Velocity Distribution in Laminar Flow

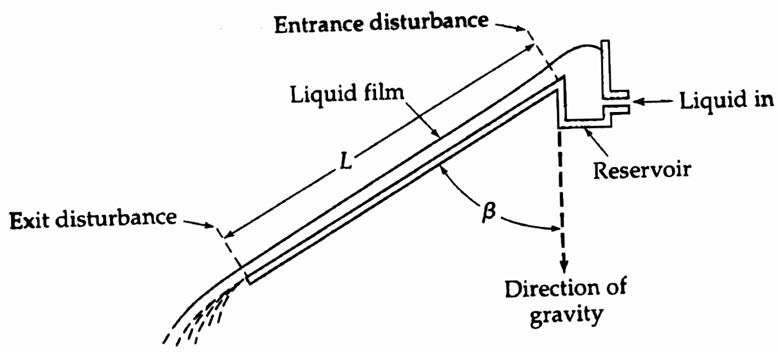
The problems treated in ch2 are for cases when Reynolds number is small enough such that the flow is laminar

Also, the fluid is assumed to be Newtonian such that the Newton's law of viscosity is applicable:

$$\underline{\underline{\tau}} = -\mu \cdot \underline{\underline{\dot{\gamma}}}$$

$$\underline{\underline{\dot{\gamma}}} = \nabla V + \nabla V^T$$

Example 1 Falling Film Flow



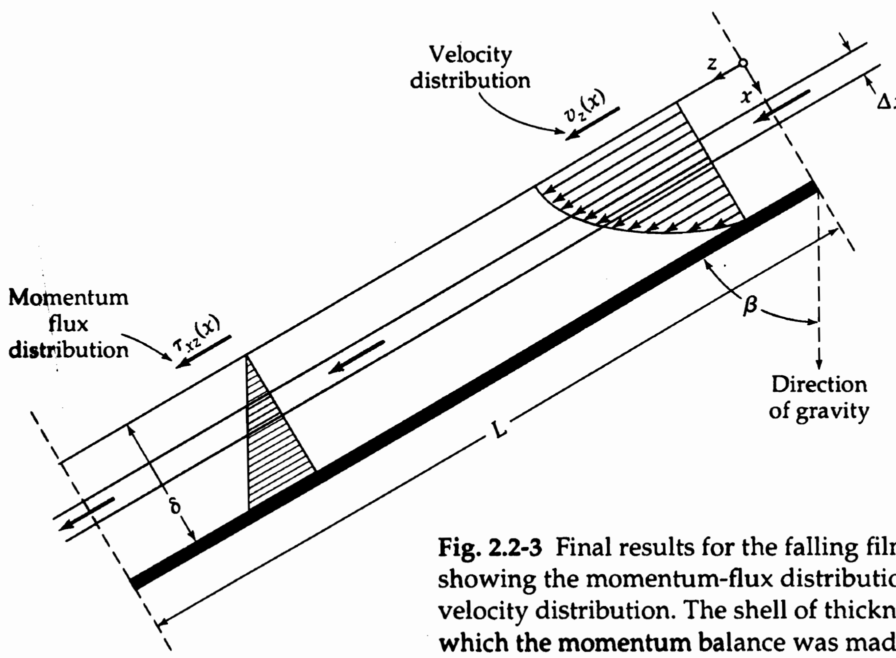


Fig. 2.2-3 Final results for the falling film problem, showing the momentum-flux distribution and the velocity distribution. The shell of thickness Δx, over which the momentum balance was made, is also shown.

Consider a Newtonian fluid flowing over an inclined plane. The fluid forms a film of thickness δ along the x-direction and moves by gravity along the z-direction.

For steady, laminar and incompressible ③
flow, there will be a single velocity
component V_z and $V_x = V_y = 0$.

The y-direction (along the width) is wide
and neutral, hence the velocity and pressure
are not functions of y.

<u>Continuity Equation</u>

$$(B.4-1) \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V_x) + \frac{\partial}{\partial y}(\rho V_y) + \frac{\partial}{\partial z}(\rho V_z) = 0$$

for incompressible fluid $\rho = \text{const.}$

$$\Rightarrow \quad \underbrace{\cancel{\frac{\partial V_x}{\partial x}}}_{\substack{0 \\ (V_x = 0)}} + \underbrace{\cancel{\frac{\partial V_y}{\partial y}}}_{(V_y = 0)} + \frac{\partial V_z}{\partial z} = 0.$$

