Microscopic Mass Balance

CHE204

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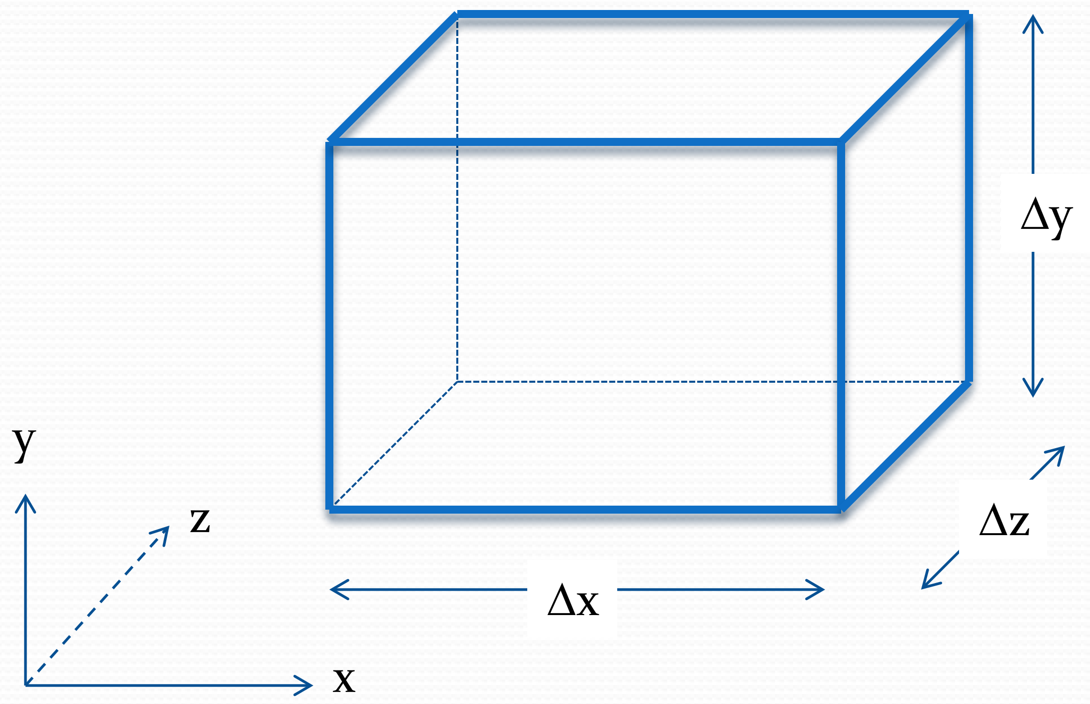
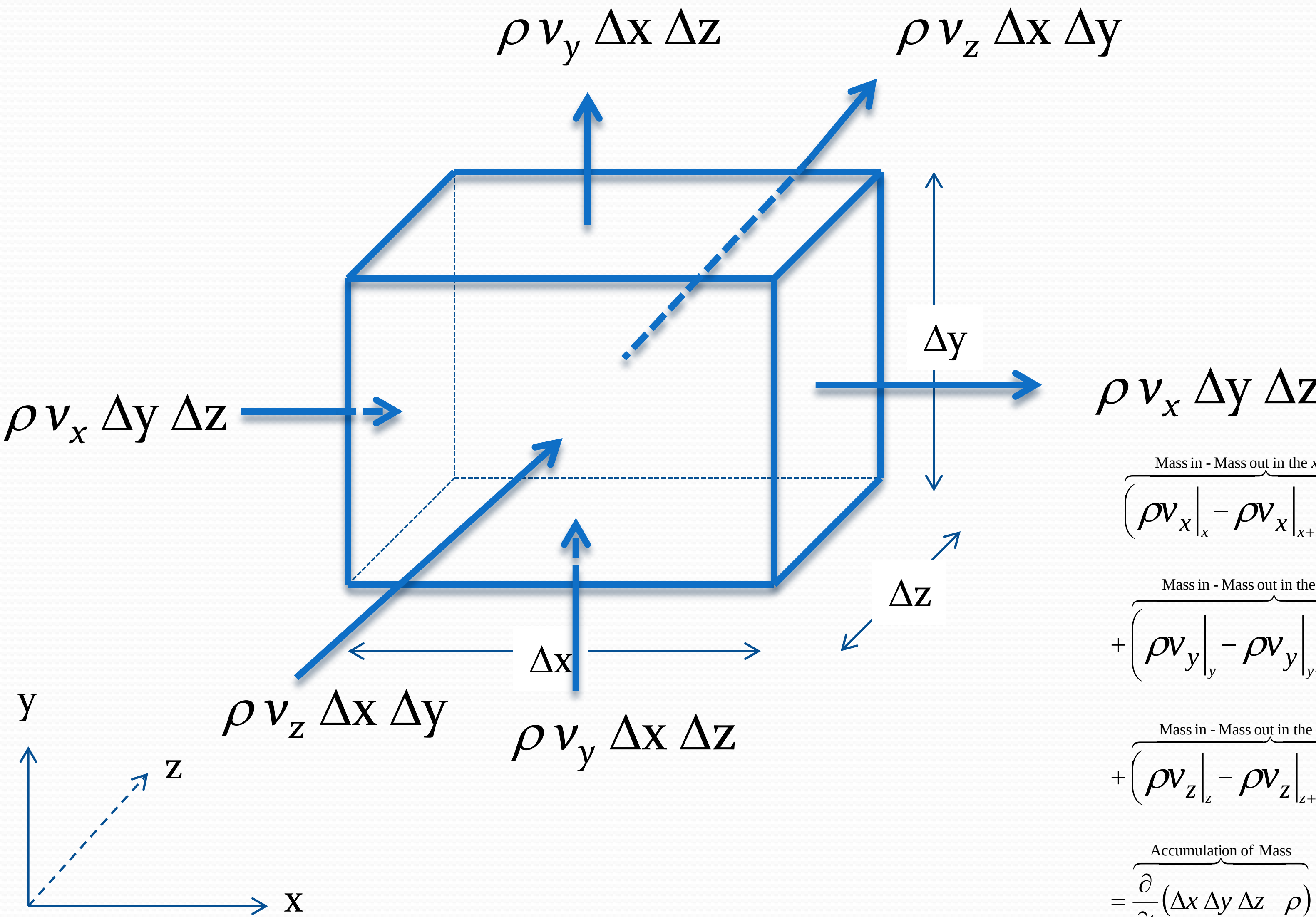
Introduce the x y z coordinate system

