Example: Tank Draining

A tank is draining a liquid through an orifice of cross-sectional area A at its base. Calculate the liquid drainage rate at point 2.

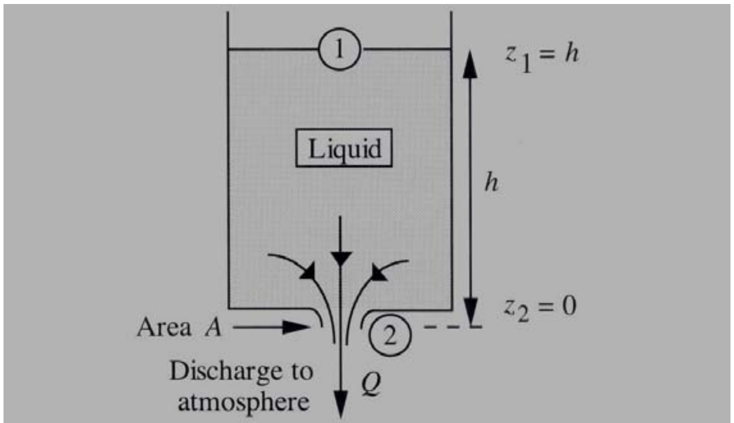


Fig. 2.7 Tank draining through a rounded orifice.

Solution:

(a) Since shaft work and frictional losses are negligible (), apply Bernoulli's equation between points 1 and 2:

$$\frac{u_1^2}{2} + g\, z_1 + \frac{P_1}{\rho} = \frac{u_2^2}{2} + g\, z_2 + \frac{P_2}{\rho} \qquad \left(u_1 \approx 0,\ P_1 = P_2 = P_{atm}, z_1 = h, z_2 = 0\right)$$

$$\frac{0}{2} + g\, h + \frac{P_{atm}}{\rho} = \frac{u_2^2}{2} + g\,(0) + \frac{P_{atm}}{\rho}$$

Solving for u_2: $u_2 = \sqrt{2\, g\, h}$

Flow Rate Calculation:

Case 1: Assume Exit at point 2 is smooth and well rounded

$$v = u_2\, A_2 = A_2 \sqrt{2\, g\, h}$$

Case 2: Exit at point 2 is sharp and rough

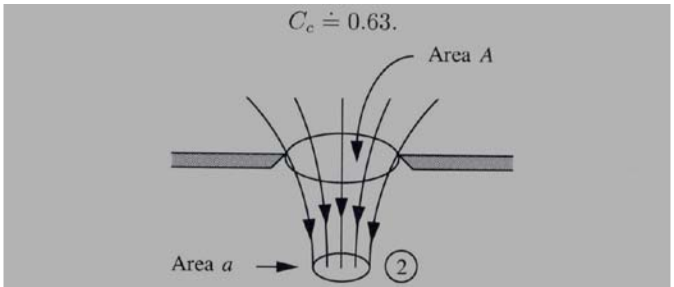


Fig. 2.8 Contraction of the jet through a sharp-edged orifice.

$$v = C_c\, u_2\, A_2 = C_c\, A_2 \sqrt{2\, g\, h}$$

where C_c is the coefficient of discharge usually $C_c = 0.63$

Example: Moving Pitot Tube

The figure below shows a moving pitot tube used to measure the velocity of a moving boat.

Derive an equation for measuring the velocity at the stagnation point.

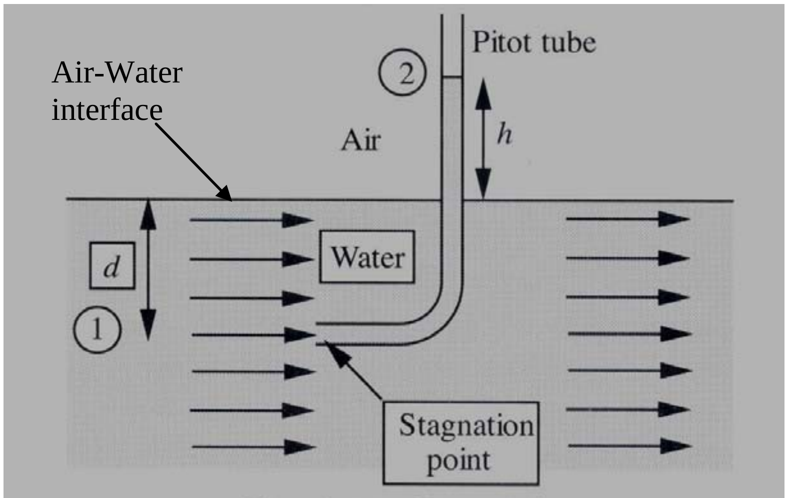


Fig. 2.11 Pitot tube

Solution:

Apply Bernoulli's equation between points 1 and 2:

$$\frac{u_1^2}{2} + g\,z_1 + \frac{P_1}{\rho} = \frac{u_2^2}{2} + g\,z_2 + \frac{P_2}{\rho} \qquad \left(u_2 = 0,\, P_2 = P_{atm},\, P_1 = P_2 + \rho\,g\,d,\, z_1 = 0,\, z_2 = h + d\right)$$

$$\frac{u_1^2}{2} + g(0) + \frac{P_2 + \rho\,g\,d}{\rho} = \frac{0}{2} + g\,(h + d) + \frac{P_2}{\rho}$$

Solving for u_1: $\quad u_1 = \sqrt{2\,g\,h}$

Example: Static Pitot Tube

The figure below shows a static pitot tube used to measure the local velocity at different radial locations in a pipe. Derive an equation for measuring the local fluid velocity.

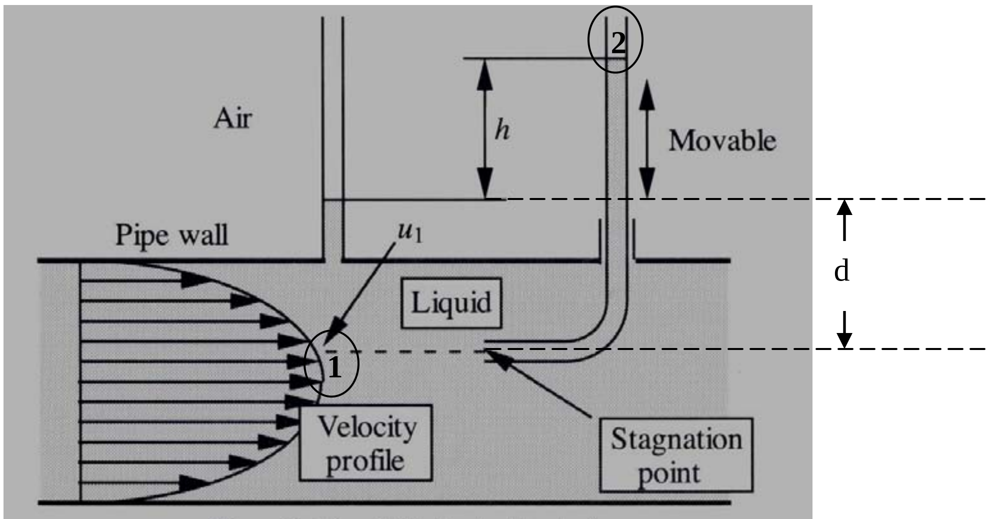


Fig. 2.12 Pitot-static tube.

Solution:

Apply Bernoulli's equation between points 1 and 2:

$$\frac{u_1^2}{2} + g\, z_1 + \frac{P_1}{\rho} = \frac{u_2^2}{2} + g\, z_2 + \frac{P_2}{\rho} \qquad \left(u_2 = 0,\, P_2 = P_{atm},\; P_1 = P_2 + \rho\, g\, d,\, z_1 = 0,\, z_2 = h + d\right)$$

$$\frac{u_1^2}{2} + g(0) + \frac{P_2 + \rho\, g\, d}{\rho} = \frac{0}{2} + g\,(h + d) + \frac{P_2}{\rho}$$

Solving for u_{1}: $u_1 = \sqrt{2\, g\, h}$

Example: Tank Filling

The figure below shows a tank that is being filled with water from an adjacent river. The level of the reviver H = 10 ft above the base of the tank. A short pipe connecting the river to the tank at a height D = 4 ft above the base of the tank. The cross-sectional area of the pipe is a = 0.01 ft^2 and that of the tank is A = 1000 ft^2. Derive and expression for the time t needed to fill the tank and then evaluate it for the specified conditions.