$$\Rightarrow \quad V_z \neq f(z)$$

Conclusions from the treatment above

$V_z = f(x)$ only and $V_x = V_y = 0$

Navier-Stokes Equations

$$v_x = 0 \rightarrow 0 \qquad\qquad v_x = 0 \rightarrow 0$$

$$\cancel{\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right)} = -\frac{\partial p}{\partial x} + \cancel{\mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right]} + \rho g_x \qquad \text{(B.6-1)}$$

$$v_y = 0 \rightarrow 0 \qquad P \neq y \qquad v_y = 0 \rightarrow 0 \qquad g_y = 0$$

$$\cancel{\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right)} = \cancel{-\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right]} + \cancel{\rho g_y} \qquad \text{(B.6-2)}$$

$$\cancelto{0}{\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right)} = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \cancelto{0}{\frac{\partial^2 v_z}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z \qquad \text{(B.6-3)}$$

$$v_x = v_y = 0 \;\&\; v_z \neq f(z) \qquad\qquad v_z \neq f(y, z)$$

summary of simplified equations

$$0 = -\frac{\partial P}{\partial x} + \rho g_x .$$

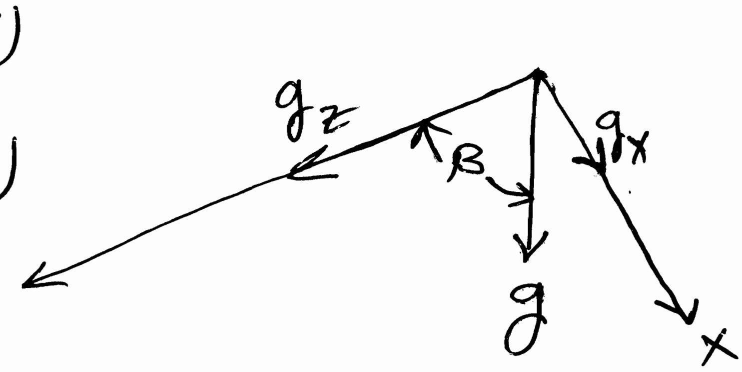
$$0 = 0 .$$

$$0 = -\cancelto{0}{\frac{\partial P}{\partial z}} + \mu \frac{d^2 v_z}{dx^2} + \rho g_z$$

Note this is a gravity driven flow and pressure does not change along direction of flow

$$g_z = +g \cos(\beta)$$

$$g_x = +g \sin(\beta)$$



integrate pressure equation.

$$P = \rho g \sin(\beta)\, x + C$$

BC $\quad x = 0 \quad P = P_{atm}. \Rightarrow C = P_{atm}.$

$$\Rightarrow \boxed{P - P_{atm} = \rho g \sin(\beta) x}$$

pressure changes due to
hydrostatic effects

integrate velocity equation

$$V_z = \frac{-\rho g}{\mu} \cos(\beta) \frac{x^2}{2} + C_1 x + C_2$$

$x = 0 \quad \tau_{xz} = 0 \Rightarrow -\mu \frac{dV_z}{dx} = 0. \Rightarrow C_1 = 0$

$x = \delta \quad V_z = 0$ (no slip condition)

$$\Rightarrow C_2 = \frac{\rho g}{\mu} \cos(\beta) \frac{\delta^2}{2}$$

$$\Rightarrow \boxed{V_z = \frac{\rho g}{\mu} \frac{\cos(\beta)\, \delta^2}{2} \left[1 - \left(\frac{x}{\delta}\right)^2\right]}$$

Once the velocity profile is evaluated, ⑥
a number of quantities of interest
can be evaluated:

Maximum Velocity at $x=0 \Rightarrow V_{z,max} = \frac{\rho g \delta^2}{2\mu}\cos(\beta)$

Average Velocity is the surface integral over the x-y plane.

$$\langle V_z \rangle = \frac{\int_0^W \int_0^\delta v_z \, dx \, dy}{\int_0^W \int_0^\delta dx \, dy} = \frac{1}{\delta}\int_0^\delta v_z \, dx$$

$$= \frac{\rho g \delta^2 \cos(\beta)}{3\mu} = \frac{2}{3} V_{z,max}$$

mass flow rate $\quad w = \rho \int_0^W \int_0^\delta v_z \, dx \, dy$

$$= \frac{\rho^2 g W \delta^3 \cos(\beta)}{3\mu}$$

$$= \rho W \delta \langle V_z \rangle$$

Film thickness

recall $\langle v_z \rangle = \dfrac{\rho g \delta^2 \cos(\beta)}{3\mu}$

$$\Rightarrow \delta = \sqrt{\frac{3\mu \langle v_z \rangle}{\rho g \cos(\beta)}}$$

