Handout_20
Boundary Layer Flow Over a Flat Plate
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The momentum balance on a control volume of the boundary layer leads to the following equation

$$\boxed{V_{x\infty} \frac{d}{dx} \int_0^\delta V_x \, dy = \frac{\tau_w}{\rho} + \frac{d}{dx} \int_0^\delta V_x^2 \, dy}$$

The approximate methode of boundary layer analysis can be summarized as follows:

1. assume $V_x = f(x, y)$
2. Solve $\int_0^\delta V_x \, dy$ and $\int_0^\delta V_x^2 \, dy$.
3. solve $\tau_w = \left.\frac{dV_x}{dy}\right|_{y=0} * \mu$.
4. Substitute in the above momentum balance and get δ & τ_w as function of x and fluid properties (ρ, μ, $V_{x\infty}$)

How to assume $V_x = f(x, y)$?

$V_x = f(x,y)$ can be assumed as follows:

$$\frac{V_x}{V_{x\infty}} = f\left(\xi\right)$$

where $\xi \equiv \dfrac{y}{\delta(x)}$

$f(\xi)$ has to satisfy the following properties:

1. $f(0) = 0$ $\quad (V_x = 0 \text{ @ } y=0)$

2. $f(1) = 1$ $\quad (V_x = V_{x\infty} \text{ @ } y=\delta)$

3. $\left.\dfrac{df}{d\xi}\right|_{\xi=1} = 0$ $\quad \left(\left.\dfrac{dV_x}{dy}\right|_{y=\delta} = 0 \right)$

4. $\left.\dfrac{d^2f}{d\xi^2}\right|_{\xi=0} = 0$

5. $\left.\dfrac{d^2f}{d\xi^2}\right|_{\xi=1} = 0$

Laminar Boundary Layer:

① Assume $\frac{V_x}{V_{x\infty}} = \underbrace{\sin\left(\frac{\pi}{2}\frac{\delta}{\delta}\right)}$

↑ satisfies conditions listed before

$$\Rightarrow V_x = V_{x\infty} \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

② evaluate:
$$\int_0^\delta V_x\,dy = \int_0^\delta V_{x\infty} \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) dy$$
$$= - V_{x\infty}\frac{2}{\pi}\delta \left[\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]_0^\delta$$
$$= \boxed{\frac{2}{\pi} V_{x\infty}\,\delta}$$

evaluate:
$$\int_0^\delta v_x^2\,dy = \int_0^\delta V_{x\infty}^2 \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) dy$$
$$= V_{x\infty}^2 \int_0^\delta \left[1 - \cos^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] dy$$
$$= V_{x\infty}^2 \left[\int_0^\delta dy - \int_0^\delta \cos^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) dy\right]$$

See Appendix A

$$= V_{x\infty}^2 \left[\left[y\right]_0^\delta - \frac{2\delta}{\pi}\frac{1}{2}\left[\frac{\pi}{2}\frac{y}{\delta} + \frac{1}{2}\sin\left(2\,\frac{\pi}{2}\frac{y}{\delta}\right)\right]_0^\delta \right]$$

$$= V_{x\infty}^2 \left[\delta - \frac{\delta}{\pi}\left[\frac{\pi}{2} + 0 - 0 - 0\right] \right].$$

$$= V_{x\infty}^2 \left[\delta - \frac{\delta}{2} \right] = \boxed{V_{x\infty}^2 \frac{\delta}{2}}$$

(3) $\tau_w = \mu \left.\frac{dV_x}{dy}\right|_{y=0}$

$$\frac{dV_x}{dy} = \frac{d}{dy}\left(V_{x\infty} \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right)$$

$$= V_{x\infty} \frac{\pi}{2}\frac{1}{\delta} \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

$$\tau_w = \left[\mu\, V_{x\infty} \frac{\pi}{2}\frac{1}{\delta} \cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]_{y=0}.$$

$$= \boxed{\mu V_{x\infty} \pi / (2\delta)}$$

(4) Substitute in momentum balance.

