11.5 − 1

Rotating Rod

$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{bmatrix}$$

$$\underline{\underline{\dot\gamma}} = \begin{bmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \\ & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta} \\ \text{Symmetric} & & 2\frac{\partial v_z}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & r\frac{d}{dr}\left(\frac{v_\theta}{r}\right) & 0 \\ r\frac{d}{dr}\left(\frac{v_\theta}{r}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

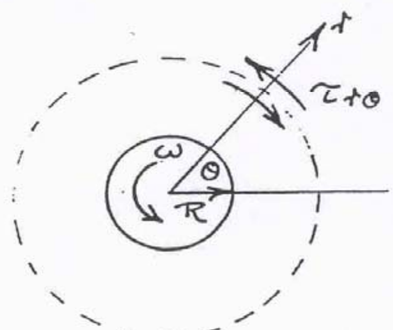
Assuming $v_r = v_z = 0$, $\frac{\partial}{\partial\theta} = 0$

Strain rate $\dot\gamma = \sqrt{\frac{1}{2} I_2} = \sqrt{\frac{1}{2}\, 2\left[r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right]^2} = \left| \underbrace{r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)}_{\text{negative}} \right|$

θ – Momentum, steady state, constant pressure

$$\frac{\partial}{\partial r}\left(r^2 \tau_{r\theta}\right) = 0$$

$$\underline{\underline{\tau}} = K\dot\gamma^{n-1}\underline{\underline{\dot\gamma}} \quad \text{or} \quad \tau_{r\theta} = K\dot\gamma^{n-1}\dot\gamma_{r\theta}$$



$$\tau_{r\theta} = K\left| r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right|^{n-1} \underbrace{r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)}_{\text{negative}}$$

$$= -K\Big[\underbrace{-r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)}_{\text{positive}}\Big]^n$$

11.5-2

$$\frac{d}{dr}\left\{-r^2 K\left[-r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right]^n\right\} = 0$$

$$-r^2 K\left[-r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right]^n = c_1$$

$$-r\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \left(-\frac{c_1}{r^2 K}\right)^{1/n} = \left(-\frac{c_1}{K}\right)^{1/n}\frac{1}{r^{2/n}}$$

$$\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = -\left(-\frac{c_1}{K}\right)^{1/n}\frac{1}{r^{2/n+1}}$$

Integrates to
$$\frac{v_\theta}{r} = \frac{n}{2}\left(-\frac{c_1}{K}\right)^{1/n}\frac{1}{r^{2/n}} + c_2 \qquad (1)$$

Boundary condition

$r \to \infty$: $v_\theta \to 0$: $\quad c_2 = 0$

$r = R \quad v_\theta = \omega R$:

$$\frac{\omega R}{R} = \frac{n}{2}\left(-\frac{c_1}{K}\right)^{1/n}\frac{1}{R^{2/n}} \qquad (2)$$

Divide (1) by (2) to eliminate c_1:

$$\frac{v_\theta}{\omega r} = \left(\frac{R}{r}\right)^{2/n}$$

Limiting case of large n (dilatant behavior)

$$\left(\frac{R}{r}\right)^{2/n \to \infty} \to 1 \quad \text{so} \quad v_\theta = \omega r$$

Fluid behaves as a solid-body rotation.

✓

11.5-3

Shear Stress

$$\tau_{r\theta} = -K\left[-r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right]^n$$

$$= -K\left[-r\,\omega R^{2/n}\left(-\frac{2}{n}\right)\frac{1}{r^{2/n+1}}\right]^n$$

$$= -K\omega^n R^2\left(\frac{2}{n}\right)^n\frac{1}{r^2} = -K\left(\frac{2\omega}{n}\right)^n\frac{R^2}{r^2}$$

Torque needed to Rotate Rod

$$(\tau_{r\theta})_{r=R} = -K\left(\frac{2\omega}{n}\right)^n$$

That is, the fluid exerts a stress on the rod in the negative θ direction — opposite to the direction of rotation.

Hence torque in the θ direction to rotate rod is

$$T_\theta = \Big[\underbrace{2\pi RL}_{\text{area}}\,K\left(\frac{2\omega}{n}\right)^n\Big]R$$ lever arm radius

$$\underline{\underline{T_\theta = 2\pi R^2 L K\left(\frac{2\omega}{n}\right)^n}}$$

✓

11.6-1

Shear-Thinning Wire Coating

Power-law fluid

$$\tau_{rz} = \eta\,\dot{\gamma}_{rz} = K\,\dot{\gamma}^{n-1}\dot{\gamma}_{rz} = -K\left(-\frac{dv_z}{dr}\right)^{n-1}\left(-\frac{dv_z}{dr}\right) = -K\left(-\frac{dv_z}{dr}\right)^{n}$$

Momentum balance

$$\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = 0\,, \qquad \frac{d}{dz}\left[-Kr\left(-\frac{dv_z}{dr}\right)^{n}\right] = 0$$

$$-Kr\left(-\frac{dv_z}{dr}\right)^{n} = -c\,, \qquad -\frac{dv_z}{dr} = \left(\frac{c}{K}\right)^{1/n}\frac{1}{r^{1/n}}$$

