Pressure Drop in Pipe

If frictional losses are important, the pressure drop in a pipe ΔP depends on fluid velocity u, fluid density ρ, fluid viscosity μ, pipe diameter D and pipe length L.

$$\Delta P = f_1(u, \rho, \mu, D, L)$$

Use dimensional analysis to derive the dimensionless groups in this problem.

Solution:

	Variable	Dimension
1	ΔP	M/(L T^2)
2	u	L/T
3	ρ	M/L^3
4	μ	M/(L T)
5	D	L
6	L	L

π-Theorem: n = 6 m = 3 p = n – m = 3

$$\pi_1 = (\rho)^{a1}(D)^{a2}(u)^{a3} \qquad \Delta P$$
$$\pi_2 = (\rho)^{b1}(D)^{b2}(u)^{b3} \qquad L$$
$$\pi_3 = (\rho)^{c1}(D)^{c2}(u)^{c3} \qquad \mu$$

Check rules for selecting repeated variables:

1. Repeated variables individually have different dimensions:

   $\rho = \frac{\mathrm{M}}{\mathrm{L}^3}$ $D = \mathrm{L}$ $u = \frac{\mathrm{L}}{\mathrm{T}}$ ➔ Yes satisfied

2. Repeated variables as a group contain all basic dimensions:

   $\frac{\mathrm{M}}{\mathrm{L}^3}$ L $\frac{\mathrm{L}}{\mathrm{T}}$ ➔ Yes satisfied they contain mass length and time

3. Repeated variable contain a velocity scale and a length scale:

   Yes they contain length scale D and velocity scale u

$$\pi_1 = (\rho)^{a1}(D)^{a2}(u)^{a3} \qquad \Delta P \qquad (\mathrm{M\,L\,T})^0 = \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{a1}(\mathrm{L})^{a2}\left(\frac{\mathrm{L}}{\mathrm{T}}\right)^{a3} \qquad \frac{\mathrm{M}}{\mathrm{L\;T}^2}$$

$$\left.\begin{array}{lll} \mathrm{M}\;: & 0 = \mathrm{a1} + 1 & \Rightarrow \quad \mathrm{a1} = -1 \\ \mathrm{L}\;: & 0 = -3\;\mathrm{a1} + \mathrm{a2} + \mathrm{a3} - 1 & \Rightarrow \quad \mathrm{a2} = 0 \\ \mathrm{T}\;: & 0 = -\,\mathrm{a3} - 2 & \Rightarrow \quad \mathrm{a3} = -2 \end{array}\right\} \Rightarrow \quad \pi_1 = \frac{\Delta P}{\rho\, u^2} = \frac{\dfrac{\mathrm{kg}}{\mathrm{m\;s}^2}}{\dfrac{\mathrm{kg}}{\mathrm{m}^3}\left(\dfrac{\mathrm{m}}{\mathrm{s}}\right)^2}$$

$$\pi_2 = (\rho)^{b1}(D)^{b2}(u)^{b3} \qquad L \qquad (\mathrm{M\,L\,T})^0 = \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{b1}(\mathrm{L})^{b2}\left(\frac{\mathrm{L}}{\mathrm{T}}\right)^{b3} \qquad \mathrm{L}$$

$$\left.\begin{array}{lll} \mathrm{M}\;: & 0 = \mathrm{b1} & \Rightarrow \quad \mathrm{b1} = 0 \\ \mathrm{L}\;: & 0 = -3\;\mathrm{b1} + \mathrm{b2} + \mathrm{b3} + 1 & \Rightarrow \quad \mathrm{b2} = -1 \\ \mathrm{T}\;: & 0 = -\,\mathrm{b3} & \Rightarrow \quad \mathrm{b3} = 0 \end{array}\right\} \Rightarrow \quad \pi_2 = \frac{L}{D}$$

$$\pi_3 = (\rho)^{c1}(D)^{c2}(u)^{c3} \qquad \mu \qquad (\mathrm{M\,L\,T})^0 = \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{c1}(\mathrm{L})^{c2}\left(\frac{\mathrm{L}}{\mathrm{T}}\right)^{c3} \qquad \frac{\mathrm{M}}{\mathrm{L\;T}}$$

$$\left.\begin{array}{lll} \mathrm{M}\;: & 0 = \mathrm{c1} + 1 & \Rightarrow \quad \mathrm{c1} = -1 \\ \mathrm{L}\;: & 0 = -3\;\mathrm{c1} + \mathrm{c2} + \mathrm{c3} + 1 & \Rightarrow \quad \mathrm{c2} = -1 \\ \mathrm{T}\;: & 0 = -\,\mathrm{c3} - 1 & \Rightarrow \quad \mathrm{c3} = -1 \end{array}\right\} \Rightarrow \quad \pi_3 = \frac{\mu}{\rho\, u\, D} = \frac{\dfrac{\mathrm{kg}}{\mathrm{m\,s}}}{\dfrac{\mathrm{kg}}{\mathrm{m}^3}\dfrac{\mathrm{m}}{\mathrm{s}}\mathrm{m}}$$

