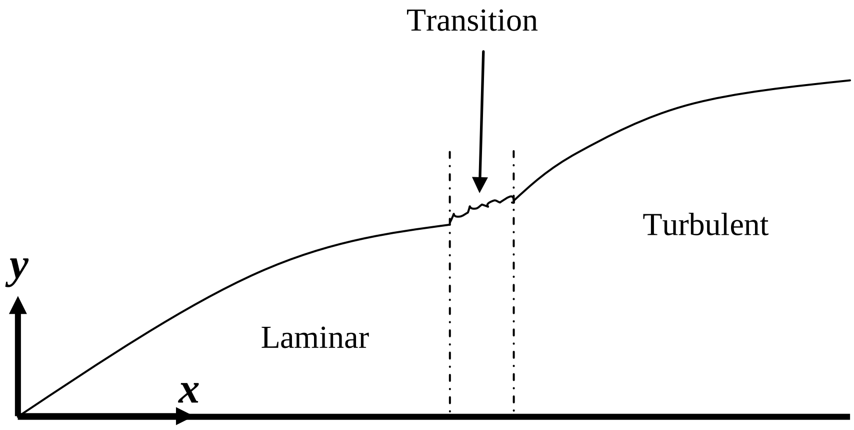
Boundary-Layer Flow over a Flat Plate

"Approximate Method"



The momentum balance on a control volume of the boundary layer leads to the following equation:

$$v_\infty \frac{d}{dx}\int_0^\delta v_x dy = \frac{\tau_w}{\rho} + \frac{d}{dx}\int_0^\delta v_x^2 dy \qquad (1)$$

The approximate method of boundary layer analysis can be summarized as follows:

(a) Assume the velocity profile, v_x, as a function of x and y.

(b) Evaluate the integrals: $\int_0^\delta v_x dy$ and $\int_0^\delta v_x^2 dy$.

(c) Evaluate the wall-shear stress: $\tau_w \equiv \mu \frac{\partial v_x}{\partial y}\Big|_{y=0}$.

(d) Substitute in the above momentum balance and get δ and τ_w as functions of the distance, x, the velocity far from the plate, $v_{x\infty}$ and the fluid properties (ρ, μ).

The question now, how to assume v_x? v_x can be assume as follows:

$$\frac{v_x}{v_{x\infty}} = f(\zeta) \text{ where } \zeta \equiv \frac{y}{\delta(x)}$$

The function $f(\zeta)$ must satisfy the following properties:

1. $f(0) = 0$ $\quad (v_x = 0 \ @ \ y = 0)$
2. $f(1) = 1$ $\quad (v_x = v_\infty \ @ \ y = \delta)$
3. $\left.\frac{df}{d\zeta}\right|_{\zeta=1} = 0$ $\quad \left(\left.\tau_{yx}\right|_{y=\delta} = \left.\mu\frac{\partial v_x}{\partial y}\right|_{y=\delta} = 0\right)$; 4. $\left.\frac{d^2 f}{d\zeta^2}\right|_{\zeta=0} = 0$, 5. $\left.\frac{d^2 f}{d\zeta^2}\right|_{\zeta=1} = 0$

Laminar Boundary Layer

Now let us follow the summary outlined previously for the approximate method of boundary layer analysis:

(a) The following velocity profile $\frac{v_x}{v_\infty} = \text{Sin}\left(\frac{\pi}{2}\zeta\right)$ satisfies the properties of the function $f(\zeta)$ listed before, hence it can be assumed.

(b) Evaluate: $$\int_0^\delta v_x dy = \int_0^\delta v_{x\infty} \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) dy = -v_{x\infty}\frac{2}{\Pi}\delta\left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]_0^\delta = \frac{2}{\pi}v_{x\infty}\delta$$

Also, $$\int_0^\delta v_x^2 dy = \int_0^\delta v_{x\infty}^2 Sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) = v_{x\infty}^2\int_0^\delta\left[1 - Cos^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]dy$$

$$= v_{x\infty}^2\left[\int_0^\delta dy - \int_0^\delta Cos^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)dy\right]$$ (using integration formula in appendix A)

$$= v_{x\infty}^2\left[[y]_0^\delta - \frac{2\delta}{\pi}\frac{1}{2}\left[\frac{\pi}{2}\frac{y}{\delta} + \frac{1}{2}Sin\left(2\frac{\pi}{2}\frac{y}{\delta}\right)\right]_0^\delta\right] = v_{x\infty}^2\frac{\delta}{2}$$