$$V_{x\infty} \frac{d}{dx}\left(\frac{2}{\pi} V_{x\infty}\delta\right) = \frac{\mu V_{x\infty}\pi}{2\delta\rho} + \frac{d}{dx}\left(V_{x\infty}^2 \frac{\delta}{2}\right)$$

$$\vdots$$

$$\frac{2}{\pi} V_{x\infty}^{\cancel{2}} \frac{d\delta}{dx} = \frac{\mu \cancel{V_{x\infty}}\pi}{2\delta\rho} + \frac{V_{x\infty}^{\cancel{2}}}{2}\frac{d\delta}{dx}$$

$$\vdots$$

$$V_{x\infty}\underbrace{\left(\frac{2}{\pi}-\frac{1}{2}\right)}_{0.137}\frac{d\delta}{dx} = \frac{\mu\pi}{2\delta\rho}$$

$$0.137\, V_{x\infty}\frac{d\delta}{dx} = \frac{\mu\pi}{2\delta\rho} \quad \text{--- solve}$$

$$\int_0^\delta \delta\, d\delta = \frac{\mu\pi}{(2)(0.137)V_{x\infty}\rho}\int_0^x dx$$

$$\frac{\delta^2}{\cancel{2}} = \frac{\mu\,\pi\, x}{\cancel{(2)}(0.137)\,V_{x\infty}\,\rho} \qquad (\div x^2)$$

$$\frac{\delta^2}{x^2} = \frac{\mu\pi}{0.137\,\rho\, V_{x\infty}\, x}$$

$$= \frac{\pi}{0.137}\,\frac{1}{Re_x}$$

$$\frac{\delta}{x} = \sqrt{\frac{\pi}{0.137}} \; \frac{1}{\sqrt{Re_x}} = \frac{4.79}{\sqrt{Re_x}}$$

Recall, $$\tau_w = \mu \left.\frac{dV_x}{dy}\right|_{y=0} = \frac{\mu V_{x\infty} \pi}{2\delta}$$

$$\tau_w = \frac{\mu V_{x\infty} \pi}{(2)(4.79)x} \sqrt{Re_x}$$

$$= \frac{\pi}{(2)(4.79)} \; \frac{\mu V_{x\infty}}{x} \left(\frac{\rho V_{x\infty} x}{\mu} \right)^{1/2}$$

$$= \frac{\pi}{(2)(4.79)} \; \frac{\mu^{1/2} V_{x\infty}^{3/2} \rho^{1/2}}{x^{1/2}}$$

$$\boxed{\tau_w = 0.328 \, V_{x\infty} \sqrt{\mu \rho V_{x\infty}} \; x^{-\frac{1}{2}}}$$

<u>drag Coefficient:</u> $$C_f \equiv \frac{\tau_w}{\frac{1}{2} \rho V_{x\infty}^2}$$

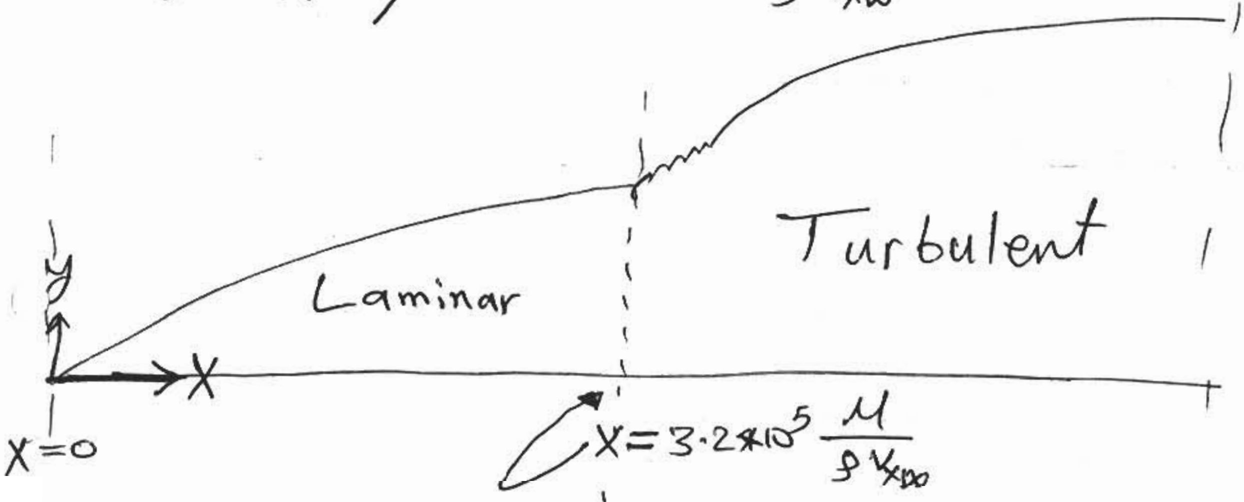
$$C_f = \frac{0.328 \, \cancel{V_{x\infty}} \sqrt{\mu \rho V_{x\infty}}}{\frac{1}{2} \rho V_{x\infty}^{\cancel{2}} \; x^{1/2}} = 2(0.328) * \frac{\mu^{1/2}}{\rho^{1/2} V_{x\infty}^{1/2} x^{1/2}}$$