Example 2.3—Tank Filling

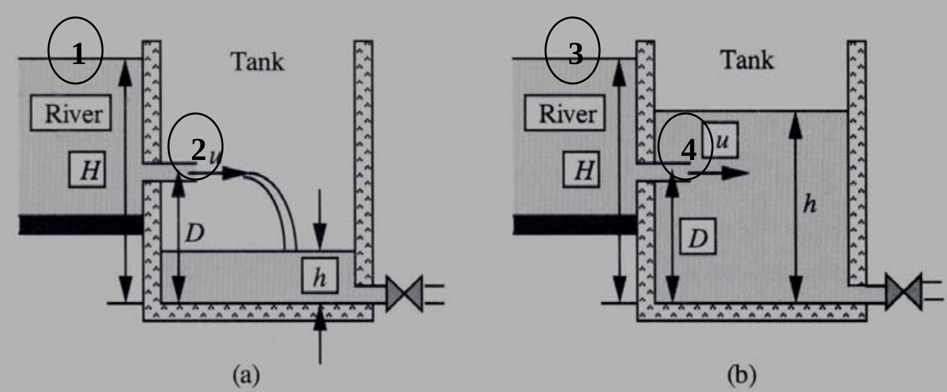


Fig. E2.3 Tank filling from river: (a) before pipe is submerged; (b) after pipe is submerged.

<u>Solution:</u>

<u>Step 1</u>: Time required to filling the tank until height D, $\boldsymbol{t_1}$:

Apply Bernoulli's equation between points 1 and 2:

$$\frac{u_1^2}{2} + g\, z_1 + \frac{P_1}{\rho} = \frac{u_2^2}{2} + g\, z_2 + \frac{P_2}{\rho} \qquad \left(u_1 \approx 0,\ P_1 = P_2 = P_{atm}, z_1 = H, z_2 = D\right)$$

$$\frac{0}{2} + g\, H + \frac{P_{atm}}{\rho} = \frac{u_2^2}{2} + g\, D + \frac{P_{atm}}{\rho}$$

Solving for u_2: $\quad u_2 = \sqrt{2\, g\,(H\text{-}D)}$

$$t_1 = \frac{\text{Volume of tank until height } D}{\text{Volumetric flow rate of water}}$$

$$= \frac{A\ D}{a\ u_2} = \frac{A\ D}{a\ \sqrt{2\, g\,(H\text{-}D)}} \qquad \left(\text{Assuming smooth exit at the end of the pipe}\right)$$

Step 2: Time required to reach height h such that $h > D$, t_2:

Apply Bernoulli's equation between points 3 and 4:

$$\frac{u_3^2}{2} + g\,z_3 + \frac{P_3}{\rho} = \frac{u_4^2}{2} + g\,z_4 + \frac{P_4}{\rho} \qquad \left(u_3 \approx 0,\; P_3 = P_{atm}, P_4 = P_{atm} + \rho g(h-D), z_3 = H, z_4 = D\right)$$

$$\frac{0}{2} + g\,H + \frac{P_{atm}}{\rho} = \frac{u_4^2}{2} + g\,D + \frac{P_{atm} + \rho g(h-D)}{\rho}$$

Solving for u_4: $\qquad u_4 = \sqrt{2\,g\,(H\text{-}h)}$

Note: In this case is u_4 a function of h, $u_4 = f(h)$, hence, u_4 is not constant!

$$\text{time required} \neq \frac{\text{Volume of tank}}{\text{Volumetric flow rate of water}} = \frac{\text{not constant } f(h)}{\text{not constant } f(h)}$$

Mass Balance: $\qquad m_{\text{in}} - m_{\text{out}} = \frac{d}{dt}\left(\text{M}_{\text{syst.}}\right), \qquad \left(m_{\text{out}} = 0,\; m_{\text{in}} = \rho\,u_4\,a\right)$

$$\rho\,u_4 a - 0 = \frac{d}{dt}(\rho\,A\,h)$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{a}{A}\sqrt{2\,g\,(H\text{-}h)}$$

$$\int_D^H \frac{A}{a\sqrt{2\,g\,(H\text{-}h)}}\,dh = \int_0^{t_2} dt$$

$$t_2 = \frac{A}{a}\sqrt{\frac{2(H\text{-}D)}{g}} \qquad \text{(do it yourself !)}$$

Total time required to reach height H in the tank = $t_1 + t_2$

$$t = \frac{A\,D}{a\sqrt{2\,g\,(H\text{-}D)}} + \frac{A}{a}\sqrt{\frac{2(H\text{-}D)}{g}} \qquad \left(H = 10\text{ ft}, D = 4\text{ ft}, A = 1000\text{ ft}^2\ \&\ a = 0.1\text{ ft}^2\right)$$

$$= 8130\text{ s} = 2.26\text{ hr}$$

Example: Orifice Plate

An Orifice Plate is a devise used to measure fluid flow rate in a pipe. It consists of a circular disc with a central hole of area A_o is mounted between the flanges on two sections of pipe of cross-sectional area A_1, see the figure below. Given the pressure drop across the orifice, P_2-P_1, derive an equation for the fluid volumetric flow rate.