(c) Evaluate: $$\tau_w = \left.\mu\frac{\partial v_x}{\partial y}\right|_{y=0}$$

the derivative: $\dfrac{\partial v_x}{\partial y} = \dfrac{\partial}{\partial y}\left(v_{x\infty} Sin\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)\right) = v_{x\infty}\dfrac{\pi}{2}\dfrac{1}{\delta}Cos\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)$

and therefore: $\tau_w = \left[\mu v_{x\infty}\dfrac{\pi}{2}\dfrac{1}{\delta}Cos\left(\dfrac{\pi}{2}\dfrac{y}{\delta}\right)\right]_{y-0} = \dfrac{\mu\, v_{x\infty}\,\pi}{2\,\delta}$

(d) Substitute in momentum balance, equation (1):

$$v_\infty \frac{d}{dx}\left(\frac{2}{\pi}v_\infty \delta\right) = \frac{\mu\, v_\infty \pi}{2\,\delta\,\rho} + \frac{d}{dx}\left(v_\infty^2 \frac{\delta}{2}\right) \qquad \Rightarrow \frac{2}{\pi}v_\infty^2\frac{d\delta}{dx} = \frac{\mu v_\infty \pi}{2\,\delta\,\rho} + \frac{v_\infty^2}{2}\frac{d\delta}{dx}$$

$$v_\infty \overbrace{\left(\frac{2}{\pi} - \frac{1}{2}\right)}^{=0.137}\frac{d\delta}{dx} = \frac{\mu\pi}{2\rho}\frac{1}{\delta} \qquad \Rightarrow 0.137 v_\infty \frac{d\delta}{dx} = \frac{\mu\pi}{2\rho}\frac{1}{\delta}$$

Integrate: $\displaystyle\int_0^\delta \delta \; d\delta = \frac{\mu\pi}{2\rho(0.137 v_\infty)}\int_0^x dx$

$$\frac{\delta^2}{2} = \frac{\mu\pi}{2\rho(0.137 v_\infty)}x \qquad \left(\div x^2\right)$$

$$\frac{\delta^2}{x^2} = \frac{\mu\pi}{\rho(0.137 v_\infty)}\frac{1}{x} = \frac{\pi}{0.137}\frac{\mu}{\rho\, v_\infty x}$$

Define Reynolds number for boundary layer flow: $\mathrm{Re}_x = \dfrac{\rho\, v_\infty x}{\mu}$

$$\frac{\delta^2}{x^2} = \frac{\pi}{0.137}\frac{1}{\mathrm{Re}_x} \qquad \boxed{\Rightarrow \frac{\delta}{x} = \frac{4.79}{\sqrt{\mathrm{Re}_x}}}$$

The above equation provides the thickness of the boundary layer as a function of x and Re_x. Now we evaluate the shear stress at the flat plate:

Recall, $\tau_w = \mu \left.\dfrac{dv_x}{dy}\right|_{y=0} = \dfrac{\mu v_\infty \pi}{2\delta}$

$$\Rightarrow \tau_w = \frac{\mu v_\infty \pi}{2} \; \frac{\sqrt{\text{Re}_x}}{4.79\, x} = \frac{\mu v_\infty \pi}{2} \; \frac{\sqrt{\frac{\rho v_\infty x}{\mu}}}{4.79\, x} = \overbrace{\frac{\pi}{(2)(4.79)}}^{0.328} \; \frac{\mu^{1/2} v_\infty^{3/2} \rho^{1/2}}{x^{1/2}}$$

Recall the definition of Drag Coefficient: $C_f \equiv \dfrac{\tau_w}{\frac{1}{2}\rho v_\infty^2}$

$$C_f = 0.328 \; \frac{\mu^{1/2} v_\infty^{3/2} \rho^{1/2}}{x^{1/2}} \; \frac{1}{\frac{1}{2}\rho v_\infty^2} = (2)(0.328) \frac{\mu^{1/2}}{v_\infty^{1/2} \rho^{1/2} x^{1/2}} \qquad \boxed{\Rightarrow C_f = \frac{0.656}{\sqrt{\text{Re}_x}}}$$

