

11.5-1

Rotating Rod

$$\underline{\tau} = \begin{bmatrix} \tau_{tt} & \tau_{t\theta} & \tau_{tz} \\ \tau_{\theta t} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zt} & \tau_{z\theta} & \tau_{zz} \end{bmatrix}$$

$$\underline{\dot{\gamma}} = \begin{bmatrix} r \frac{\partial v_t}{\partial t} + \frac{\partial}{\partial t} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_t}{\partial \theta} & \frac{\partial v_z}{\partial t} + \frac{\partial v_t}{\partial z} \\ r \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_t}{r} \right) & \frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \\ \text{Symmetric} & r \frac{\partial v_z}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & + \frac{d}{dt} \left(\frac{v_\theta}{r} \right) & 0 \\ + \frac{d}{dt} \left(\frac{v_\theta}{r} \right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Assuming } v_t = v_z = 0$$

$\frac{\partial}{\partial \theta} = 0$

Strain rate $\dot{\gamma} = \sqrt{\frac{1}{2} \underline{\dot{\gamma}}_2} = \sqrt{\frac{1}{2} 2 \left[+ \frac{d}{dt} \left(\frac{v_\theta}{r} \right) \right]^2} = \left| + \frac{d}{dt} \left(\frac{v_\theta}{r} \right) \right|$
negative

θ - Momentum, steady state, constant pressure

$$\frac{\partial}{\partial t} (r^2 \tau_{t\theta}) = 0$$

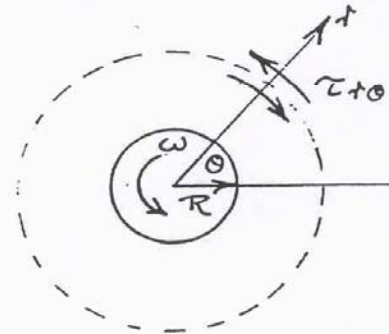
$$\underline{\tau} = K \dot{\gamma}^{n-1} \dot{\gamma} \quad \text{or} \quad \tau_{t\theta} = K \dot{\gamma}^{n-1} \dot{\gamma}_{t\theta}$$

$$\tau_{t\theta} = K \left| + \frac{d}{dt} \left(\frac{v_\theta}{r} \right) \right|^{n-1} + \frac{d}{dt} \left(\frac{v_\theta}{r} \right)$$

negative

$$= -K \left[- + \frac{d}{dt} \left(\frac{v_\theta}{r} \right) \right]^n$$

positive



11.5-2

$$\frac{d}{dt} \left\{ -r^2 K \left[-r \frac{d}{dt} \left(\frac{v_\theta}{r} \right) \right]^n \right\} = 0$$

$$-r^2 K \left[-r \frac{d}{dt} \left(\frac{v_\theta}{r} \right) \right]^n = c_1$$

$$-r \frac{d}{dt} \left(\frac{v_\theta}{r} \right) = \left(\frac{-c_1}{r^2 K} \right)^{1/n} = \left(\frac{-c_1}{K} \right)^{1/n} \frac{1}{r^{2/n}}$$

$$\frac{d}{dt} \left(\frac{v_\theta}{r} \right) = - \left(\frac{-c_1}{K} \right)^{1/n} \frac{1}{r^{2/n+1}}$$

Integrates to

$$\frac{v_\theta}{r} = \frac{n}{2} \left(\frac{-c_1}{K} \right)^{1/n} \frac{1}{r^{2/n}} + c_2 \quad (1)$$

Boundary condition

$$r \rightarrow \infty: v_\theta \rightarrow 0: c_2 = 0$$

$$r = R \quad v_\theta = \omega R:$$

$$\frac{\omega R}{R} = \frac{n}{2} \left(\frac{-c_1}{K} \right)^{1/n} \frac{1}{R^{2/n}} \quad (2)$$

Divide (1) by (2) to eliminate c_1 :

$$\frac{v_\theta}{\omega r} = \left(\frac{R}{r} \right)^{2/n}$$

Limiting case of large n (dilatant behavior)

$$\left(\frac{R}{r} \right)^{2/n \rightarrow \infty} \rightarrow 1 \quad \text{so} \quad v_\theta = \omega r$$

Fluid behaves as a solid-body rotation.

