\frac{1}{n}+1} + C_2$$

$v_z = 0 \qquad r = R$

$$\Rightarrow C_2 = \left(\frac{P_0 - P_L}{2mL}\right)^{\frac{1}{n}} \frac{R^{\frac{1}{n}+1}}{\frac{1}{n}+1}$$

$$v_z = \left(\frac{(P_0 - P_L)R}{2mL}\right)^{\frac{1}{n}} \frac{R}{\frac{1}{n}+1}\left[1 - \left(\frac{r}{R}\right)^{\frac{1}{n}+1}\right]$$

$$\omega = \rho\, 2\pi \int_0^R v_z\, r\, dr = \frac{\pi R^3 \rho}{\frac{1}{n}+3}\left(\frac{(P_0 - P_L)R}{2mL}\right)^{\frac{1}{n}}$$

for $n = 1$, $m = \mu$ $\Rightarrow$ Newtonian Fluid