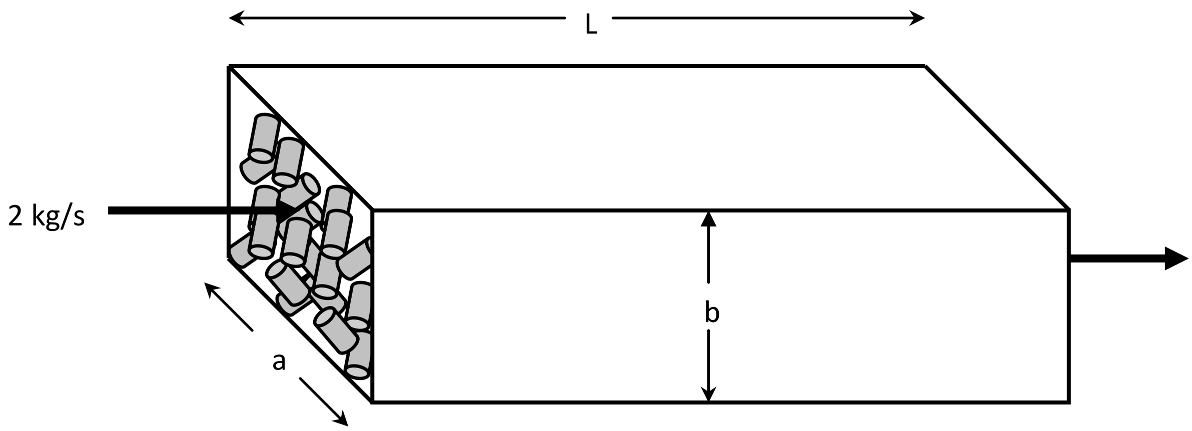
Example 1

Water (ρ = 1000 kg/m^3, μ = 1*10^{-3} kg/ m s) flows through a horizontal duct that has a rectangular cross section at a mass flow rate 2 kg/s, see figure below. The duct is packed with 9,000,000 small cylinders each with diameter D_{cylinder} = 1 mm and length L_{cylinder} = 2 mm. Calculate the pressure drop across the duct.



L = 1 m
a = 0.2 m
b = 0.1 m

Solution:

First of all we have to calculate the superficial velocity $\boldsymbol{u_0}$, effective particle diameter $\boldsymbol{D_P}$ and void fraction $\boldsymbol{\varepsilon}$. For details see Table 4.2 of the textbook:

$$u_0 = \frac{Q}{A} = \frac{m/\rho}{A} = \frac{2/1000}{(0.1)(0.2)} = 0.1 \quad \text{m/s}$$

$$a_{\text{v}} = \frac{\text{total external surface area of particle}}{\text{volume of particle}}$$

$$= \frac{2\left(\frac{\pi}{4} D^2_{\text{particle}}\right)2 + \pi\, D_{\text{particle}} L_{\text{particle}}}{\frac{\pi}{4} D^2_{\text{particle}}\; L_{\text{particle}}} = \frac{2\left(\frac{\pi}{4}(0.001)^2\right) + \pi(0.001)(0.002)}{\frac{\pi}{4}(0.001)^2 * (0.002)} = 5000 \;\; \frac{1}{\text{m}}$$

D_P (effective particle diameter) = 6/a$_{\text{v}}$ = 6/5000 = 1.2 x 10^{-3} m

$$\varepsilon = \frac{\text{Volume of Duct - Valume of all Particles}}{\text{Volume of Duct}}$$

$$= \frac{(0.1)(0.2)(1) - 9{,}000{,}000 \frac{\pi}{4} D^2_{\text{cylinder}}\; L_{\text{cylinder}}}{(0.1)(0.2)(1)} = 0.293$$

For horizontal packed beds the Ergun equation [4.26] of the textbook is applicable:

$$-\frac{\Delta P}{\rho u_0^2}\frac{D_P}{L}\frac{\varepsilon^3}{1-\varepsilon}=\left[\frac{150}{\text{Re}}+1.75\right]$$

