Particle Settling Velocity

The momentum balance for a spherical particle settling freely in a stagnant fluid:

$$\underbrace{\frac{\pi D^3 \rho_s}{6}\frac{du}{dt}}_{\text{Mass} \times \text{Acceleration of Particle}} = \overbrace{\underbrace{\frac{\pi D^3}{6}\rho_s g}_{\text{Gravity Force}} \quad \underbrace{- \frac{\pi D^3}{6}\rho_f g}_{\text{Buoyancy Force}} \quad \underbrace{- F_D}_{\text{Grag Force}}}^{\text{Sum of Forces Acting on Particle}} \tag{1}$$

Recall the drag coefficient on the particle:

$$C_D = \frac{F_D / A_P}{\frac{1}{2}\rho_f u^2} \tag{2}$$

where A_P is the projected surface area of the particle, and for the spherical case it is the area of a circle having the same diameter of the particle:

$$A_P = \frac{\pi}{4}D^2 \tag{3}$$

Therefore drag force on the particle is equal to:

$$F_D = \frac{\pi}{8}D^2 \rho_f u^2 C_D \tag{4}$$

And the momentum balance is written as:

$$\frac{\pi D^3 \rho_s}{6}\frac{du}{dt} = \frac{\pi D^3}{6}\left(\rho_s - \rho_f\right)g - \frac{\pi}{8}D^2 \rho_f u^2 C_D \tag{5}$$

The particle velocity u depends on the particle diameter D, particle density ρ_s, fluid density ρ_f, gravitational acceleration constant g, and drag coefficient C_D.

Steady State Analysis:

At steady state the particle acceleration is equal to zero:

$$\frac{\pi D^3 \rho_s}{6}\frac{du}{dt} = 0 \tag{6}$$

In this case the sum of gravity, buoyancy and drag forces on the particle is equal to zero:

$$\frac{\pi D^3}{6}\left(\rho_s - \rho_f\right)g - \frac{\pi}{8}D^2 \rho_f u_t^2 C_D = 0 \tag{7}$$

Note the particle steady state velocity is known as terminal velocity u_t. Rearranging and solving for the drag coefficient, the following equation can be derived:

$$C_D = \frac{4}{3}\frac{g\,D}{u_t^2}\frac{\left(\rho_s - \rho_f\right)}{\rho_f} \tag{8}$$

The above equation is equivalent to equation (4.16) of the textbook.

Unsteady State Analysis:

Recall the unsteady state momentum balance on the particle:

$$\frac{\pi\,D^3\,\rho_s}{6}\frac{du}{dt} = \frac{\pi\,D^3}{6}\left(\rho_s - \rho_f\right)g - \frac{\pi}{8}D^2\rho_f u^2 C_D \tag{5}$$

Rearranging:

$$\frac{du}{dt} = \left(1 - \frac{\rho_f}{\rho_s}\right)g - \frac{3\rho_f C_D}{4D\,\rho_s}u^2 \tag{9}$$

In order to simplify the work, let:

$$G = \left(1 - \frac{\rho_f}{\rho_s}\right)g \tag{10-a}$$

$$c = \frac{3\rho_f C_D}{4D\,\rho_s} \tag{10-b}$$

Therefore,

$$\frac{du}{dt} = G - c\,u^2 \tag{11}$$

Equation (11) is a first order ordinary differential equation. Integrating:

$$\int_0^u \frac{du}{G - c\,u^2} = \int_0^t dt \tag{12}$$

From appendix A of the textbook the following integration formula is available:

$$\int \frac{dx}{a - b\,x^2} = \frac{1}{2\sqrt{a\,b}} \ln\left(\frac{\sqrt{a} + x\sqrt{b}}{\sqrt{a} - x\sqrt{b}}\right) \qquad \left\{\begin{matrix} a = G \\ b = c \\ x = u \end{matrix}\right\}$$

Applying the integration formula to eq. (12) yields the following:

$$\left[\frac{1}{2\sqrt{G\,c}} \ln\left(\frac{\sqrt{G} + u\sqrt{c}}{\sqrt{G} - u\sqrt{c}}\right)\right]_0^u = [t]_0^t$$

Solving for *t*:

$$t = \frac{1}{2\sqrt{G\,c}} \ln\left(\frac{\sqrt{G} + u\sqrt{c}}{\sqrt{G} - u\sqrt{c}}\right) \tag{13}$$

And rearranging to solve for *u*:

$$u = \sqrt{\frac{G}{c}}\,\frac{e^{\left(2\sqrt{Gc}\right)t} - 1}{e^{\left(2\sqrt{Gc}\right)t} + 1} \tag{14}$$

The above equation for the particle unsteady state velocity is similar to equation (E4.2.5) of the textbook, however, it is more general because in the textbook it is assumed that the fluid density is very small compared to the particle density: $G = \left(1 - \rho_f / \rho_s\right)g \approx g$.

Distant Travelled by Particle:

Noting that the velocity is the rate of change of distant with respect to time:

$$u = \frac{dx}{dt} \tag{15}$$

where *x* is the distant travelled by the settling sphere in time *t*. Substituting in equation (14):

$$\frac{dx}{dt} = \sqrt{\frac{G}{c}}\,\frac{e^{\left(2\sqrt{Gc}\right)t} - 1}{e^{\left(2\sqrt{Gc}\right)t} + 1} \tag{16}$$

Integrating:

$$\int_0^x dx = \int_0^t \sqrt{\frac{G}{c}}\,\frac{e^{\left(2\sqrt{Gc}\right)t} - 1}{e^{\left(2\sqrt{Gc}\right)t} + 1}\,dt = \sqrt{\frac{G}{c}}\left[\int_0^t \frac{e^{\left(2\sqrt{Gc}\right)t}}{e^{\left(2\sqrt{Gc}\right)t} + 1}\,dt - \int_0^t \frac{1}{e^{\left(2\sqrt{Gc}\right)t} + 1}\,dt\right] \tag{17}$$

And by using integration by substitution yields the following formula for the distant travelled by particle as a function of time:

$$x = \frac{1}{2c}\left[2\ln\left(\frac{e^{\left(2\sqrt{Gc}\right)t}+1}{2}\right) - \left(2\sqrt{Gc}\right)t\right] \tag{18}$$

The above equation for the for the distant travelled by particle as function of time is similar to equation (E4.2.7) of the textbook, however, it is more general because in the textbook it is assumed that the fluid density is very small compared to the particle density: $G = \left(1 - \rho_f / \rho_s\right)g \approx g$.