Turbulent Boundary Layer

The preceding analysis was for laminar B.L. However, it is found experimentally that when $\text{Re}_x > 3.2 \text{ x } 10^5$, the boundary layer becomes turbulent over a flat plate:

$\text{Re}_x < 3.2 \times 10^5$ Laminar boundary layer

$\text{Re}_x > 3.2 \times 10^5$ Turbulent boundary layer

What is the value of x at which the boundary layer undergoes transition to turbulent flow?

$$\frac{\rho v_\infty x}{\mu} = 3.2 \times 10^5 \quad \Rightarrow \quad x = 3.2 \times 10^5 \, \frac{\mu}{\rho v_\infty}$$

The velocity profile for turbulent boundary layer is based on experience from experimental data:

From experience: $\dfrac{v_x}{v_\infty} = \left(\dfrac{y}{\delta}\right)^{1/7}$ for turbulent B.L.

$$\int_0^\delta v_x \, dy = \int_0^\delta v_\infty \left(\frac{y}{\delta}\right)^{1/7} dy = \frac{7}{8} v_\infty \, \delta$$

$$\int_0^{\delta} v_x^2\, dy = \int_0^{\delta} v_\infty^2 \left(\frac{y}{\delta}\right)^{2/7} dy = \frac{7}{9} v_\infty^2\, \delta$$

$$\tau_w = \mu \frac{dv_x}{dy}\bigg|_{y=0} = \mu\left[\frac{d}{dy}\left(v_\infty \left(\frac{y}{\delta}\right)^{1/7}\right)\right]_{y=0} = \mu\, \delta^{-\frac{1}{7}} \frac{1}{7}\left[\frac{1}{y^{-\frac{6}{7}}}\right]_{y=0} = \mu\, \delta^{-\frac{1}{7}} \frac{1}{7}\left[\frac{1}{0}\right] = \infty!$$

Therefore, we have a problem that the turbulent velocity profile that is based on experience $\frac{v_x}{v_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$, predicts infinite shear stress τ_w. However, from experiment:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho v_\infty^2} = 0.045\left(\frac{\mu}{\rho v_\infty \delta}\right)^{\frac{1}{4}}$$

$$\Rightarrow \tau_w = 0.045 \frac{1}{2} \rho v_\infty^2 \left(\frac{\mu}{\rho v_\infty \delta}\right)^{\frac{1}{4}}$$

Recall, the momentum balance on a control volume of the boundary layer:

$$v_\infty \frac{d}{dx}\int_0^{\delta} v_x dy = \frac{\tau_w}{\rho} + \frac{d}{dx}\int_0^{\delta} v_x^2 dy$$

$$\Rightarrow v_\infty \frac{d}{dx}\left(\frac{7}{8} v_\infty\, \delta\right) = \frac{1}{\rho}\left(0.045 \frac{1}{2} \rho v_\infty^2 \left(\frac{\mu}{\rho v_\infty \delta}\right)^{\frac{1}{4}}\right) + \frac{d}{dx}\left(\frac{7}{9} v_\infty^2\, \delta\right)$$

$$\Rightarrow \left(\frac{7}{8} - \frac{7}{9}\right) v_\infty^2 \frac{d\delta}{dx} = \frac{0.045}{2} v_\infty^2 \left(\frac{\mu}{\rho v_\infty \delta}\right)^{\frac{1}{4}}$$

Integrating and simplifying:

$$\boxed{\Rightarrow \frac{\delta}{x} = \frac{0.376}{(\mathrm{Re}_x)^{\frac{1}{5}}}} \quad \text{and} \quad \boxed{C_f = 0.045\left(\frac{\mu}{\rho v_\infty \delta}\right)^{\frac{1}{4}} = \frac{0.0576}{(\mathrm{Re}_x)^{\frac{1}{5}}}}$$

Summary

Laminar B.L.