$$= \frac{0.656}{\sqrt{Re_x}}$$

Turbulent Boundary Layer

The preceding analysis was for laminar B.L. However, it is found experimentally that wehen $Re_x > 3.2 * 10^5$, the boundary layer becomes turbulent over a flat plate. $\Rightarrow$ $\frac{\rho V_{x\infty} X}{\mu} > 3.2 * 10^5$

$$\Rightarrow X > 3.2 * 10^5 \frac{\mu}{\rho V_{x\infty}}$$



From experience $\frac{V_x}{V_{x\infty}} = \left(\frac{y}{\delta}\right)^{1/7}$ for turbulent flow. $\Rightarrow$ $V_x = V_{x\infty}\left(\frac{y}{\delta}\right)^{1/7}$ Turbulent Boundary Layer.

$$\int_0^\delta V_x \, dy = \frac{V_{x\infty}}{\delta^{1/7}} \frac{7}{8}\left[\left(y\right)^{\frac{8}{7}}\right]_0^\delta = \frac{7}{8} V_{x\infty} \delta$$

$$\int_0^\delta V_x^2 \, dy = \frac{V_{x\infty}^2}{\delta^{2/7}} \frac{7}{9}\left[y^{\frac{9}{7}}\right]_0^\delta = \frac{7}{9} V_{x\infty}^2 \delta$$

$$\tau_w = \mu \left.\frac{dV_x}{dy}\right|_{y=0} = \mu \; \delta^{-\frac{1}{7}} \; \frac{1}{7}\left[y^{-\frac{6}{7}}\right]\Big|_{y=0}.$$

$$= \frac{\mu \, \delta^{-\frac{1}{7}} \frac{1}{7}}{(0)^{\frac{6}{7}}} = \infty$$

Therefore, we have a problem that

$\frac{V_x}{V_{x\infty}} = \left(\frac{y}{\delta}\right)^{1/7}$ pridicts infinit τ_w

However, from experiment;

$$\tau_w / \tfrac{1}{2}\rho V_{x\infty}^2 = 0.045\left(\frac{\mu}{\rho V_{x\infty}\delta}\right)^{1/4}$$

$$\Rightarrow \tau_w = \frac{1}{2} \rho V_{x\infty}^2 \quad 0.045 \left(\frac{\mu}{\rho V_{x\infty} \delta}\right)^{1/4}$$

Recall, $\quad V_{x\infty} \frac{d}{dx} \int_0^\delta V_x \, dy = \frac{\tau_w}{\rho} + \frac{d}{dx} \int_0^\delta V_x^2 \, dy.$

$$\Rightarrow \cancel{V_{x\infty}} \; \frac{7}{8} \cancel{V_{x\infty}} \frac{d\delta}{dx} = \cancel{V_{x\infty}^2} \frac{0.045}{2} \left(\frac{\mu}{\rho V_{x\infty}}\right)^{1/4} \frac{1}{\delta^{1/4}} + \frac{7}{9} \cancel{V_{x\infty}^2} \frac{d\delta}{dx}.$$