Note $\pi_3 = \dfrac{\mu}{\rho\, u\, D}$ is dimensionless

But also $\pi_3^{'} = \dfrac{1}{\pi_3} = \dfrac{\rho\, u\, D}{\mu} = \text{Reynolds' number } \mathrm{Re}$

$$\Rightarrow \quad \frac{\Delta P}{\rho\, u^2} = f\left(\frac{L}{D},\;\; Re\right)$$

Let us violate the rules:

$$\begin{aligned}
\pi_1 &= (\rho)^{a1}(D)^{a2}(L)^{a3} && \Delta P \\
\pi_2 &= (\rho)^{b1}(D)^{b2}(L)^{b3} && u \\
\pi_3 &= (\rho)^{c1}(D)^{c2}(L)^{c3} && \mu
\end{aligned}$$

$$\pi_1 = (\rho)^{a1}(D)^{a2}(L)^{a3} \qquad \Delta P \qquad\qquad (\mathrm{M\,L\,T})^0 = \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{a1}(\mathrm{L})^{a2}(\mathrm{L})^{a3} \qquad \frac{\mathrm{M}}{\mathrm{L\ T}^2}$$

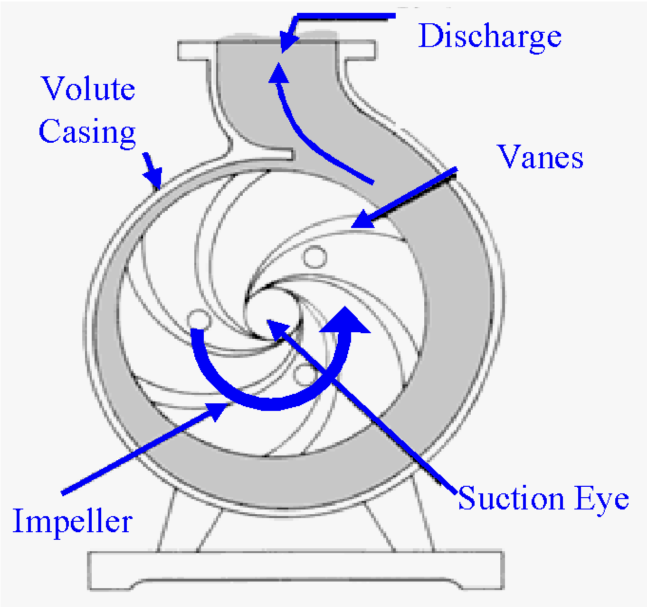
$$\left.\begin{aligned}
&\mathrm{M} : \quad 0 = \mathrm{a1} + 1 \\
&\mathrm{L} : \quad 0 = -3\ \mathrm{a1} + \mathrm{a2} + \mathrm{a3} - 1 \\
&\mathrm{T} : \quad 0 = -2 \ \text{(this is not acceptable)}
\end{aligned}\right\} \Rightarrow \quad \pi_1 = ?$$

Power for Centrifugal Pump

The power $\boldsymbol{P_W}$ required to drive a centrifugal pump depends on liquid volumetric flow rate $\boldsymbol{Q}$, liquid density $\boldsymbol{\rho}$, angular speed of impeller $\boldsymbol{N}$ and diameter of impeller $\boldsymbol{D}$.

$$\boldsymbol{P_W} = f_1(\boldsymbol{Q}, \boldsymbol{\rho}, \boldsymbol{N}, \boldsymbol{D})$$

Use dimensional analysis to derive the dimensionless groups in this problem.



