Microscopic Momentum Balance

CHE204

Prepared By:
Dr. Usamah Al-Mubaiyedh

Rate of Momentum $_{in}$ - Rate of Momentum $_{out}$ + Sum of Forces = Accumulation

$$\dot{M}_{in} - \dot{M}_{out} + \left(\begin{array}{c} \text{Pressure Forces} + \\ \text{Friction (Stress) Forces} + \\ \text{Body Forces} \end{array} \right) = \text{Accum.}$$

Recall that the momentum is a vector quantity:

$$\dot{M} = \overbrace{m}^{\text{Mass Scaler}} \quad \overbrace{\vec{v}}^{\text{Velocity Vector}}$$

$$\dot{M}_{in}\Big|_x - \dot{M}_{out}\Big|_x + \begin{pmatrix} \text{Pressure Forces} + \\ \text{Friction (Stress) Forces} + \\ \text{Body Forces} \end{pmatrix}_x = \text{Accumulation}\Big|_x$$

$$\dot{M} = \overbrace{m}^{\text{Mass Scaler}} \qquad \overbrace{\vec{v}}^{\text{Velocity Vector}}$$

$$m = \rho\, \vec{v}.\vec{A} = \rho\left(v_x\, \hat{i}, v_y\, \hat{j}, v_z\, \hat{k}\right) \bullet \left(\Delta y \Delta z\, \hat{i}, \Delta x \Delta z\, \hat{j}, \Delta x \Delta y\, \hat{k}\right)$$

$$\dot{M}_x = \rho\left(v_x\, \hat{i}, v_y\, \hat{j}, v_z\, \hat{k}\right) \bullet \left(\Delta y \Delta z\, \hat{i}, \Delta x \Delta z\, \hat{j}, \Delta x \Delta y\, \hat{k}\right) v_x$$

$$\dot{M}_y = \rho\left(v_x\, \hat{i}, v_y\, \hat{j}, v_z\, \hat{k}\right) \bullet \left(\Delta y \Delta z\, \hat{i}, \Delta x \Delta z\, \hat{j}, \Delta x \Delta y\, \hat{k}\right) v_y$$

$$\dot{M}_z = \rho\left(v_x\, \hat{i}, v_y\, \hat{j}, v_z\, \hat{k}\right) \bullet \left(\Delta y \Delta z\, \hat{i}, \Delta x \Delta z\, \hat{j}, \Delta x \Delta y\, \hat{k}\right) v_z$$

$$\dot{M} = \overbrace{m}^{\text{Mass Scaler}} \quad \overbrace{\vec{v}}^{\text{Velocity Vector}}$$