Rearranging to solve for the pressure drop:

$$-\Delta P=\left[\frac{150}{\text{Re}}+1.75\right]\rho u_0^2\frac{L}{D_P}\frac{1-\varepsilon}{\varepsilon^3}$$

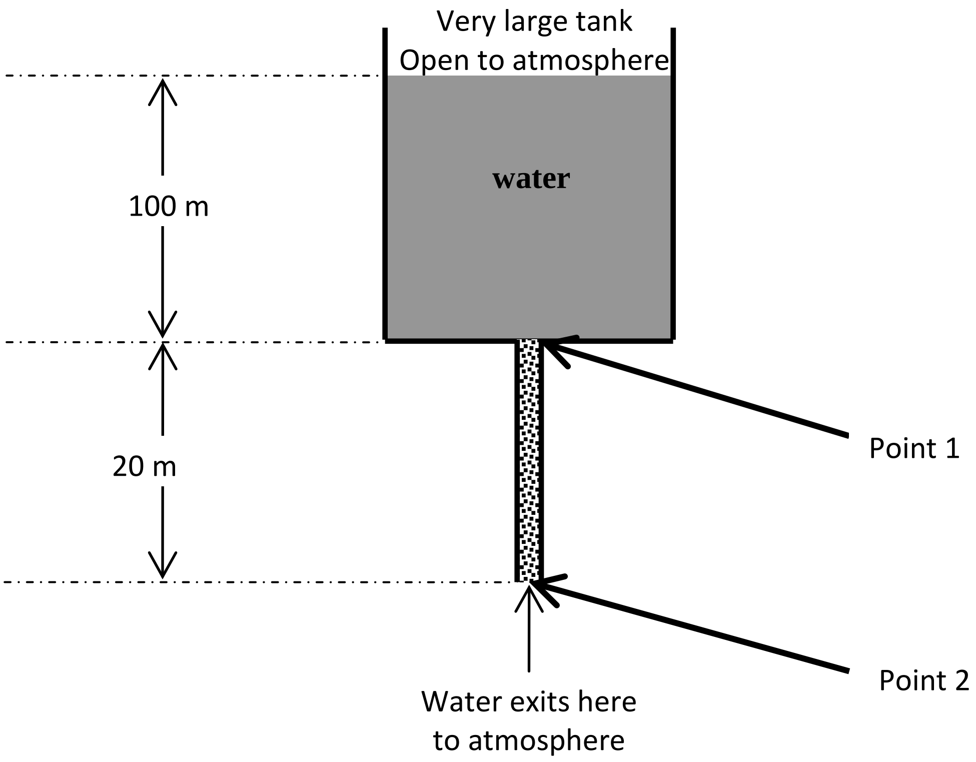
$$\text{Re}=\frac{\rho u_0 D_P}{(1-\varepsilon)\mu}=\frac{(1000)(0.1)(1.2*10^{-3})}{(1-0.293)(1*10^{-3})}=169.7$$

$$-\Delta P=\left[\frac{150}{169.7}+1.75\right](1000)(0.1)^2\frac{1}{1.2*10^{-3}}\frac{1-0.293}{(0.293)^3}+(1000)(9.8)(0)$$

$$=616{,}931.1\ \frac{\text{N}}{\text{m}^2}\ (\text{Pascals})=6.09\ \text{atm}$$

Example 2

A tank drains water by gravity through an exit pipe of diameter 1 m, see shown figure below. The exit pipe is packed with sand particles (there is no sand in the tank itself). The sand particles are uniform, with an average diameter of 0.2 mm and void fraction of 0.3. Calculate the volumetric flow rate of water in gal/min (gpm).