Integrate from wire to radius r

$$\underbrace{-\int_{U}^{v_z} dv_z}_{U - v_z} = \left(\frac{c}{K}\right)^{1/n}\int_{r_1}^{r}\frac{dr}{r^{1/n}} = \left(\frac{c}{K}\right)^{1/n}\frac{1}{q}\left(r^{q} - r_1^{q}\right)$$

where $q = 1 - \frac{1}{n}$

$(n \neq 1)$

$$v_z = U - \left(\frac{c}{K}\right)^{1/n}\frac{1}{q}\left(r^{q} - r_1^{q}\right)$$

$n < 1$ for Shear-Thinning

Hence q is negative, and both numerator & denominator are positive

At the surface of the die, $r = r_2$, $v_z = 0$

Hence $\left(\frac{c}{K}\right)^{1/n}\frac{1}{q} = \frac{U}{r_2^{q} - r_1^{q}}$

Velocity at any point is then

$$v_z = U - \frac{U\left(r^{q} - r_1^{q}\right)}{r_2^{q} - r_1^{q}} = U\,\frac{r^{q} - r_2^{q}}{r_1^{q} - r_2^{q}}$$

✓

11.6-2

Volumetric flow rate

$$Q = \int_{r_1}^{r_2} 2\pi r \, dr \; U \frac{r^q - r_2^q}{r_1^q - r_2^q}$$

$$= \frac{2\pi U}{r_1^q - r_2^q} \int_{r_1}^{r_2} \left(r^{q+1} - r r_2^q \right) dr$$

$$= \frac{2\pi U}{r_1^q - r_2^q} \left[\frac{r_2^{q+2} - r_1^{q+2}}{q+2} - \frac{r_2^q}{2} \left(r_2^2 - r_1^2 \right) \right]$$

Coating Thickness

From continuity, since velocity is uniformly U

$$Q = U \pi \left[(r_1 + \delta)^2 - r_1^2 \right]$$

$$Q = \pi U \left(2 r_1 \delta + \delta^2 \right)$$

✓

11.15-1

Determination of Constitutive Equation

Bingham Plastic

$$Q = \frac{\pi a^4}{2\eta}\left(-\frac{dp}{dz}\right)\left(\frac{1}{4} - \frac{\beta}{3} + \frac{\beta^4}{12}\right), \quad \beta = \frac{2\tau_0}{a(-dp/dz)}$$

Power-Law Fluid

$$Q = \left(-\frac{dp}{dz}\,\frac{1}{2K}\right)^{1/n} \frac{\pi n\, a^{3+\frac{1}{n}}}{1 + 3n}$$

Spreadsheet solution for best fit

Calculate predicted Q based on above two equations for assumed (τ_0, η) or (n, K). Then use the Solver to minimize sum of squared differences between observed and predicted Q's.

Power-law model is clearly the better fit, with

$$n = 0.437$$

$$K = 6.709 \ \frac{kg}{m\,s}\, s^{n-1}$$

$$= 67.09 \ P\, s^{n-1}$$

11.15-2

Test for Bingham Plastic			
a =	0.01	m	
tau_0 =	77.55	N/m^2	
eta =	0.03265	kg/m s	
pi =	3.14159		
factor =	15,510		
	Observed	Predicted	Squared
(-dp/dz)	Q	Q	Differences
Pa/m = N/m^3	m^3/s	m^3/s	
100,000	1.11E-02	9.54E-03	2.42E-06
90,000	9.12E-03	8.34E-03	6.07E-07
80,000	7.13E-03	7.14E-03	9.34E-11
70,000	4.87E-03	5.94E-03	1.14E-06
60,000	3.36E-03	4.74E-03	1.91E-06
50,000	2.35E-03	3.55E-03	1.43E-06
40,000	1.29E-03	2.36E-03	1.15E-06
30,000	6.89E-04	1.21E-03	2.68E-07
20,000	2.64E-04	2.08E-04	3.11E-09
10,000	5.37E-05	1.04E-03	9.64E-07
		Sum =	9.89E-06

Test for Power-Law Fluid			
n =	0.437	-	
K =	6.709	(kg/m s) s^(n-1)	
const =	1.5914E-11		
[pi*n*a^(1/n + 3)/(1 + 3*n)]			
	Observed	Predicted	Squared
(-dp/dz)	Q	Q	Differences
Pa/m = N/m^3	m^3/s	m^3/s	
100,000	1.11E-02	1.13E-02	5.51E-08
90,000	9.12E-03	8.91E-03	4.47E-08
80,000	7.13E-03	6.81E-03	1.05E-07
.70,000	4.87E-03	5.02E-03	2.11E-08
60,000	3.36E-03	3.53E-03	2.74E-08
50,000	2.35E-03	2.32E-03	6.79E-10
40,000	1.29E-03	1.40E-03	1.11E-08
[illegible]	[illegible]	[illegible]	1.15E-09