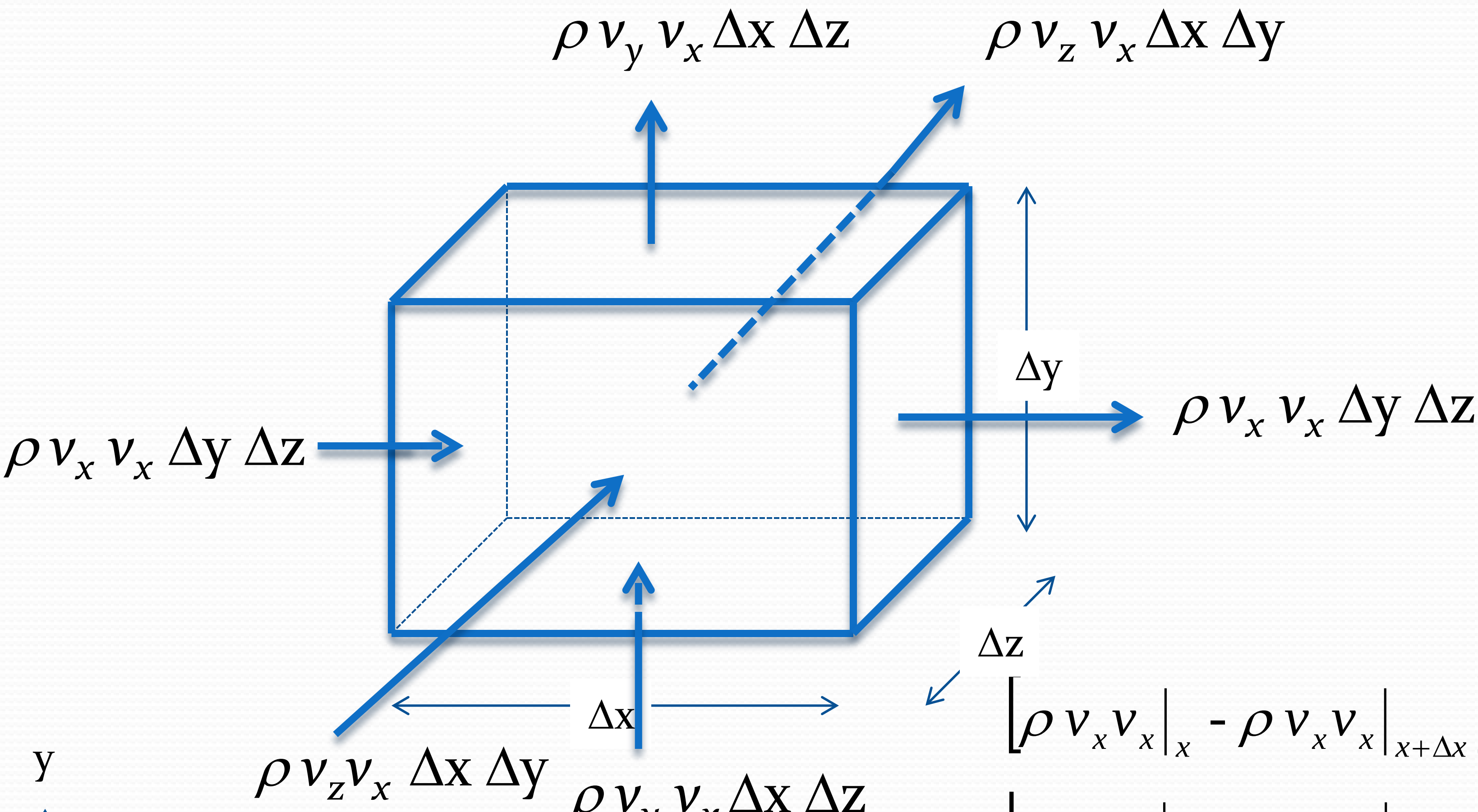
$$m = \rho\, \vec{v}.\vec{A} = \rho\left(v_x\, \hat{i}, v_y\, \hat{j}, v_z\, \hat{k}\right) \bullet \left(\Delta y \Delta z\, \hat{i}, \Delta x \Delta z\, \hat{j}, \Delta x \Delta y\, \hat{k}\right)$$

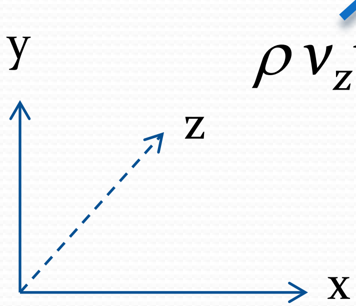
$$\dot{M}_x = \rho\left(v_x\, \hat{i}, v_y\, \hat{j}, v_z\, \hat{k}\right) \bullet \left(\Delta y \Delta z\, \hat{i}, \Delta x \Delta z\, \hat{j}, \Delta x \Delta y\, \hat{k}\right) v_x$$

$$\dot{M}_y = \rho\left(v_x\, \hat{i}, v_y\, \hat{j}, v_z\, \hat{k}\right) \bullet \left(\Delta y \Delta z\, \hat{i}, \Delta x \Delta z\, \hat{j}, \Delta x \Delta y\, \hat{k}\right) v_y$$

$$\dot{M}_z = \rho\left(v_x\, \hat{i}, v_y\, \hat{j}, v_z\, \hat{k}\right) \bullet \left(\Delta y \Delta z\, \hat{i}, \Delta x \Delta z\, \hat{j}, \Delta x \Delta y\, \hat{k}\right) v_z$$

Consider the x-component of the momentum that enters and leaves out of the six surfaces of the control volume