Introduce a control volume (microscopic system) of dimensions $\Delta x\ \Delta y\ \Delta z$



Consider the mass that enters and leaves out of the six surfaces of the control volume



$$\overbrace{\left(\rho v_x\big|_x - \rho v_x\big|_{x+\Delta x}\right)\Delta y\ \Delta z}^{\text{Mass in - Mass out in the } x \text{ direction}}$$

$$+\overbrace{\left(\rho v_y\big|_y - \rho v_y\big|_{y+\Delta y}\right)\Delta x\ \Delta z}^{\text{Mass in - Mass out in the } y \text{ direction}}$$

$$+\overbrace{\left(\rho v_z\big|_z - \rho v_z\big|_{z+\Delta z}\right)\Delta x\ \Delta y}^{\text{Mass in - Mass out in the } z \text{ direction}}$$

$$=\overbrace{\frac{\partial}{\partial t}\left(\Delta x\ \Delta y\ \Delta z\ \ \rho\right)}^{\text{Accumulation of Mass}}$$

Mass Balance on Control Volume:

Mass $_{in}$ – Mass $_{out}$ = Accumulation of Mass

$$\overbrace{\left(\rho v_x\big|_x - \rho v_x\big|_{x+\Delta x}\right)\Delta y\,\Delta z}^{\text{Mass in - Mass out in the } x \text{ direction}} + \overbrace{\left(\rho v_y\big|_y - \rho v_y\big|_{y+\Delta y}\right)\Delta x\,\Delta z}^{\text{Mass in - Mass out in the } y \text{ direction}} + \overbrace{\left(\rho v_z\big|_z - \rho v_z\big|_{z+\Delta z}\right)\Delta x\,\Delta y}^{\text{Mass in - Mass out in the } z \text{ direction}} = \overbrace{\frac{\partial}{\partial t}\left(\Delta x\,\Delta y\,\Delta z\;\;\rho\right)}^{\text{Accumulation of Mass}}$$