Recall the Ergun equation for non-horizontal packed beds:

$$-\frac{\Delta P}{\rho u_0^2}\frac{D_P}{L}\frac{\varepsilon^3}{1-\varepsilon}=\left[\frac{150}{\text{Re}}+1.75\right]+\frac{1}{u_0^2}\frac{D_P}{L}\frac{\varepsilon^3}{1-\varepsilon}g\,\Delta z$$

The above equation is similar to equation [4.26] of the textbook, however, it is more general to account for non-horizontal beds. Hence, equation [4.26] is for the special case when $\Delta z = 0$. Rearranging the Ergun equation, the following equation can be written for the pressure drop across non-horizontal packed beds:

$$-\frac{\Delta P}{\rho}=\overbrace{\left[\frac{150}{\text{Re}}+1.75\right]u_0^2\frac{L}{D_P}\frac{1-\varepsilon}{\varepsilon^3}}^{\text{Pressure drop due to friction}}+\overbrace{g\,\Delta z}^{\text{Pressure drop due to gravity}}$$

Once again the above equation is similar to equation [4.29] of the textbook, however, it is more general to account for non-horizontal beds. Hence, equation [4.29] is for the special case when $\Delta z = 0$.

$$\text{Re} = \frac{\rho\, u_0\, D_P}{(1-\varepsilon)\mu}$$

Substitute in the Ergun equation

$$-\frac{\Delta P}{\rho\, u_0^2}\frac{D_P}{L}\frac{\varepsilon^3}{1-\varepsilon} = \left[\frac{150(1-\varepsilon)\mu}{\rho\, u_0\, D_P} + 1.75\right] + \frac{1}{u_0^2}\frac{D_P}{L}\frac{\varepsilon^3}{1-\varepsilon}\, g\, \Delta z$$

Multiply the above equation by u_0^2:

$$-\frac{\Delta P}{\rho}\frac{D_P}{L}\frac{\varepsilon^3}{1-\varepsilon} = \left[\frac{150(1-\varepsilon)\mu}{\rho\ D_P} u_0 + 1.75 u_0^2\right] + \frac{D_P}{L}\frac{\varepsilon^3}{1-\varepsilon}\, g\, \Delta z$$

and rearrange:

$$(1.75)u_0^2 + \left(\frac{150(1-\varepsilon)\mu}{\rho\ D_P}\right)u_0 + \left(\frac{\Delta P}{\rho} + g\ \Delta z\right)\frac{D_P}{L}\frac{\varepsilon^3}{1-\varepsilon} = 0$$

Calculate the change pressure change across the bed:

$$\Delta P = P_2 - P_1 = (P_{\text{atm}}) - (P_{\text{atm}} + \rho\, g\, 100) = -9.8 * 10^5 \ \text{ Pascals}$$

Substituting:

$$(1.75)u_0^2 + \left(\frac{150(1-0.3)(0.001)}{(1000)\ (0.2*10^{-3})}\right)u_0 + \left(\frac{-9.8*10^5}{(1000)} + (9.8)(-20)\right)\frac{0.2*10^{-3}}{20}\frac{0.3^3}{1-0.3} = 0$$

Simplifying:

$$(1.75)u_0^2 + (0.525)u_0 - \left(4.54*10^{-4}\right) = 0$$

$$\Rightarrow u_0 = \frac{-0.525 \pm \sqrt{0.525^2 - 4(1.75)\left(-4.54*10^{-4}\right)}}{2(1.75)} = \frac{-0.525 \pm 0.528}{2(1.75)}$$

$$u_0 = 8.57*10^{-4} \ \text{ m/s} \quad \text{or} \quad -0.3 \ \text{ m/s} \ \ (\text{reject negative value})$$

$$Q = \frac{\pi}{4} D^2\, u_0 = \frac{\pi}{4}(1)^2\left(8.57*10^{-4}\right) = 6.73*10^{-4} \ \text{ m}^3\text{/s} = 10.67 \ \ \text{gpm}$$