Solution:

	Variable	Dimension
1	$\boldsymbol{P_W}$	$M\,L^2/T^3$
2	$\boldsymbol{Q}$	L^3/T
3	$\boldsymbol{\rho}$	M/L^3
4	$\boldsymbol{N}$	$1/T$
5	$\boldsymbol{D}$	L

$$\text{Power, } P_W = \overbrace{\frac{\text{kg}}{\text{s}}}^{\text{mass flow rate}} \overbrace{\overbrace{\frac{\text{kg m}}{\text{s}^2}}^{\text{force}} \overbrace{\text{m}}^{\text{length}} \overbrace{\frac{1}{\text{kg}}}^{\text{per mass}}}^{\text{energy/mass}} = \frac{\text{kg m}^2}{\text{s}^3} = \frac{\text{M L}^2}{\text{T}^3}$$

π-Theorem: n = 5 m = 3 p = n – m = 2

$$\Rightarrow \quad f(\pi_1, \pi_2) = 0, \quad \text{or} \quad \pi_1 = f(\pi_2)$$

$$\pi_1 = (\rho)^{a1}(D)^{a2}(N)^{a3} \qquad P$$

$$\pi_2 = (\rho)^{b1}(D)^{b2}(N)^{b3} \qquad Q$$

Check rules for selecting repeated variables:

4. Repeated variables individually have different dimensions:

$$\rho = \frac{\mathrm{M}}{\mathrm{L}^3} \qquad D = \mathrm{L} \qquad N = \frac{1}{\mathrm{T}} \qquad \Rightarrow \qquad \text{Yes satisfied}$$

5. Repeated variables as a group contain all basic dimensions:

$$\frac{\mathrm{M}}{\mathrm{L}^3} \quad \mathrm{L} \quad \frac{1}{\mathrm{T}} \quad \Rightarrow \quad \text{Yes satisfied they contain mass length and time}$$

6. Repeated variable contain a velocity scale and a length scale:

   Yes they contain length scale D and velocity scale $D \times N$

$$\pi_1 = (\rho)^{a1}(D)^{a2}(N)^{a3} \qquad P \qquad\qquad (\mathrm{M\,L\,T})^0 = \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{a1}(\mathrm{L})^{a2}\left(\frac{1}{\mathrm{T}}\right)^{a3} \quad \frac{\mathrm{M\ L}^2}{\mathrm{T}^3}$$

$$\left.\begin{array}{lll} \mathrm{M}: & 0 = a1 + 1 & \Rightarrow \quad a1 = -1 \\ \mathrm{L}: & 0 = -3\ a1 + a2 + 2 & \Rightarrow \quad a2 = -5 \\ \mathrm{T}: & 0 = -a3 - 3 & \Rightarrow \quad a3 = -3 \end{array}\right\} \Rightarrow \quad \pi_1 = \frac{P_W}{\rho\ D^5\ N^3}$$

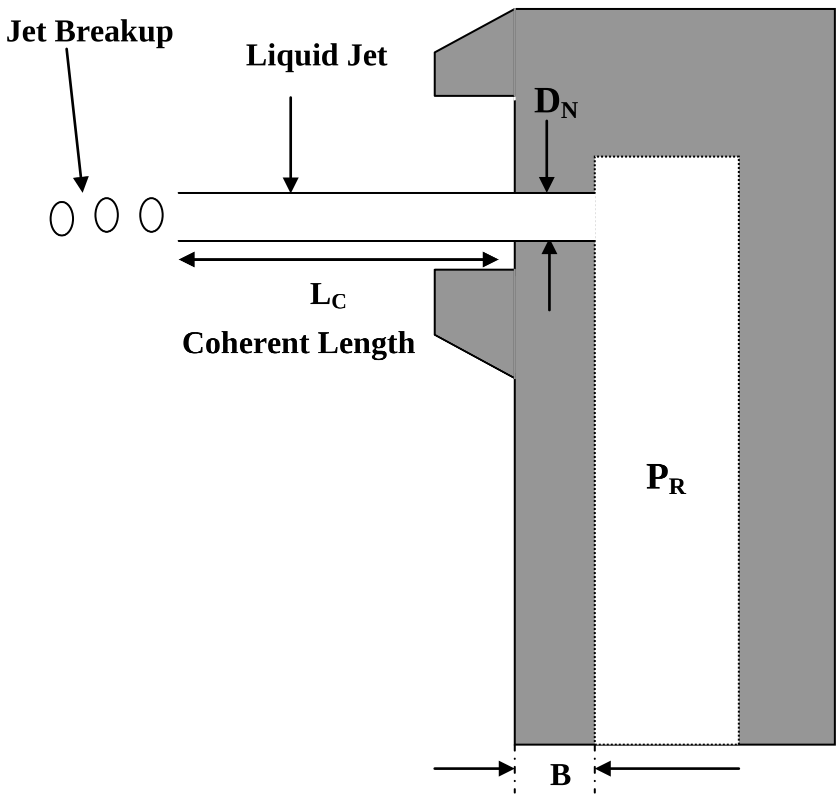
$$\pi_2 = (\rho)^{b1}(D)^{b2}(N)^{b3} \qquad Q \qquad\qquad (\mathrm{M\,L\,T})^0 = \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{b1}(\mathrm{L})^{b2}\left(\frac{1}{\mathrm{T}}\right)^{b3} \quad \frac{\mathrm{L}^3}{\mathrm{T}}$$