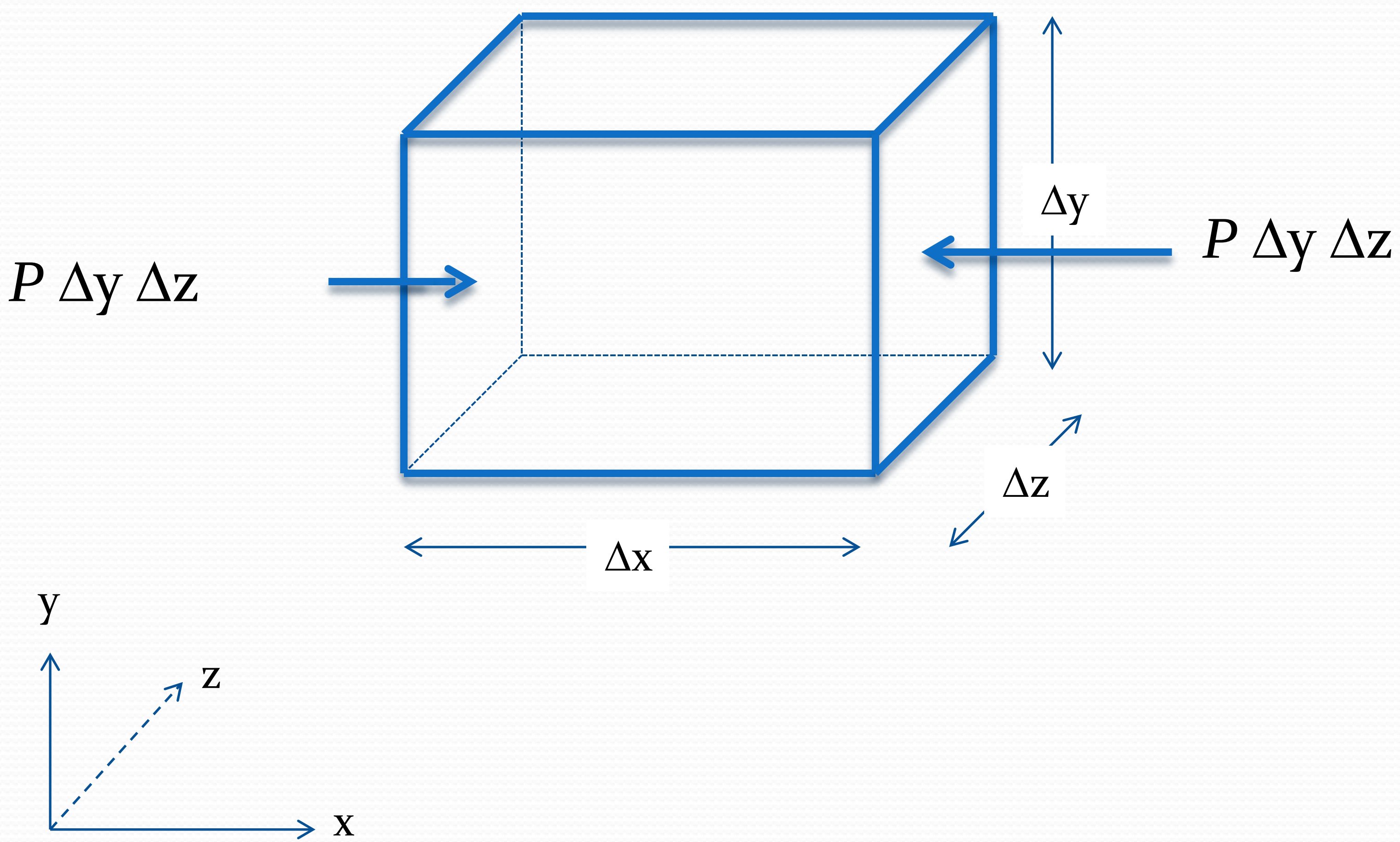
$$\left[\rho v_x v_x\big|_x - \rho v_x v_x\big|_{x+\Delta x}\right]\Delta y \Delta z +$$

$$\left[\rho v_y v_x\big|_y - \rho v_y v_x\big|_{y+\Delta y}\right]\Delta x \Delta z +$$

$$\left[\rho v_z v_x\big|_z - \rho v_z v_x\big|_{z+\Delta z}\right]\Delta x \Delta y$$