✓
11.5-3

Shear Stress

$$\begin{aligned}\tau_{+0} &= -K \left[-r \frac{d}{dr} \left(\frac{v_{\theta}}{r} \right) \right]^n \\ &= -K \left[-r \omega R^{2/n} \left(-\frac{2}{n} \right) \frac{1}{r^{2/n+1}} \right]^n \\ &= -K \omega^n R^2 \left(\frac{2}{n} \right)^n \frac{1}{r^2} = -K \left(\frac{2\omega}{n} \right)^n \frac{R^2}{r^2}\end{aligned}$$

Torque needed to Rotate Rod

$$(\tau_{+0})_{r=R} = -K \left(\frac{2\omega}{n} \right)^n$$

That is, the fluid exerts a stress on the rod in the negative θ direction — opposite to the direction of rotation.

Hence torque in the θ direction to rotate rod is

$$T_{\theta} = \underbrace{[2\pi R L K \left(\frac{2\omega}{n} \right)^n]}_{\text{area}} R \quad \leftarrow \text{lever arm radius}$$

$$\underline{\underline{T_{\theta} = 2\pi R^2 L K \left(\frac{2\omega}{n} \right)^n}}$$

11.6-1

Shear-Thinning Wite Coating

Power-law fluid

$$\tau_{rz} = \eta \dot{\gamma}_{rz} = K \dot{\gamma}^{n-1} \dot{\gamma}_{rz} = -K \left(-\frac{dv_z}{dr} \right)^{n-1} \left(-\frac{dv_z}{dr} \right) = -K \left(-\frac{dv_z}{dr} \right)^n$$

Momentum balance

$$\frac{1}{r} \frac{d}{dr} (r \tau_{rz}) = 0, \quad \frac{d}{dz} \left[-K r \left(-\frac{dv_z}{dr} \right)^n \right] = 0$$

$$-K r \left(-\frac{dv_z}{dr} \right)^n = -c, \quad -\frac{dv_z}{dr} = \left(\frac{c}{K} \right)^{1/n} \frac{1}{r^{1/n}}$$

Integrate from wite to radius r

$$-\int_{v_z}^{u} dv_z = \left(\frac{c}{K} \right)^{1/n} \int_{r_1}^r \frac{dr}{r^{1/n}} = \left(\frac{c}{K} \right)^{1/n} \frac{1}{q} (r^q - r_1^q)$$

$$\underbrace{u}_{u-v_z}$$

$$\text{where } q = 1 - \frac{1}{n} \quad (n \neq 1)$$

$$v_z = u - \left(\frac{c}{K} \right)^{1/n} \frac{1}{q} (r^q - r_1^q)$$

$n < 1$ for
Shear-Thinning

At the surface of the die, $r = r_2, v_z = 0$

$$\text{Hence } \left(\frac{c}{K} \right)^{1/n} \frac{1}{q} = \frac{u}{r_2^q - r_1^q}$$

Hence q is
negative, and both
numerator & denominator
are positive

Velocity at any point is then

$$v_z = u - \frac{u (r^q - r_1^q)}{r_2^q - r_1^q} = u \frac{r^q - r_2^q}{r_1^q - r_2^q}$$

✓

11.6-2Volumetric flow rate

$$\begin{aligned}
 Q &= \int_{r_1}^{r_2} 2\pi r \, dr \, u \frac{r^q - r_2^q}{r_1^q - r_2^q} \\
 &= \frac{2\pi u}{r_1^q - r_2^q} \int_{r_1}^{r_2} (r^{q+1} - r r_2^q) \, dr \\
 &= \frac{2\pi u}{r_1^q - r_2^q} \left[\frac{r^{q+2} - r_1^{q+2}}{q+2} - \frac{r_2^q}{2} (r_2^2 - r_1^2) \right]
 \end{aligned}$$