also

$$\delta = \sqrt[3]{\frac{3\mu w}{\rho^2 g W \cos(\beta)}}$$

Force exerted on Plate

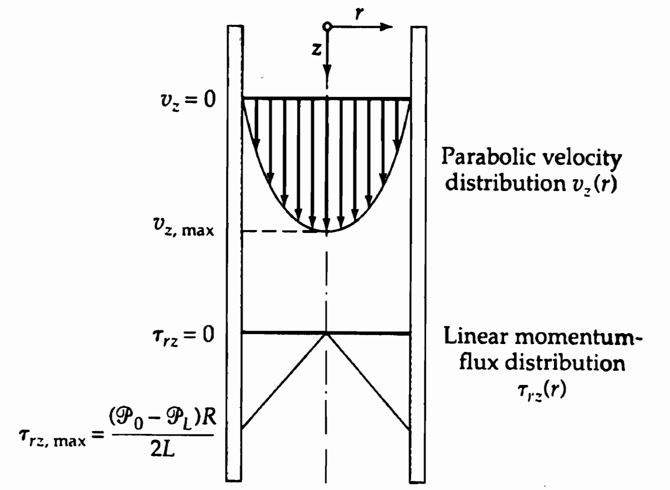
$$F_z = \int_0^L \int_0^W \left[\tau_{xz} \Big|_{x=\delta} \right] dy\, dz$$

$$\tau_{xz} = -\mu \frac{dv_z}{dx}$$

$$\rightarrow F_z = \rho g \delta L W \cos(\beta)$$

For falling films $\quad Re = \dfrac{4\delta \langle v_z \rangle \rho}{\mu}$

Example 2 Flow through a circular and vertical tube



Consider the flow of a fluid downwards through a circular tube as shown above. In this problem there is a single velocity v_z & $v_r = v_\theta = 0$. Moreover, this problem is axisymmetric, hence, the velocity and pressure are not functions of the azimuthal direction $\theta \Rightarrow \frac{\partial}{\partial \theta} = 0$.

Continuity Eq.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

for incompressible fluid: $\underbrace{\cancel{\frac{1}{r}\frac{\partial}{\partial r}(r v_r)}}_{v_r = 0 \;\to\; 0} + \underbrace{\cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial r}}}_{v_\theta = 0 \;\to\; 0} + \frac{\partial v_z}{\partial z} = 0$

$\Rightarrow v_z \neq f(z)$

N. S. Eq.

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \qquad \text{(B.6-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \qquad \text{(B.6-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad \text{(B.6-6)}$$

taking into account that $V_r = V_\theta = 0$ and ($V_z = f(r)$ only and $g_r = g_\theta = 0$) axisymmetry $\left(\frac{\partial}{\partial \theta} = 0\right)$, the above equation are simplified as follows:

$$\frac{\partial P}{\partial r} = 0$$

$$0 = 0$$

$$0 = -\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) + \rho g_z$$

Note in this problem, the flow is induced by two factors:

1. pressure gradient $\frac{dP}{dz}$
2. gravity

Defining the modified pressure

$\mathcal{P} \equiv p - \rho g z$ and rearranging.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{\partial P}{\partial z}$$

in this case pressure changes linearly along direction of flow

$$\Rightarrow \quad \frac{\partial P}{\partial z} = \frac{\Delta P}{\mu L} = \frac{P|_{z=L} - P|_{z=0}}{L}.$$

integrating twice: $V_z = \frac{\Delta P}{4\mu L} r^2 + C_1 \ln(r) + C_2$

Boundary Conditions

$BC_1 \quad r=0 \quad \frac{dV_z}{dr}=0$ (symmetry)

$BC_2. \quad r=R \quad V_z = 0$ (no-slip condition)

$BC1 \Rightarrow C_1 = 0 \qquad BC2 \Rightarrow C_2 = -\frac{\Delta P}{4\mu L}R^2.$

substituting $\Rightarrow V_z = -\frac{\Delta P R^2}{4\mu L}\left[1-\left(\frac{r}{R}\right)^2\right]$