Consider the forces that act in the x-direction

Pressure Forces in the x-direction



Stress Tensor

$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

$$\tau \underbrace{y}_{\text{Plane}} \underbrace{x}_{\text{Direction}}$$

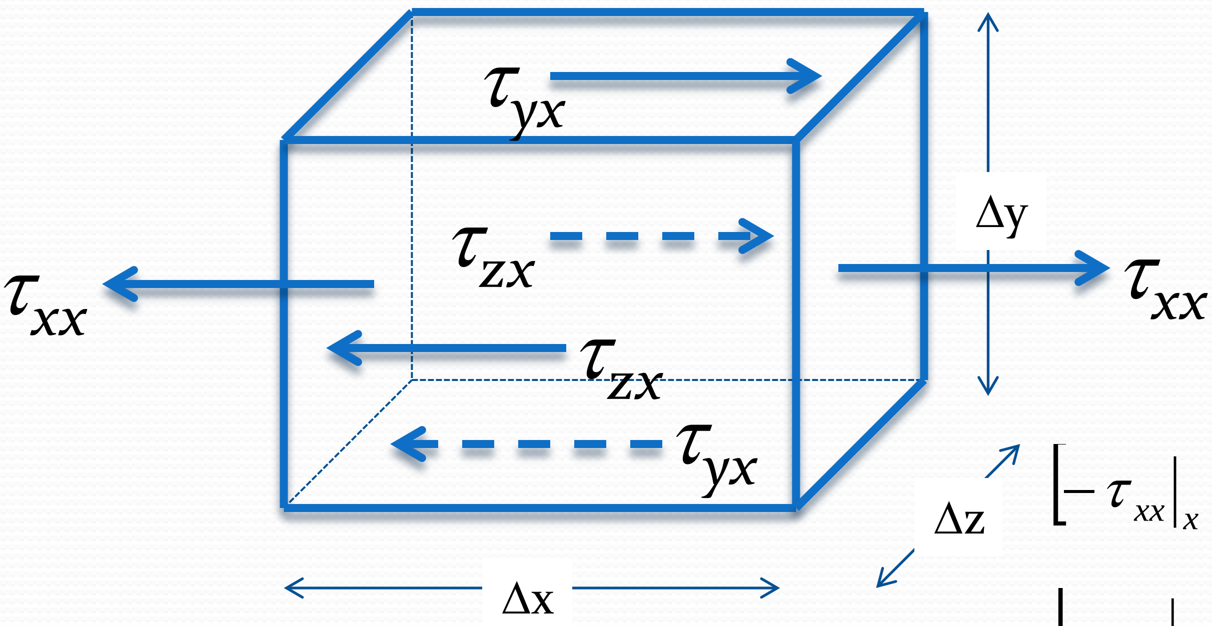
τ_{yx} acts in a plane normal to the y axis and it is pointing in the x-direction

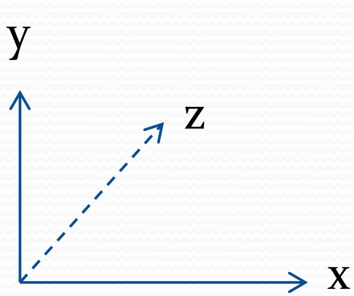
$$\tau_{yx} = \tau_{yx} \quad , \quad \tau_{zx} = \tau_{xz} , \ldots$$