$$\left.\begin{array}{lll} \mathrm{M}: & 0 = b1 + 0 & \Rightarrow \quad b1 = 0 \\ \mathrm{L}: & 0 = -3\ b1 + b2 + 3 & \Rightarrow \quad b2 = -3 \\ \mathrm{T}: & 0 = -b3 - 1 & \Rightarrow \quad b3 = -1 \end{array}\right\} \Rightarrow \quad \pi_2 = \frac{Q}{D^3\ N}$$

$$\Rightarrow \quad \frac{P_W}{\rho\ D^5\ N^3} = f\left(\frac{Q}{D^3\ N}\right)$$

Coherent Jet Length of Insect Killer

You are developing a product consisting of a reservoir (a can) of insect killer under pressure. A nozzle-pressure system has been designed that permits the user to produce a coherent jet of liquid and aim it at the target insect(s). The system is more effective if the target is struck by a coherent jet. In particular, the more coherent the jet, the more the distance from the target can be increased, with resulting safety for the user.



You are responsible for an experimental program aimed at providing experimental data for the dependence of the coherent length $\mathbf{L_C}$ as a function of liquid density ρ, liquid surface tension σ, nozzle design ($\mathbf{D_N}$ and $\mathbf{B}$) and reservoir pressure $\mathbf{P_R}$. Use dimensional analysis to relate $\mathbf{L_C}$ (m) to ρ (kg/m^3), σ (kg/s^2), $\mathbf{D_N}$ (m), $\mathbf{B}$ (m) and $\mathbf{P_R}$ (kg/(m s^2)).

$$L_C = f_1(\rho,\ \sigma,\ D_N,\ B,\ P_R)$$

Solution:

	Variable	Dimension
1	L_C	L
2	ρ	M/L^3
3	σ	M/T^2
4	D_N	L
5	B	L
6	P_R	M/(L T^2)

π-Theorem: n = 6 m = 3 p = n – m = 3

$$\pi_1 = (\rho)^{a1}(D_N)^{a2}(\sigma)^{a3} \qquad L_C$$
$$\pi_2 = (\rho)^{b1}(D_N)^{b2}(\sigma)^{b3} \qquad B$$
$$\pi_3 = (\rho)^{c1}(D_N)^{c2}(\sigma)^{c3} \qquad P_R$$

$$\pi_1 = (\rho)^{a1}(D_N)^{a2}(\sigma)^{a3} \qquad L_C \qquad (\mathrm{M\,L\,T})^0 = \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{a1}(\mathrm{L})^{a2}\left(\frac{\mathrm{M}}{\mathrm{T}^2}\right)^{a3} \qquad \mathrm{L}$$

$$\left.\begin{array}{lll} \mathrm{M}: & 0 = a1 + a3 & \Rightarrow \quad a1 = -a3 \\ \mathrm{L}: & 0 = -3\,a1 + a2 + 1 & \Rightarrow \quad a2 = 3a1 - 1 \\ \mathrm{T}: & 0 = -2\,a3 & \Rightarrow \quad a3 = 0 \end{array}\right\} \Rightarrow \quad \pi_1 = \frac{L_C}{D_N}$$

$$\pi_2 = (\rho)^{b1}(D_N)^{b2}(\sigma)^{b3} \qquad B \qquad (\mathrm{M\,L\,T})^0 = \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{b1}(\mathrm{L})^{b2}\left(\frac{\mathrm{M}}{\mathrm{T}^2}\right)^{b3} \qquad \mathrm{L}$$

$$\left.\begin{array}{lll} \mathrm{M}: & 0 = b1 + b3 & \Rightarrow \quad b1 = -b3 \\ \mathrm{L}: & 0 = -3\,b1 + b2 + 1 & \Rightarrow \quad b2 = 3b1 - 1 \\ \mathrm{T}: & 0 = -2\,b3 & \Rightarrow \quad b3 = 0 \end{array}\right\} \Rightarrow \quad \pi_2 = \frac{B}{D_N}$$

$$\pi_3 = (\rho)^{c1}(D_N)^{c2}(\sigma)^{c3} \qquad P_R \qquad (\mathrm{M\,L\,T})^0 = \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{c1}(\mathrm{L})^{c2}\left(\frac{\mathrm{M}}{\mathrm{T}^2}\right)^{c3} \qquad \frac{\mathrm{M}}{\mathrm{L\ T}^2}$$

