Example 6.1—Flow Between Parallel Plates

Fig. E6.1.1 shows a fluid of viscosity μ that flows in the x direction between two rectangular plates, whose width is very large in the z direction when compared to their separation in the y direction. Such a situation could occur in a die when a polymer is being extruded at the exit into a sheet, which is subsequently cooled and solidified. Determine the relationship between the flow rate and the pressure drop between the inlet and exit, together with several other quantities of interest.

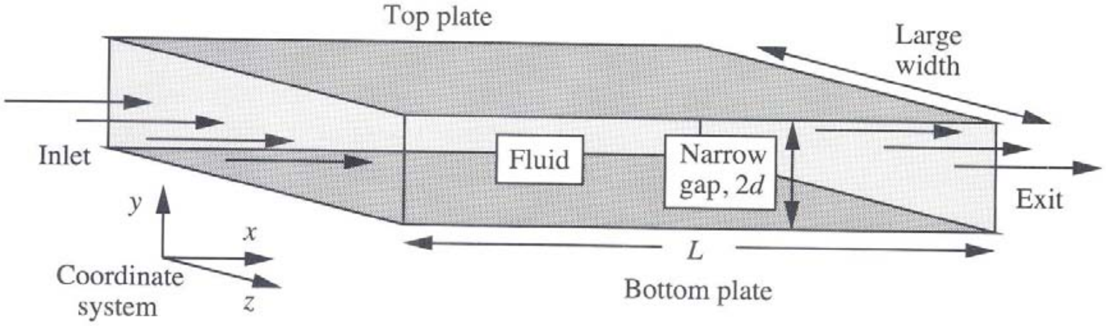


Fig. E6.1.1 Geometry for flow through a rectangular duct. The spacing between the plates is exaggerated in relation to their length.

Solution:

First we start with the continuity equation in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_x)}{\partial x} + \frac{\partial (\rho v_y)}{\partial y} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

For incompressible fluid, the density is constant:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Since the flow is in the x-direction only, then we have only one component of the velocity $v_x \neq 0$ and $v_y = v_z = 0$, continuity equation simplifies to:

$$\frac{\partial v_x}{\partial x} = 0$$

Conclusion the simplified continuity equation implies that v_x is not a function of x

$$v_x \neq f(x)$$

Also, for wide plates in z-direction, $v_x \neq f(z)$

Second we use the Navier-Stokes equations in Cartesian coordinates:

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x,$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y,$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z.$$

To simplify Navier-Stokes equations we can utilize the following results:

1. Steady state: $\frac{\partial(\text{any thing})}{\partial t} = 0$
2. Plates are wide in *z*-direction: $\frac{\partial(\text{any thing})}{\partial z} = 0$
3. We have one component of the velocity $v_x \neq 0$ and $v_y = v_z = 0$
4. $v_x \neq f(x, z)$ it is only a function of *y*: $v_x = f(y)$.
5. $g_x = g_z = 0$. Only we have gravity in y direction $g_y = -g$.

Therefore the N-S equations simplify to:

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{d^2 v_x}{dy^2} \qquad (1)$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \qquad (2)$$

$$0 = 0 \qquad (3)$$

Pressure Profile:

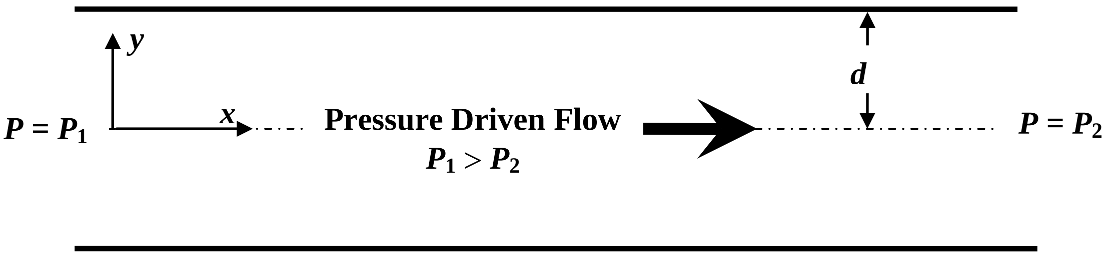
Integrate equation (2):

$$\frac{\partial p}{\partial y} = -\rho g \qquad \Rightarrow \qquad p = -\rho\, g\, y + f(x)$$

Note for we integrated the pressure with respect to y, hence the constant of integration is not a function of y but could be a function of x. In pressure driven flows like this problem the pressure changes linearly along the direction of the flow:

$$f(x) = a\ x + b$$

$$\Rightarrow p = -\rho\, g\, y + (a\ x + b)$$



Now we apply boundary conditions for the pressure:

$$@\ \ x = 0 \text{ and } y = 0 \ \rightarrow \quad p = p_1$$
$$@\ \ x = L \text{ and } y = 0 \ \rightarrow \quad p = p_2$$

$$\left.\begin{array}{l} p_1 = -\rho\, g\,(0) + (a\ 0 + b) \\ p_2 = -\rho\, g\,(0) + (a\ L + b) \end{array}\right\} \Rightarrow a = \frac{p_2 - p_1}{L} = \frac{\Delta P}{L} \quad \text{and} \quad b = p_1$$

Therefore, the pressure profile is given by:

$$p = -\rho\, g\, y + \left(\frac{\Delta P}{L}x + p_1\right) \qquad (4)$$

Velocity Profile:

Recall equation (1):