$$\Delta x\,\Delta y\,\Delta z = \text{Constant}$$

$$\left(\rho v_x\big|_x - \rho v_x\big|_{x+\Delta x}\right)\Delta y\,\Delta z + \left(\rho v_y\big|_y - \rho v_y\big|_{y+\Delta y}\right)\Delta x\,\Delta z + \left(\rho v_z\big|_z - \rho v_z\big|_{z+\Delta z}\right)\Delta x\,\Delta y = \Delta x\,\Delta y\,\Delta z\frac{\partial}{\partial t}(\rho)$$

$$\left(\rho v_x\Big|_x - \rho v_x\Big|_{x+\Delta x}\right)\Delta y\,\Delta z + \left(\rho v_y\Big|_y - \rho v_y\Big|_{y+\Delta y}\right)\Delta x\,\Delta z + \left(\rho v_z\Big|_z - \rho v_z\Big|_{z+\Delta z}\right)\Delta x\,\Delta y = \Delta x\,\Delta y\,\Delta z\frac{\partial}{\partial t}(\rho)$$

Divide the whole equation by Δx Δy Δz
then take the limit when Δx Δy Δz approaches zero

$$\lim_{\Delta x\,\Delta y\,\Delta z\to 0}\left[\frac{\left(\rho v_x\Big|_x - \rho v_x\Big|_{x+\Delta x}\right)}{\Delta x} + \frac{\left(\rho v_y\Big|_y - \rho v_y\Big|_{y+\Delta y}\right)}{\Delta y} + \frac{\left(\rho v_z\Big|_z - \rho v_z\Big|_{z+\Delta z}\right)}{\Delta z} = \frac{\partial}{\partial t}(\rho)\right]$$

$$-\frac{\partial}{\partial x}(\rho v_x) - \frac{\partial}{\partial y}(\rho v_y) - \frac{\partial}{\partial z}(\rho v_z) = \frac{\partial}{\partial t}(\rho)$$

Divergence Operator

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Velocity vector $\vec{v} = \left(v_x\,\hat{i} + v_y\,\hat{j} + v_z\,\hat{k}\right)$

Dell operator $\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$

Divergence $\vec{\nabla}$. Vector (Dot product between two vectors)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, v) = 0$$

Microscopic Continuity Equation For Incompressible Fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