$$\left.\begin{array}{lll} \mathrm{M}: & 0 = c1 + c3 + 1 & \Rightarrow \quad c1 = -1 - c3 \\ \mathrm{L}: & 0 = -3\,c1 + c2 - 1 & \Rightarrow \quad c2 = 3c1 + 1 \\ \mathrm{T}: & 0 = -2\,c3 - 2 & \Rightarrow \quad c3 = -1 \end{array}\right\} \Rightarrow \quad \pi_3 = \frac{P_R\ D_N}{\sigma} = \frac{\dfrac{\mathrm{kg}}{\mathrm{m\ s^2}}\ \mathrm{m}}{\dfrac{\mathrm{kg}}{\mathrm{s^2}}}$$

$$\Rightarrow \quad \frac{L_C}{D_N} = f\left(\frac{B}{D_N},\quad \frac{P_R\ D_N}{\sigma}\right)$$

Similitude of Airplane Model

The landing behavior of an airplane scale model is to be tested. Since this is a low-speed problem, the Reynolds number is to be held constant between the model and full-scale airplane. The model is to be one-tenth of the length of the airplane with the same shape. The landing speed of the full-size airplane is 1.61 km/h.

(a) How fast should the air in the wind tunnel move past the model to have the same Reynolds number?
(b) Suppose we could test the model in water. What should the water velocity be for the same Reynolds numbers?

Solution:

(a) $$\mathrm{Re}_{\text{Full Scale}} = \frac{\rho_{\text{Air}}\, u_{\text{Full Scale}}\, D_{\text{Full Scale}}}{\mu_{\text{Air}}} \qquad \mathrm{Re}_{\text{Model}} = \frac{\rho_{\text{Air}}\, u_{\text{Model}}\, D_{\text{Model}}}{\mu_{\text{Air}}}$$

$$\frac{\mathrm{Re}_{\text{Full Scale}}}{\mathrm{Re}_{\text{Model}}} = 1 = \frac{\dfrac{\rho_{\text{Air}}\, u_{\text{Full Scale}}\, D_{\text{Full Scale}}}{\mu_{\text{Air}}}}{\dfrac{\rho_{\text{Air}}\, u_{\text{Model}}\, D_{\text{Model}}}{\mu_{\text{Air}}}}, \qquad D_{\text{Model}} = \frac{D_{\text{Full Scale}}}{10}$$

$$\frac{u_{\text{Full Scale}}\, D_{\text{Full Scale}}}{u_{\text{Model}}\, \dfrac{D_{\text{Full Scale}}}{10}} = 1 \qquad \Rightarrow u_{\text{Model}} = 10\ u_{\text{Full Scale}} = 16.1\ \text{km/hr}$$

(b) $$\mathrm{Re}_{\text{Model-Air}} = \frac{\rho_{\text{Air}}\, u_{\text{Model-Air}}\, D_{\text{Model}}}{\mu_{\text{Air}}} \qquad \mathrm{Re}_{\text{Model-Water}} = \frac{\rho_{\text{Water}}\, u_{\text{Model-Water}}\, D_{\text{Model}}}{\mu_{\text{Water}}}$$

$$\frac{\mathrm{Re}_{\text{Model-Air}}}{\mathrm{Re}_{\text{Model-Water}}} = 1 = \frac{\dfrac{\rho_{\text{Air}}\, u_{\text{Model-Air}}\, D_{\text{Model}}}{\mu_{\text{Air}}}}{\dfrac{\rho_{\text{Water}}\, u_{\text{Model-Water}}\, D_{\text{Model}}}{\mu_{\text{Water}}}} = \frac{u_{\text{Model-Air}}}{u_{\text{Model-Water}}}\frac{\rho_{\text{Air}}}{\rho_{\text{Water}}}\frac{\mu_{\text{Water}}}{\mu_{\text{Air}}}$$

$$\Rightarrow u_{\text{Model-Water}}\ \mu_{\text{Air}}\ \rho_{\text{Water}} = u_{\text{Model-Air}}\ \rho_{\text{Air}}\ \mu_{\text{Water}}$$

$$u_{\text{Model-Water}}\ (0.0171\ \text{cP})\left(1000\ \frac{\text{kg}}{\text{m}^3}\right) = 16.1\left(1.29\ \frac{\text{kg}}{\text{m}^3}\right)(1\ \text{cP})$$

$$u_{\text{Model-Water}} = 16.1\,\frac{\left(1.29\ \dfrac{\text{kg}}{\text{m}^3}\right)(1\ \text{cP})}{(0.0171\ \text{cP})\left(1000\ \dfrac{\text{kg}}{\text{m}^3}\right)} = 1.21\ \ \text{km/hr}$$