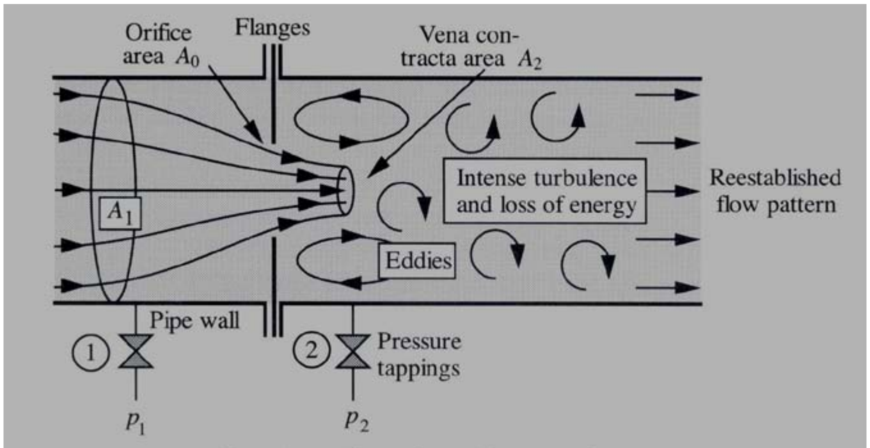


Fig. 2.9 Flow through an orifice plate.

__Solution:__

Note that the frictional losses after the orifice are significant due to intense mixing (turbulence) and cannot be neglected. However, <u>between points 1 and 2, as shown in the figure, frictional losses are minor and can be neglected</u>.

Apply Bernoulli's equation between points 1 and 2:

$$\frac{u_1^2}{2} + g\,z_1 + \frac{P_1}{\rho} = \frac{u_2^2}{2} + g\,z_2 + \frac{P_2}{\rho} \qquad (z_1 = z_2)$$

Continuity equation between points 1 and 2:

$$\rho\,u_1\,A_1 = \rho\,u_2\,A_2 \qquad \Rightarrow \qquad u_2 = \frac{A_1}{A_2}u_1$$

Substitute in BE:

$$\frac{u_1^2}{2} + \frac{P_1}{\rho} = \frac{\left(\frac{A_1}{A_2}u_1\right)^2}{2} + \frac{P_2}{\rho} \qquad (z_1 = z_2)$$

Solving for u_1: $$u_1 = \sqrt{\frac{2(P_1 - P_2)}{\rho\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right)}}$$

Volumetric flow rate: $$v = u_1 A_1 = A_1 \sqrt{\frac{2(P_1 - P_2)}{\rho\left(\left(\frac{A_1}{A_2}\right)^2 - 1\right)}}$$

Note: The actual cross-sectional area of the jet leaving the orifice A_2 is not equal to the area of the orifice A_o. $A_2 \neq A_o$

Therefore, apply a correction factor: $$v = C_D\, u_1 A_1 = C_D\, A_1 \sqrt{\frac{2(P_1 - P_2)}{\rho\left(\left(\frac{A_1}{A_o}\right)^2 - 1\right)}} \qquad (*)$$