<u>Maximum velocity</u> @ $r=0$

$$\Rightarrow V_{z\,max} = -\frac{\Delta P R^2}{4\mu L}$$

Average velocity is the surface integral of the velocity over the area of the crosssection

$$\langle V_z \rangle = \frac{\int_0^{2\pi}\int_0^{R} V_z \, r\, dr\, d\theta}{\int_0^{2\pi}\int_0^{R} r\, dr\, d\theta}$$

$$= \frac{\int_0^{R} V_z r\, dr}{\int_0^{R} r\, dr} = \frac{-\frac{\Delta P R^2}{4\mu L}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R}{\left[\frac{r^2}{2}\right]_0^R}$$

$$= \frac{-\frac{\Delta P R^2}{4\mu L}\left[\frac{R^2}{2} - \frac{R^2}{4}\right]}{\frac{R^2}{2}}.$$

$$= \frac{-\Delta P R^2}{8\mu L} = \frac{1}{2} V_{z,max}.$$

Mass flow rate: $\omega = \rho \pi R^2 \langle V_z \rangle$

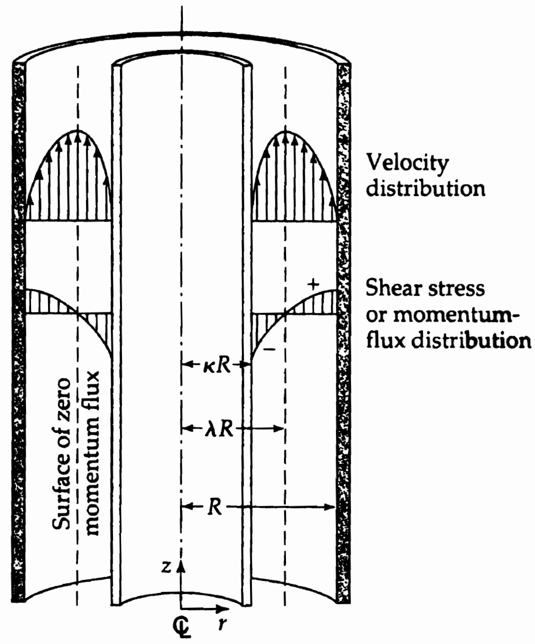
Hagen-Poiseuille equation

Force Exerted on the wall

$$F_z = 2\pi R L\left[\tau_{rz}\big|_{r=R}\right] = 2\pi R L \; -\mu \frac{dV_z}{dr}\bigg|_{r=R}$$

$$= \pi R^2 \Delta P$$

Example 3
Flow Through an Annulus



This problem is similar to the previous example except that:

1. the flow is upwards.
2. the flow is through an annulus between concentric cylinders.

recall,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{d\mathcal{P}}{dz}$$

Note in this case $\mathcal{P} \equiv p + \rho g z$

recall also, $V_z = \frac{\Delta P}{4\mu L} r^2 + C_1 \ln(r) + C_2$

Boundar conditions

$r = kR \qquad V_z = 0$

$r = R \qquad V_z = 0.$

$$0 = \frac{\Delta P}{4\mu L} k^2 R^2 + C_1 \ln(kR) + C_2 \quad (1)$$

$$0 = \frac{\Delta P}{4\mu L} R^2 + C_1 \ln R + C_2 . \quad (2)$$

$$(1) - (2) \Rightarrow C_1 = -\frac{\frac{\Delta P R^2}{4\mu L}(k^2-1)}{\ln(k)}$$

$$\Rightarrow C_2 = -\frac{\Delta P R^2}{4\mu L} + \frac{\Delta P R^2 (k^2-1)}{4\mu L} \frac{\ln R}{\ln(k)}$$

$$= \frac{-\Delta P R^2}{4\mu L}\left[1 - (k^2-1)\frac{\ln R}{\ln k}\right].$$

$$V_z = \frac{\Delta P}{4\mu L} r^2 + \frac{-\Delta P R^2}{4\mu L}(k^2-1)\frac{\ln(r)}{\ln(k)} + \frac{-\Delta P R^2}{4\mu L} - \frac{-\Delta P R^2}{4\mu L}(k^2-1)\frac{\ln R}{\ln(k)}$$

$$V_z = \frac{\Delta P R^2}{4\mu L}\left[\left(\frac{r}{R}\right)^2 - 1\right] + \frac{-\Delta P R^2 (\kappa^2 - 1)}{4\mu L \ln(\kappa)} \ln\left(\frac{r}{R}\right)$$

$$V_z = \frac{-\Delta P R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2 - \frac{(1-\kappa^2)}{\ln \kappa} \ln\left(\frac{r}{R}\right)\right]$$