$\mathrm{Re}_{cr} = 3.2 \times 10^5$ $\Rightarrow x_{cr} = 3.2\times10^5 \dfrac{\mu}{\rho v_\infty}$ $\mathrm{Re}_x \le \mathrm{Re}_{cr}$ $\Rightarrow x \le x_{cr}$

$\dfrac{v_x}{v_\infty} = f(\zeta), \quad \zeta = \dfrac{y}{\delta(x)}$ $C_f = \dfrac{\tau_w}{\frac{1}{2}\rho v_\infty^2}$

$f(\zeta)$	$\dfrac{\delta}{x}$	C_f
Exact Solution	Numerical See Fig 8.5	$\dfrac{0.664}{\sqrt{\mathrm{Re}_x}}$
$\sin\left(\frac{\pi}{2}\zeta\right)$	$\dfrac{4.79}{\sqrt{\mathrm{Re}_x}}$	$\dfrac{0.656}{\sqrt{\mathrm{Re}_x}}$
ζ	Do it yourself !	$\dfrac{0.578}{\sqrt{\mathrm{Re}_x}}$
$2\zeta - \zeta^2$	Do it yourself !	$\dfrac{0.730}{\sqrt{\mathrm{Re}_x}}$
$\frac{3}{2}\zeta - \frac{1}{2}\zeta^3$	Do it yourself !	$\dfrac{0.646}{\sqrt{\mathrm{Re}_x}}$
$2\zeta - 2\zeta^3 + \zeta^4$	Do it yourself !	$\dfrac{0.686}{\sqrt{\mathrm{Re}_x}}$

Turbulent B.L.

$\mathrm{Re}_L > \mathrm{Re}_{cr}$ $C_f = \dfrac{\tau_w}{\frac{1}{2}\rho v_\infty^2} = 0.045\left(\dfrac{\mu}{\rho v_\infty \delta}\right)^{\frac{1}{4}}$

$\dfrac{\delta}{x} = \dfrac{0.376}{(\mathrm{Re}_x)^{1/5}}$ equation (8.61 of textbook)

$C_f \equiv \dfrac{\tau_w}{\frac{1}{2}\rho v_\infty^2} = \dfrac{0.0576}{(\mathrm{Re}_x)^{1/5}}$ equation (8.62 of textbook)

Calculation of Drag Force Over a Flat Plate

Drag Force: $F_D = A\,\tau_w$

However, $A = f(x)$ and $\tau_w = f(x)$

$$\Rightarrow F_D = \int \tau_w \, dA; \qquad dA = dx \ \ (\text{Area per unit width})$$

$$\Rightarrow F_D = \int_0^L \tau_w dx \qquad (\text{Drag Force per unit width})$$

$$x_{cr} = 3.2\times 10^5 \frac{\mu}{\rho v_\infty} \qquad \Rightarrow F_D = \int_0^{x_{cr}} (\tau_w)_{\text{Laminar}} dx + \int_{x_{cr}}^{L} (\tau_w)_{\text{Turbulent}} dx$$

$$F_D = \left\{ \int_0^{x_{cr}} (\tau_w)_{\text{Laminar}} dx + \int_{x_{cr}}^{L} (\tau_w)_{\text{Turbulent}} dx \right\} \times \frac{\frac{1}{2}\rho v_\infty^2}{\frac{1}{2}\rho v_\infty^2} = \frac{1}{2}\rho v_\infty^2 \left[\int_0^{x_{cr}} (C_f)_{\text{Laminar}} dx + \int_{x_{cr}}^{L} (C_f)_{\text{Turbulent}} dx \right]$$

$$= \frac{1}{2}\rho v_\infty^2 \left[\int_0^{x_{cr}} \frac{0.656}{\sqrt{\text{Re}_x}} dx + \int_{x_{cr}}^{L} \frac{0.0576}{(\text{Re}_x)^{1/5}} dx \right] = \frac{1}{2}\rho v_\infty^2 \left[\frac{0.656}{\sqrt{\frac{\rho v_\infty}{\mu}}} \int_0^{x_{cr}} x^{-\frac{1}{2}} dx + \frac{0.0576}{\left(\frac{\rho v_\infty}{\mu}\right)^{1/5}} \int_{x_{cr}}^{L} x^{-\frac{1}{5}} dx \right]$$

$$F_D = \frac{1}{2}\rho v_\infty^2 \left[\frac{0.656}{\sqrt{\frac{\rho v_\infty}{\mu}}} (2)\left(x_{\text{cr}}^{\frac{1}{2}} \right) + \frac{0.0576}{\left(\frac{\rho v_\infty}{\mu}\right)^{1/5}} \left(\frac{5}{4}\right)\left(L^{\frac{4}{5}} - x_{\text{cr}}^{\frac{4}{5}} \right) \right]$$

$$F_D = \frac{1}{2}\rho v_\infty^2 \left[0.656(2)\frac{x_{\text{cr}}}{\sqrt{\text{Re}_{\text{cr}}}} + 0.0576\left(\frac{5}{4}\right)\left(\frac{L}{(\text{Re}_L)^{1/5}} - \frac{x_{\text{cr}}}{(\text{Re}_{\text{cr}})^{1/5}} \right) \right]$$