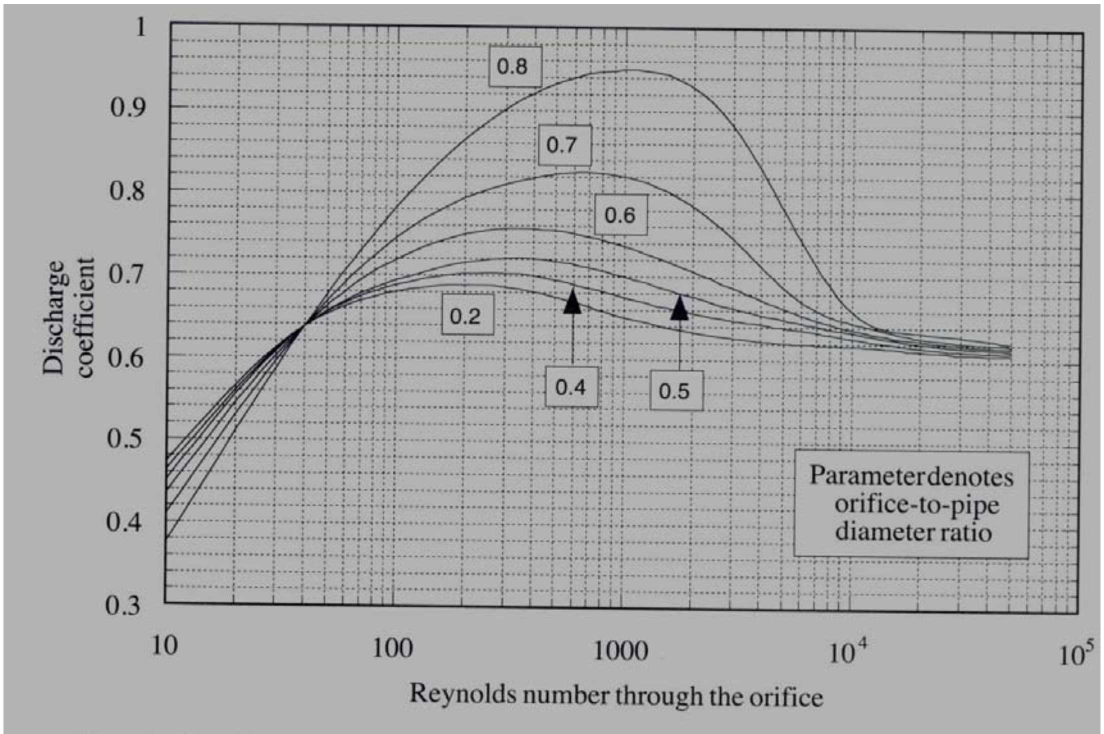


Fig. 2.10 Discharge coefficient for orifice plates. Based on values from G.G. Brown et al., Unit Operations, *John Wiley & Sons, New York, 1950.*

Solution Procedure:

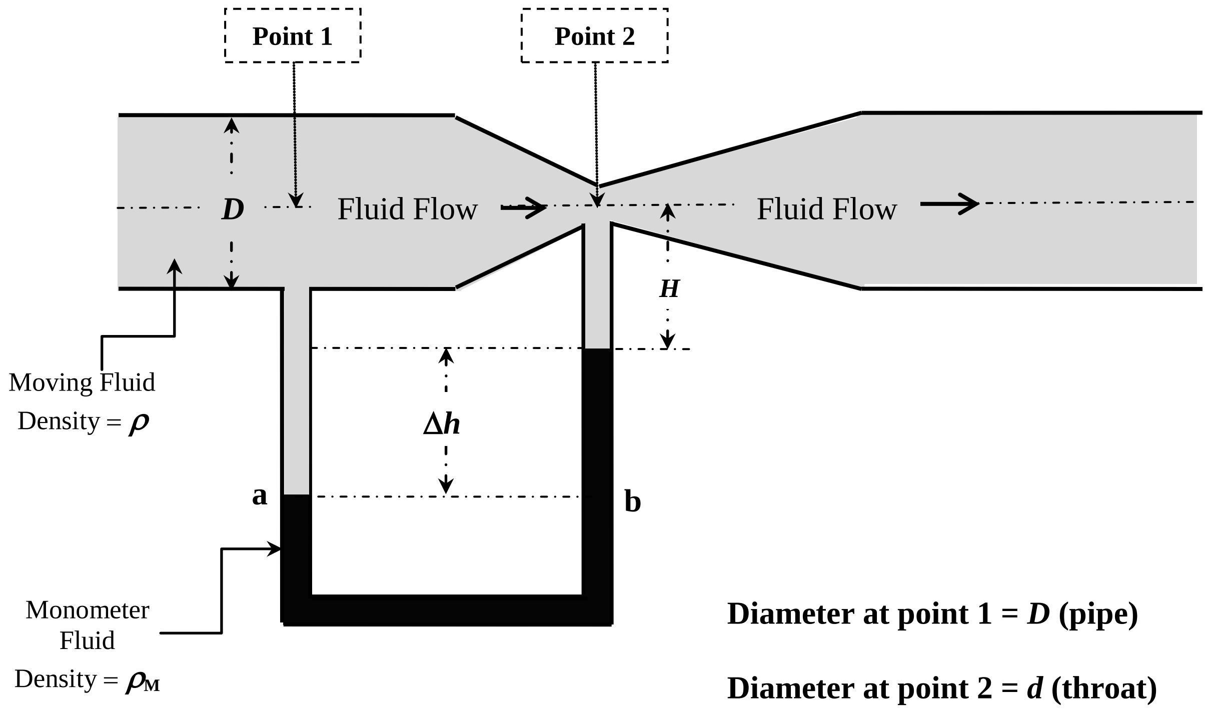
Calculate Reynolds number through orifice: $\mathrm{Re}_{\text{orifice}} = \dfrac{\rho\ u_2\ D_O}{\mu}$ But u_2 is unknown!