one can show at $r = \lambda R \Rightarrow \frac{dV_z}{dr} = 0$

$$\Rightarrow 2\lambda^2 = \frac{1-\kappa^2}{\ln(1/\kappa)}$$

$$\Rightarrow V_{z,max} = V_z\Big|_{r=\lambda R}$$

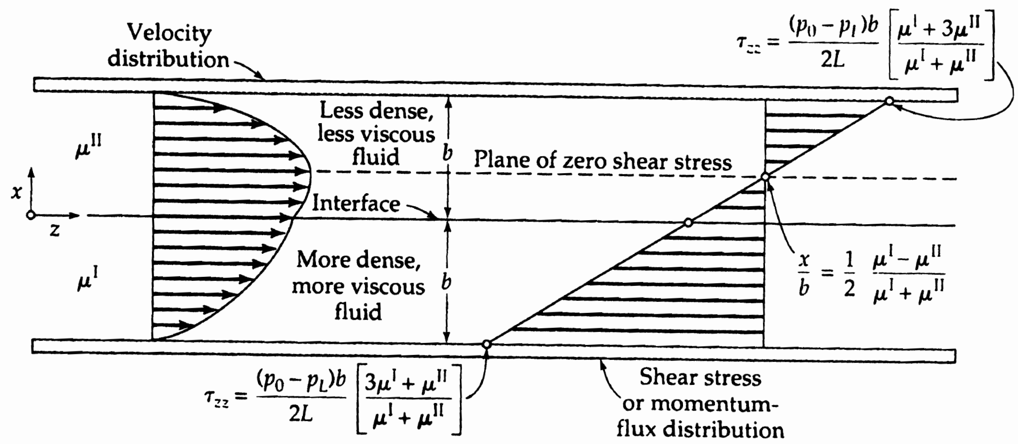
$$= -\frac{\Delta P R^2}{4\mu L}\left[1 - \lambda^2(1 - \ln \lambda^2)\right]$$

$$\langle V_z \rangle = \frac{-\Delta P R^2}{4\mu L} \frac{1-\kappa^4}{1-\kappa^2} - \frac{1-\kappa^2}{\ln(1/\kappa)}$$

$$F_z = 2\pi \kappa R L \left[+\tau_{rz}\Big|_{r=\kappa R}\right] + 2\pi R L \left[+\tau_{rz}\Big|_{r=R}\right]$$

Example 4 Pressure driven flow of two immiscible fluids

Two immiscible, incompressible liquids are flowing in the z direction in a horizontal thin slit of length L and width W under the influence of a horizontal pressure gradient $(p_0 - p_L)/L$. The fluid flow rates are adjusted so that the slit is half filled with fluid I (the more dense phase) and half filled with fluid II (the less dense phase). The fluids are flowing sufficiently slowly that no instabilities occur—that is, that the interface remains exactly planar.



Flow in z direction, hence, $V_x = V_y = 0$.

also, by continuity equation one can show that $\frac{dV_z}{dz} = 0$. Also, $V_z \neq f(y)$ because we have wide plates in y-direction.

Therefore, we only have $V_z = f(x)$

Recall the N.S. Equations and apply them to fluid I & II after simplification:

$$0 = -\frac{dP}{dz} + \mu^{I} \frac{d^2 V_z^{I}}{dx^2}$$

$$0 = -\frac{dP}{dz} + \mu^{II} \frac{d^2 V_z^{II}}{dx^2}$$

recall, $\frac{dP}{dz} = \frac{\Delta P}{L}$.

$$\Rightarrow V_z^{I} = \frac{\Delta P}{2\mu^{I} L} x^2 + C_1 x + C_2$$

$$V_z^{II} = \frac{\Delta P}{2\mu^{II} L} x^2 + C_3 x + C_4$$

Boundary Conditions:

$V_z^{I} = 0$ $\quad x = -b$

$V_z^{I} = V_z^{II}$ $\quad x = 0$.

$\mu^{I} \frac{dV_z^{I}}{dx} = \mu^{II} \frac{dV_z^{II}}{dx}$ $\quad x = 0$.

$V_z^{II} = 0$ $\quad x = b$, one can show.

$$V_z^{I} = \frac{-\Delta P b^2}{2\mu^{I} L} \cdot \left[\frac{2\mu^{I}}{\mu^{I} + \mu^{II}} + \left(\frac{\mu^{I} - \mu^{II}}{\mu^{I} + \mu^{II}} \right) \frac{x}{b} - \left(\frac{x}{b} \right)^2 \right]$$

$$V_z^{II} = \frac{-\Delta P b^2}{2\mu^{II} L} \left[\frac{2\mu^{II}}{\mu^{I} + \mu^{II}} + \left(\frac{\mu^{I} - \mu^{II}}{\mu^{I} + \mu^{II}} \right) \frac{x}{b} - \left(\frac{x}{b} \right)^2 \right].$$