Coating Thickness

From continuity, since velocity is uniform u

$$Q = u \pi [(r_1 + \delta)^2 - r_1^2]$$

$$Q = \pi u (2r_1 \delta + \delta^2)$$

✓

11.15-1

Determination of Constitutive Equation

Bingham Plastic

$$Q = \frac{\pi a^4}{2\eta} \left(-\frac{dp}{dz}\right) \left(\frac{1}{4} - \frac{\beta}{3} + \frac{\beta^4}{12}\right), \quad \beta = \frac{2\tau_0}{a(-dp/dz)}$$

Power-Law Fluid

$$Q = \left(-\frac{dp}{dz} \frac{1}{2k}\right)^{1/n} \frac{\pi n a^{3+1/n}}{1+3/n}$$

Spreadsheet solution for best fit

Calculate predicted Q based on above two equations for assumed (τ_0, η) or (n, k) . Then use the Solver to minimize sum of squared differences between observed and predicted Q 's.

Power-law model is clearly the better fit,

with

$$n = 0.437$$

$$k = 6.709 \frac{\text{kg}}{\text{m s}} s^{n-1}$$

$$= 67.09 \text{ P s}^{n-1}$$

11.15-2

| Test for Bingham Plastic | | | |
|--|-------------------|-----------------------------|-------------|
| a = | 0.01 | m | |
| tau_0 = | 77.55 | N/m ² | |
| eta = | 0.03265 | kg/m s | |
| pi = | 3.14159 | | |
| factor = | 15,510 | | |
| | | | |
| | Observed | Predicted | Squared |
| (-dp/dz) | Q | Q | Differences |
| Pa/m = N/m ³ | m ³ /s | m ³ /s | |
| | | | |
| 100,000 | 1.11E-02 | 9.54E-03 | 2.42E-06 |
| 90,000 | 9.12E-03 | 8.34E-03 | 6.07E-07 |
| 80,000 | 7.13E-03 | 7.14E-03 | 9.34E-11 |
| 70,000 | 4.87E-03 | 5.94E-03 | 1.14E-06 |
| 60,000 | 3.36E-03 | 4.74E-03 | 1.91E-06 |
| 50,000 | 2.35E-03 | 3.55E-03 | 1.43E-06 |
| 40,000 | 1.29E-03 | 2.36E-03 | 1.15E-06 |
| 30,000 | 6.89E-04 | 1.21E-03 | 2.68E-07 |
| 20,000 | 2.64E-04 | 2.08E-04 | 3.11E-09 |
| 10,000 | 5.37E-05 | 1.04E-03 | 9.64E-07 |
| | | Sum = | 9.89E-06 |
| | | | |
| Test for Power-Law Fluid | | | |
| n = | 0.437 | - | |
| K = | 6.709 | (kg/m s) s ⁽ⁿ⁻¹⁾ | |
| const = | 1.5914E-11 | | |
| [pi*n*a ^{(1/n + 3)/(1 + 3*n)}] | | | |
| | Observed | Predicted | Squared |
| (-dp/dz) | Q | Q | Differences |
| Pa/m = N/m ³ | m ³ /s | m ³ /s | |
| | | | |
| 100,000 | 1.11E-02 | 1.13E-02 | 5.51E-08 |
| 90,000 | 9.12E-03 | 8.91E-03 | 4.47E-08 |
| 80,000 | 7.13E-03 | 6.81E-03 | 1.05E-07 |
| 70,000 | 4.87E-03 | 5.02E-03 | 2.11E-08 |
| 60,000 | 3.36E-03 | 3.53E-03 | 2.74E-08 |
| 50,000 | 2.35E-03 | 2.32E-03 | 6.79E-10 |
| 40,000 | 1.29E-03 | 1.40E-03 | 1.11E-08 |
| | | | |