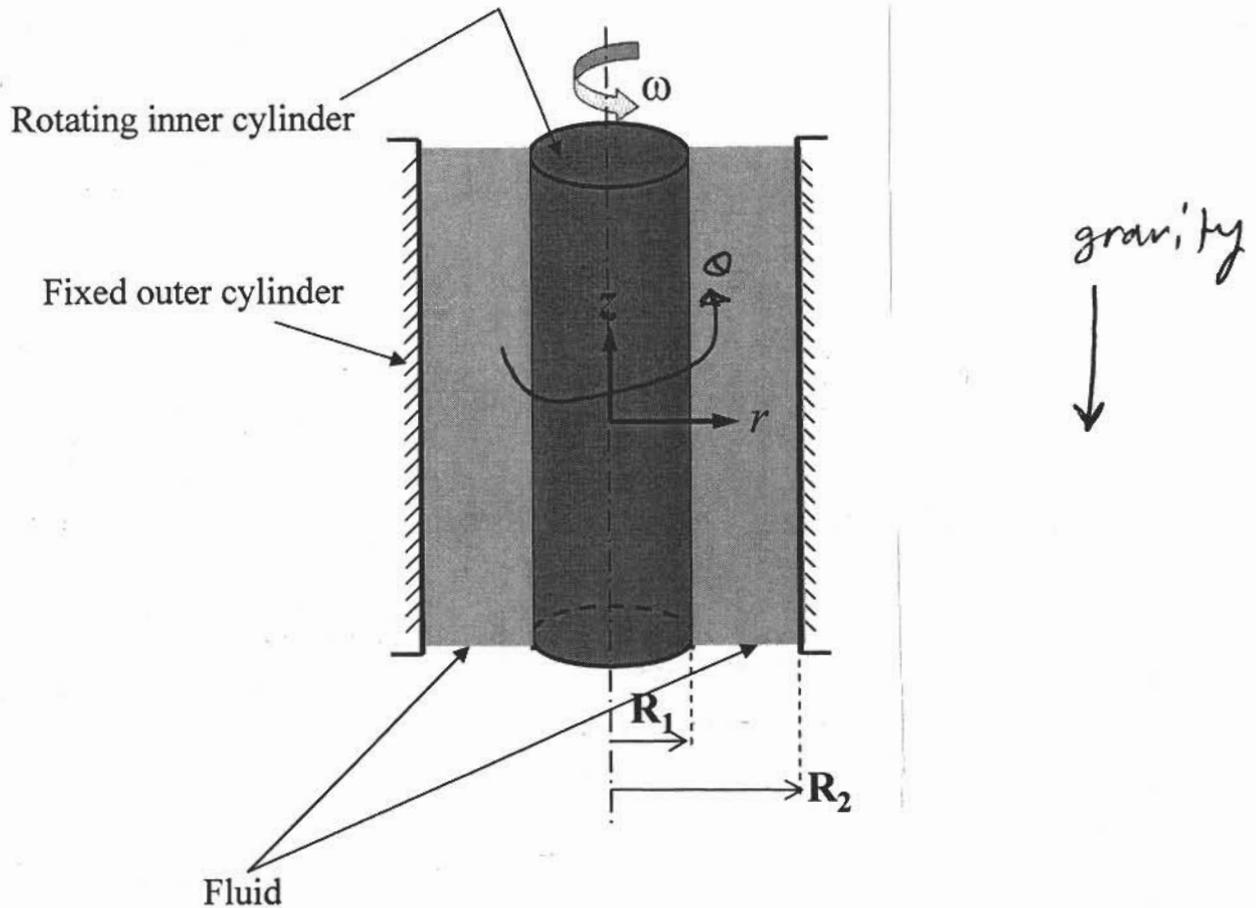


13. *Torque in a Couette viscometer*—M. Fig. P6.13 shows the horizontal cross section of a concentric cylinder or “Couette” viscometer, which is an apparatus for determining the viscosity μ of the fluid that is placed between the two vertical cylinders. The inner and outer cylinders have radii of r_1 and r_2 , respectively. If the inner cylinder is rotated with a steady angular velocity ω , and the outer cylinder is stationary, derive an expression for v_θ (the θ velocity component) as a function of radial location r .



Assumptions :

1. steady state
2. Newtonian fluid
3. There is one component of the velocity v_θ
 $v_r = v_z = 0$
4. Axisymmetric problem $\frac{\partial}{\partial \theta}(\text{any thing}) = 0$
5. $v_\theta = f(r)$ only.

SOLUTION:

Continuity equation: $\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho r v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0.$

$\Rightarrow 0 = 0$ no useful information

Navier-Stokes equations:

$V_r = V_z = 0$, $v_\theta = f(r)$ only and $\frac{\partial}{\partial \theta} () = 0$

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial(r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial(r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z$$

$$-\frac{V_\theta^2}{r} = -\frac{\partial p}{\partial r} \quad \text{--- (1)}$$

$$0 = \mu \frac{d}{dr} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) \quad \text{--- (2)}$$

$$0 = -\frac{\partial p}{\partial z} + \rho g_z \quad \text{--- (3)}$$

(3)

$$\textcircled{2} \Rightarrow \frac{d}{dr} \left(\frac{1}{r} \frac{d}{dr} (r v_\theta) \right) = 0$$

integrate once $\Rightarrow \frac{1}{r} \frac{d}{dr} (r v_\theta) = C_1$

\hookrightarrow again $\Rightarrow r v_\theta = \frac{C_1}{2} r^2 + C_2$

$$\Rightarrow v_\theta = C_1' r + \frac{C_2}{r} \quad (C_1' = \frac{C_1}{2})$$

boundary conditions:

$r = R_1$	$v_\theta = \omega R_1$
$r = R_2$	$v_\theta = 0$

BC1: $\omega R_1 = C_1' R_1 + \frac{C_2}{R_1}$

BC2: $0 = C_1' R_2 + \frac{C_2}{R_2}$

solve $\Rightarrow C_1' = \frac{-R_1 \omega}{\frac{R_2^2}{R_1} - R_1} \quad \neq \quad C_2 = \frac{R_1 \omega}{\frac{1}{R_1} - \frac{R_1}{R_2^2}}$

$$\Rightarrow v_\theta = \frac{R_1 \omega}{R_1 - \frac{R_2^2}{R_1}} r + \frac{R_1 \omega}{\frac{1}{R_1} - \frac{R_1}{R_2^2}} \frac{1}{r}$$

Stress: the is one component of the stress tensor

$$\tau_{r\theta} = \tau_{\theta r} = \mu r \frac{d}{dr} \left(\frac{v_\theta}{r} \right)$$

$$\Rightarrow \tau_{r\theta} = \frac{-2\mu R_1 \omega}{\frac{1}{R_1} - \frac{R_1}{R_2}} \frac{1}{r^2}$$

Force required to rotate inner cylinder:

$$F = \tau_{r\theta} \Big|_{r=R_1} * \underbrace{2\pi R_1 L}_{\substack{\downarrow \\ \text{Area of inner cylinder}}}$$

$$= \frac{-4\pi\mu R_1^2 \omega L}{\left(\frac{1}{R_1} - \frac{R_1}{R_2}\right)} \frac{1}{R_1^2}$$

$$\text{Torque} = F * \text{unit arm}$$

$$= F * R_1$$

$$\Rightarrow T = \frac{-4\pi\mu\omega L R_1}{\frac{1}{R_1} - \frac{R_1}{R_2}}$$

$$\Rightarrow \mu = \frac{T \left(\frac{1}{R_1} - \frac{R_1}{R_2}\right)}{-4\pi\omega L R_1}$$