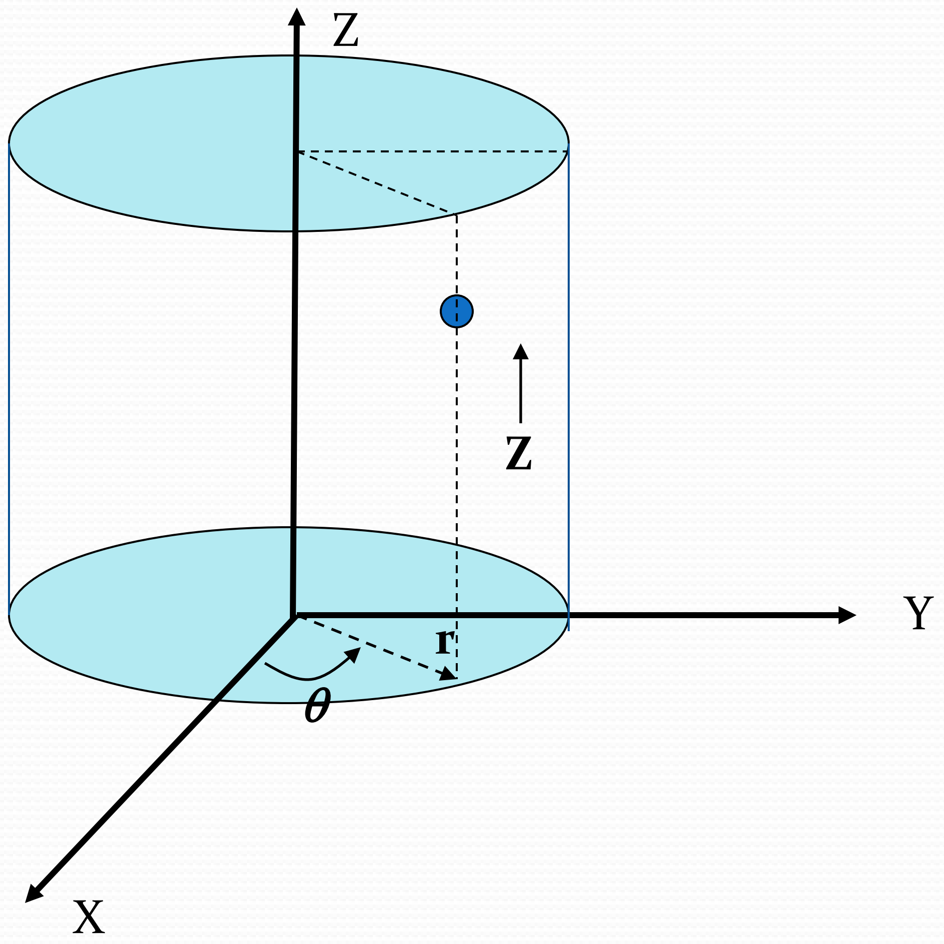
Incompressible Fluid ➔ Constant Density

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

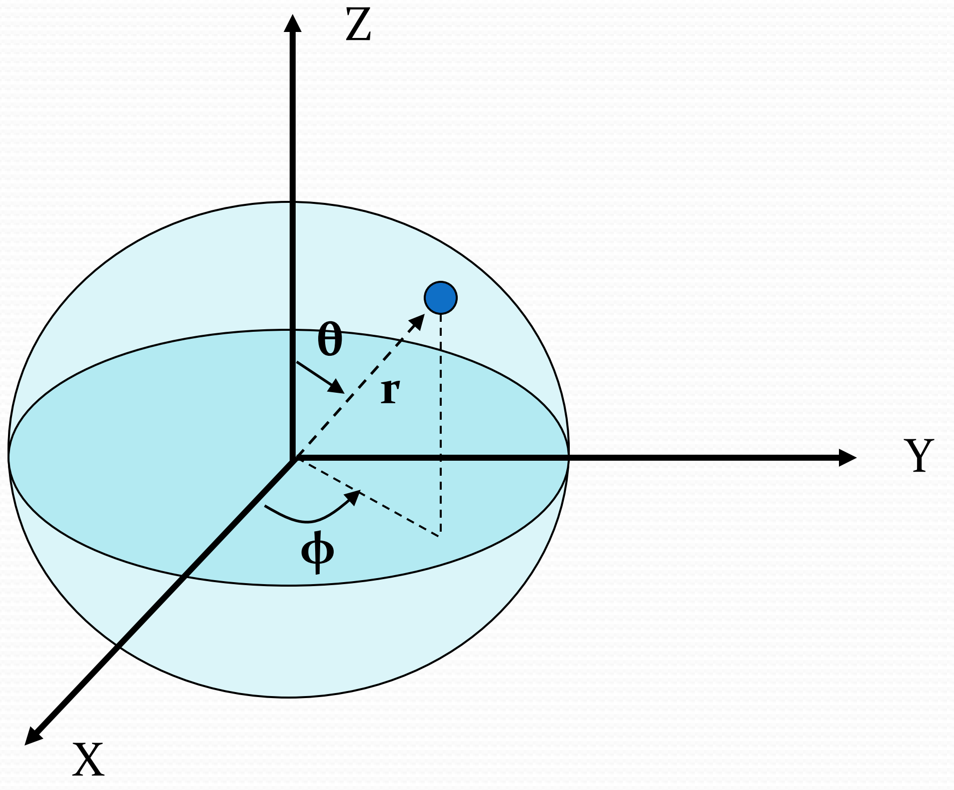
$$\nabla \cdot v = 0$$

For Incompressible Fluid:
The divergence of the velocity is equal to zero

Cylindrical Coordinates



Spherical Coordinates



Divergence Operator in Different Coordinate Systems

Rectangular Coordinates : $$\nabla \cdot v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Cylindrical Coordinates : $$\nabla \cdot v = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (r v_\theta)}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

Spherical Coordinates : $$\nabla \cdot v = \frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\,\mathrm{Sin}\theta}\frac{\partial (v_\theta \mathrm{Sin}\theta)}{\partial \theta} + \frac{1}{r\,\mathrm{Sin}\theta}\frac{\partial v_\phi}{\partial \phi}$$