Note: the above analysis for calculation of the drag force per unit width is done for the case when $L > x_{\text{cr}}$ ($Re_L > Re_{\text{cr}}$). However, for the case when $L < x_{\text{cr}}$ ($Re_L < Re_{\text{cr}}$) the flow over the entire length of the plate is laminar and the drag force per unit width:

$$F_D = \int_0^L (\tau_w)_{\text{Laminar}} dx = \frac{1}{2}\rho v_\infty^2 \int_0^L (C_f)_{\text{Laminar}} dx = \frac{1}{2}\rho v_\infty^2 \left[\int_0^L \frac{0.656}{\sqrt{\text{Re}_x}} dx \right] = \frac{1}{2}\rho v_\infty^2 \left[0.656(2)\frac{L}{\sqrt{\text{Re}_L}} \right]$$

Example

Air (ρ_{air} = 0.075 lb_m/ft^3, mair = $1.35x10^{-5}$ lb_m/ft s) flows over a flat plat with a velocity 50 ft/s. Calculate the drug force over the flat plate if:

(a) The plate is 1 ft long along the direction of flow.
(b) The plate is 5 ft long along the direction of flow.

Solution

(a) $$\mathrm{Re}_L = \frac{\rho v_\infty L}{\mu} = \frac{(0.075)(50)(1)}{1.35\times 10^{-5}} = 2.8\times 10^5 < 3.2\times 10^5$$

In this case, the flow over the entire plate is laminar:

$$F_D = \frac{1}{2}\rho\, v_\infty^2\left[0.656(2)\frac{L}{\sqrt{\mathrm{Re}_L}}\right] = \frac{1}{2}\times 0.075\times 50^2\left[0.656(2)\frac{1}{\sqrt{2.8\times 10^5}}\right]$$

$$= 0.23\frac{\mathrm{lb_m\ ft}}{\mathrm{s}^2}\ \frac{1}{\mathrm{ft}} = 7.2\times 10^{-3}\frac{\mathrm{lb_f}}{\mathrm{ft}}$$

(b) $$\mathrm{Re}_L = \frac{\rho v_\infty L}{\mu} = \frac{(0.075)(50)(5)}{1.8\times 10^{-4}(0.075)} = 1.39\times 10^6 > 3.2\times 10^5$$

In this case, the flow undergoes transition from laminar to turbulent. The transition takes place at:

$$x_{\mathrm{cr}} = 3.2\times 10^5\frac{\mu}{\rho v_\infty} = 3.2\times 10^5\frac{1.18\times 10^{-4}}{50} = 1.152\,\mathrm{ft} < 5\,\mathrm{ft}$$

$$F_D = \frac{1}{2}\rho\, v_\infty^2\left[0.656(2)\frac{x_{\mathrm{cr}}}{\sqrt{\mathrm{Re}_{\mathrm{cr}}}} + 0.0576\left(\frac{5}{4}\right)\left(\frac{L}{(\mathrm{Re}_L)^{1/5}} - \frac{x_{\mathrm{cr}}}{\sqrt{(\mathrm{Re}_{\mathrm{cr}})^{1/5}}}\right)\right]$$

$$F_D = \frac{1}{2}\left(1.8\times 10^{-4}\times 0.075\right)50^2\left[0.656(2)\frac{1.152}{\sqrt{3.2\times 10^5}} + 0.0576\left(\frac{5}{4}\right)\left(\frac{5}{(1.39\times 10^6)^{1/5}} - \frac{1.152}{(3.2\times 10^5)^{1/5}}\right)\right]$$

$$F_D = 2.53\times 10^{-4}\frac{\mathrm{lb_m\ ft}}{\mathrm{s}^2}\ \frac{1}{\mathrm{ft}} = 7.86\times 10^{-6}\frac{\mathrm{lb_f}}{\mathrm{ft}}$$