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 v_x}{dy^2}$$

From equation (4): $\frac{\partial p}{\partial x} = \frac{\Delta p}{L}$, substitute in equation (1):

$$\frac{d^2 v_x}{dy^2} = \frac{1}{\mu}\frac{\Delta p}{L}$$

Integrate the above equation once:

$$\frac{dv_x}{dy} = \frac{1}{\mu}\frac{\Delta p}{L} y + C_1$$

Integrate second time:

$$v_x = \frac{1}{2\mu}\frac{\Delta p}{L} y^2 + C_1 y + C_2$$

To find the constants of integration apply the following Boundary Conditions:

$$y = 0 \qquad \frac{dv_x}{dy} = 0 \qquad (\text{Velocity is maximum at center between the two plates})$$

$$y = d \qquad v_x = 0 \qquad (\text{No - Slip Boundary Condition})$$

$$0 = \frac{1}{\mu}\frac{\Delta p}{L} 0 + C_1 \qquad \Rightarrow \qquad C_1 = 0$$

$$0 = \frac{1}{2\mu}\frac{\Delta p}{L} d^2 + C_1 y + C_2 \qquad \Rightarrow \qquad C_2 = -\frac{1}{2\mu}\frac{\Delta p}{L} d^2$$

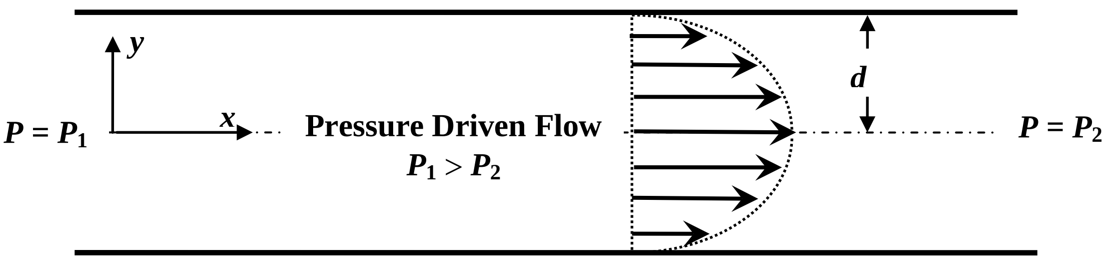
$$v_x = \frac{1}{2\mu}\frac{\Delta p}{L} y^2 - \frac{1}{2\mu}\frac{\Delta p}{L} d^2$$

Rearrange:

$$v_x = \frac{1}{2\mu}\frac{\Delta p}{L}\left(y^2 - d^2\right)$$

The above equation is similar to an equation of a parabola and hence the velocity profile is called a parabolic velocity profile, see figure below:

Parabolic Velocity Profile



Volumetric Flow Rate:

$$Q = \int_A v_x \, dA$$

In this problem:

$$dA = dy\, dz$$

$$Q = \int_0^W \int_{-d}^{+d} v_x dy\, dz$$

If we consider the volumetric flow rate per unit width $W = 1$:

$$Q = \int_{-d}^{+d} v_x dy$$

$$Q = \int_{-d}^{d} \frac{1}{2\mu}\frac{\Delta p}{L}\left(y^2 - d^2\right)dy \;=\; \frac{1}{2\mu}\frac{\Delta p}{L}\left(\frac{y^3}{3} - d^2 y\right)_{-d}^{d}$$

$$Q = \frac{2d^3}{3\mu}\frac{-\Delta p}{L}$$

Maximum Velocity:

$$v_{Max} = v_x\big|_{y=0} = \frac{1}{2\mu}\frac{-\Delta p}{L}d^2$$

Mean Velocity:

$$v_m = \frac{Q(\text{perunit width})}{A(\text{perunit width})} = \frac{\dfrac{2d^3}{3\mu}\dfrac{-\Delta p}{L}}{2d} = \frac{d^3}{3\mu}\frac{-\Delta p}{L} = \frac{2}{3}v_{Max}$$

Shear Stress:

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \mu \begin{bmatrix} \left(2\dfrac{\partial v_x}{\partial x}\right) & \left(\dfrac{\partial v_x}{\partial y} + \dfrac{\partial v_y}{\partial x}\right) & \left(\dfrac{\partial v_x}{\partial z} + \dfrac{\partial v_z}{\partial x}\right) \\ \left(\dfrac{\partial v_y}{\partial x} + \dfrac{\partial v_x}{\partial y}\right) & \left(2\dfrac{\partial v_y}{\partial y}\right) & \left(\dfrac{\partial v_y}{\partial z} + \dfrac{\partial v_z}{\partial y}\right) \\ \left(\dfrac{\partial v_z}{\partial x} + \dfrac{\partial v_x}{\partial z}\right) & \left(\dfrac{\partial v_z}{\partial y} + \dfrac{\partial v_y}{\partial z}\right) & \left(2\dfrac{\partial v_z}{\partial z}\right) \end{bmatrix}$$

Recall simplifications:

1. $\frac{\partial(\text{any thing})}{\partial z} = 0$
2. $v_x \neq 0$ and $v_y = v_z = 0$
3. $v_x \neq f(x, z)$ it is only a function of y: $v_x = f(y)$.

This leads to the following simplifications:

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \mu \begin{bmatrix} 0 & \left(\frac{\partial v_x}{\partial y}\right) & 0 \\ \left(\frac{\partial v_x}{\partial y}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the only nonzero stresses are: $\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y}\right)$.

Recall, $v_x = \frac{1}{2\mu}\frac{\Delta p}{L}\left(y^2 - d^2\right)$

$$\Rightarrow \tau_{yx} = \frac{\Delta p}{L} y$$

$$\Delta p = p_2 - p_1 = -ve$$

$$\text{if y is} + \text{ve} \Rightarrow \tau_{yx} \text{ is } -ve$$

$$\text{if y is} - \text{ve} \Rightarrow \tau_{yx} \text{ is } +ve$$

Shear Stress Profile