stress tensor is symmetrical

Consider the forces that act in the x-direction

Stress (Friction) Forces in the x-direction





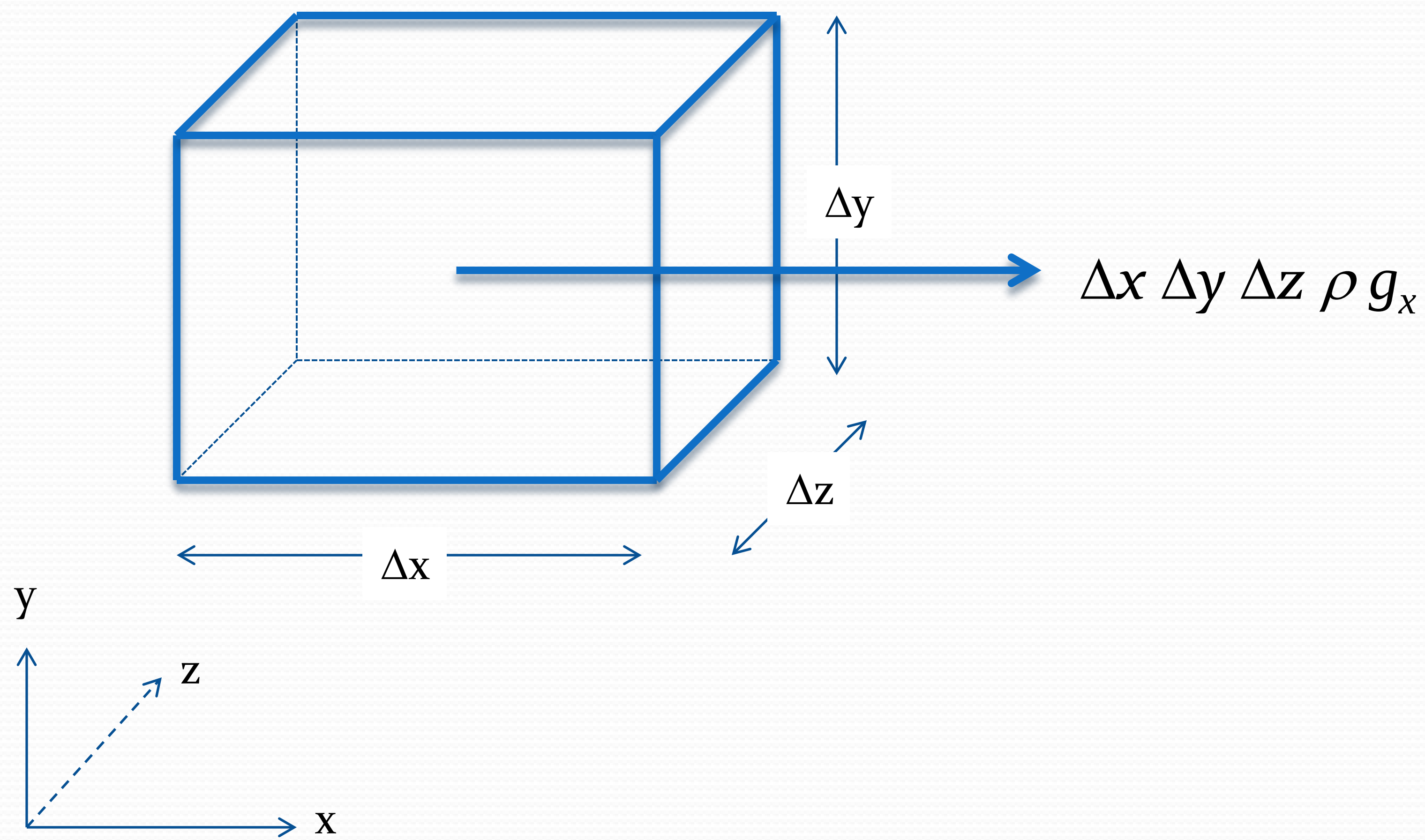
$$\left[-\tau_{xx}\big|_{x}+\tau_{xx}\big|_{x+\Delta x}\right]\Delta y\Delta z+$$

$$\left[-\tau_{yx}\big|_{y}+\tau_{yx}\big|_{y+\Delta y}\right]\Delta x\Delta z+$$