Solve by trial and error:

1. Assume $C_D \approx 0.63$
2. Calculate volumetric flow rate v from (*).
   a. Calculate: $u_2 = \dfrac{v}{\frac{\pi}{4} D_o^2}$
   b. Calculate: $\mathrm{Re}_{\text{orifice}} = \dfrac{\rho\ u_2\ D_O}{\mu}$
3. Find C_D from the chart
4. Calculate volumetric flow rate v from (*) using calculated value of C_D.
   a. Calculate: $u_2 = \dfrac{v}{\frac{\pi}{4} D_o^2}$
   b. Calculate: $\mathrm{Re}_{\text{orifice}} = \dfrac{\rho\ u_2\ D_O}{\mu}$
5. Check $\mathrm{Re}_{\text{orifice}}(\text{from step 4}) \overset{?}{\approx} \mathrm{Re}_{\text{orifice}}(\text{from step 2})$
   a. Yes → stop you have v.
   b. No go to step 2.

Example: Venturi Meter

The figure below shows a venturi meter which is used to measure flow rate of fluids in pipelines. A fluid of density ρ flows through a horizontal pipe of diameter D with a volumetric flow rate Q then passes through a contraction where the diameter at the throat is d. A monometer containing a manometer fluid of density ρ_M is connected between the upstream (point 1) and the throat (point 2) and reads a height difference Δh between the two levels. By measuring the pressure drop the flow rate is calculated. Derive an expression for Q as a function of D, d, ρ and ρ_M.



Diameter at point 1 = D (pipe)

Diameter at point 2 = d (throat)

Solution:

First derive an expression for the pressure drop ($P_1 - P_2$) measured by the monometer:

$$P_a = P_1 + \rho g(\Delta h + H)$$

$$P_b = P_2 + \rho\, gH + \rho_M\, g\Delta h$$

$$P_a = P_b$$

$$\Rightarrow \qquad P_1 + \rho g(\Delta h + H) = P_2 + \rho\, gH + \rho_M\, g\Delta h$$

$$\Rightarrow \qquad P_1 - P_2 = (\rho_M - \rho)\, g\, \Delta h$$

Now to derive an equation for $\boldsymbol{Q}$ apply Bernoulli's equation between Point 1 and Point 2:

$$\frac{u_1^2}{2} + g\,z_1 + \frac{P_1}{\rho} = \frac{u_2^2}{2} + g\,z_2 + \frac{P_2}{\rho} \qquad (z_1 = z_2)$$

Because u_2 is unknown apply mass balance (Continuity Equation) between Point 1 and Point 2:

$$\rho\,u_1 A_1 = \rho\,u_2 A_2 \qquad \Rightarrow \qquad u_2 = u_1\frac{A_1}{A_2} = u_1\frac{\frac{\pi}{4}D^2}{\frac{\pi}{4}d^2} = u_1\frac{D^2}{d^2}$$

From B.E. $\displaystyle \frac{u_1^2}{2} + g\,z_1 + \frac{P_1}{\rho} = \frac{\left(u_1\frac{D^2}{d^2}\right)^2}{2} + g\,z_2 + \frac{P_2}{\rho} \qquad (z_1 = z_2)$

Simplify $\displaystyle u_1 = \sqrt{\frac{2(P_1 - P_2)}{\rho\left[\left(\frac{D^2}{d^2}\right)^2 - 1\right]}}$

$$Q = u_1 A_1 = \frac{\pi}{4}D^2\sqrt{\frac{2(P_1 - P_2)}{\rho\left[\left(\frac{D^2}{d^2}\right)^2 - 1\right]}}$$

Correction: the actual area at Point 2, $A_2 \neq \frac{\pi}{4}d^2$, due to the contraction of the fluid jet after the throat, hence apply the following correction for the flow rate:

$$Q = C_D\quad \frac{\pi}{4}D^2\sqrt{\frac{2(P_1 - P_2)}{\rho\left[\left(\frac{D^2}{d^2}\right)^2 - 1\right]}}$$

Where C_D is the discharge coefficient for the venturi.