$$\left[-\tau_{zx}\big|_{z}+\tau_{zx}\big|_{z+\Delta z}\right]\Delta x\Delta y$$

Consider the forces that act in the x-direction

Gravity Force in the x-direction



$$\left[\rho\, v_x v_x\big|_x - \rho\, v_x v_x\big|_{x+\Delta x}\right]\Delta y\Delta z + \left[\rho\, v_y v_x\big|_y - \rho\, v_y v_x\big|_{y+\Delta y}\right]\Delta x\Delta z + \left[\rho\, v_z v_x\big|_z - \rho\, v_z v_x\big|_{z+\Delta z}\right]\Delta x\Delta y +$$

$$\left[-\tau_{xx}\big|_x + \tau_{xx}\big|_{x+\Delta x}\right]\Delta y\Delta z + \left[-\tau_{yx}\big|_y + \tau_{yx}\big|_{y+\Delta y}\right]\Delta x\Delta z + \left[-\tau_{zx}\big|_z + \tau_{zx}\big|_{z+\Delta z}\right]\Delta x\Delta y +$$

$$\left[P_x - P_{x+\Delta x}\right]\Delta y\Delta z +$$

$$\rho\,\Delta x\Delta y\Delta z\;\; g_x$$

$$= \frac{\partial}{\partial t}\left[\rho\,\Delta x\Delta y\Delta z\, v_x\right]$$

Divide the whole equation by Δx Δy Δz
then take the limit when Δx Δy Δz approaches zero

x-component of the momentum balance

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+\rho g_x$$

Similarly

y-component of the momentum balance

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+\rho g_y$$

z-component of the momentum balance

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z$$

Microscopic Continuity and Momentum Balance Equations in Cartesian Coordinates

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}=0$$

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+\rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z$$

We have four (4) equations with ten (10) unknowns!

$$p, v_x, v_y, v_z, \tau_{xx}, \tau_{xy}, \tau_{xz}, \tau_{yy}, \tau_{yz}, \tau_{zz}$$

Newton's Law of Viscosity

For a Newtonian fluid (simple fluid without microstructure):

$$\underline{\underline{\tau}} = \overbrace{\mu}^{\text{Constant Viscosity}} \; \overbrace{\underline{\underline{\dot{\gamma}}}}^{\text{Shear Rate Tensor}}$$

$$\underline{\underline{\dot{\gamma}}} = \nabla v + \nabla v^T$$

For Newtonian Fluid:
The viscosity is constant and independent of shear rate

Cartesian Coordinates

$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

$$\underline{\underline{\dot{\gamma}}} = \begin{bmatrix} \left(2\frac{\partial v_x}{\partial x}\right) & \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\ \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) & \left(2\frac{\partial v_y}{\partial y}\right) & \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \\ \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) & \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) & \left(2\frac{\partial v_z}{\partial z}\right) \end{bmatrix}$$

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \mu \begin{bmatrix} \left(2\frac{\partial v_x}{\partial x}\right) & \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\ \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) & \left(2\frac{\partial v_y}{\partial y}\right) & \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \\ \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) & \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) & \left(2\frac{\partial v_z}{\partial z}\right) \end{bmatrix}$$

Cylindrical Coordinates

$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{bmatrix}$$

$$\underline{\underline{\dot{\gamma}}} = \begin{bmatrix} \left(2\frac{\partial v_r}{\partial r}\right) & \left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) & \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \\ \left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) & \left(2\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + 2\frac{v_r}{r}\right) & \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \\ \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) & \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) & \left(2\frac{\partial v_z}{\partial z}\right) \end{bmatrix}$$

$$\begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{bmatrix} = \mu \begin{bmatrix} \left(2\frac{\partial v_r}{\partial r}\right) & \left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) & \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) \\ \left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) & \left(2\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + 2\frac{v_r}{r}\right) & \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \\ \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) & \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) & \left(2\frac{\partial v_z}{\partial z}\right) \end{bmatrix}$$

Navier Stokes
for Incompressible Newtonian Fluid
(Chartiesian Coordinates)

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x,$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y,$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_x.$$

Navier Stokes
for Incompressible Newtonian Fluid
(Cylindrical Coordinates)

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial(rv_\theta)}{\partial\theta}+\frac{\partial v_z}{\partial z}=0$$

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial x}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial y}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r,$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial x}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta,$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial x}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z.$$