0.097

$$\left(\frac{7}{8} - \frac{7}{9}\right) \frac{d\delta}{dx} = \frac{0.045}{2} \left(\frac{\mu}{\rho V_{x\infty}}\right)^{1/4} \frac{1}{\delta^{1/4}}$$

$$\int_0^\delta \delta^{1/4} \, d\delta = \frac{0.045}{(2)(0.097)} \left(\frac{\mu}{\rho V_{x\infty}}\right)^{1/4} \int_0^x dx.$$

$$\frac{4}{5} \delta^{\frac{5}{4}} = \frac{0.045}{(2)(0.097)} \left(\frac{\mu}{\rho V_{x\infty}}\right)^{1/4} x$$

$$\delta^{\frac{5}{4}} = \frac{(0.045)(5)}{(2)(4)(0.097)} \left(\frac{\mu}{\rho V_{x\infty}}\right)^{1/4} x.$$

$$= 0.29 \left(\frac{\mu}{\rho V_{x\infty}}\right)^{1/4} x \qquad \div \left(x^{\frac{5}{4}}\right)$$

$$\left(\frac{\delta}{X}\right)^{\frac{5}{4}} = 0.29 \left(\frac{\mu}{\rho V_{x\infty}}\right)^{1/4} X^{-\frac{1}{4}}$$

$$= \frac{0.29}{\left(Re_x\right)^{1/4}}$$

$$\Rightarrow \frac{\delta}{X} = \left(\frac{0.29}{Re_x^{1/4}}\right)^{\frac{4}{5}} = \frac{0.376}{Re_x^{\frac{1}{5}}}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho V_{x\infty}^2} = 0.045 \left(\frac{\mu}{\rho V_{x\infty} \frac{0.376 X}{Re_x^{1/5}}}\right)^{1/4}$$

$$= (0.045)(0.376)^{-\frac{1}{4}} \left(\frac{Re_x^{1/5}}{\underbrace{\frac{\rho V_{x\infty} X}{\mu}}_{Re_x.}}\right)^{1/4}$$

$$= \frac{0.0576}{\left(Re_x\right)^{1/5}}$$

(Summary)

Laminar B.L.

$$Re_x \leq 3.2 * 10^5$$

$$\Rightarrow X \leq 3.2 * 10^5 \frac{\mu}{\rho V_{x\infty}}$$

$$\frac{\delta}{x} = \frac{4.79}{\sqrt{Re_x}} \quad \text{---- eqn. } (8.16)$$

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho V_{x\infty}^2} = \frac{0.656}{\sqrt{Re_x}} \quad \text{----} (8.17)$$

Turbulent B.L.

$$Re_x > 3.2 * 10^5$$

$$\Rightarrow X > 3.2 * 10^5 \frac{\mu}{\rho V_{x\infty}}$$

$$\frac{\delta}{X} = \frac{0.376}{(Re_x)^{1/5}} \quad \text{-----} (8.61)$$

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho V_{x\infty}^2} = \frac{0.0576}{(Re_x)^{1/5}} \quad \text{----} (8.62)$$

Example Air flows over a flat plat with velocity $50 \frac{ft}{s}$. The kinematic viscosity $\frac{\mu}{\rho}$ of air is $1.8 \times 10^{-4} \frac{ft^2}{s}$

Calculate the drag force (F_D) over the flat plate if the plate is 5 ft long along the direction of flow. $\rho_{air} = 0.075 \frac{lbm}{ft^3}$

Solution

$$\cancel{F_D = \underbrace{A}_{\text{area per unit width}} \tau_w}$$

however, $A = f(x)$ and $\tau_w = f(x)$

$$\Rightarrow F_D = \int \tau_w \, dA \quad ; \quad dA = \underbrace{dx}_{\left(\frac{\text{area}}{\text{unit width}}\right)}$$

$$\Rightarrow F_D = \int_0^L \tau_w \, dx \quad ; \quad L = 5 \text{ ft}$$

(13)

$$F_D = \int_0^{L_t} (\tau_w)_{Laminar}\, dx + \int_{L_t}^{L} (\tau_w)_{turbulent}\, dx.$$

$L \rightarrow 5$ ft given

Length at which transition to Tubulent flow happens

$$L_t = 3.2 * 10^5 \frac{\mu}{\rho V_{x\infty}} = 3.2 * 10^5 \frac{1.8 * 10^{-4}}{50}$$

$$= 1.152 \text{ ft}$$

Note: if L_t was $> L$ then the whole B.L. is Laminar $\Rightarrow F_D = \int_0^L (\tau_w)_{laminar}\, dx$ only.

But in this problem $L_t < L$

$$\Rightarrow F_D = \int_0^{1.152} \left[\frac{1}{2}\rho V_{x\infty}^2\ 0.656 \left(\frac{\rho V_{x\infty}}{\mu}\right)^{-\frac{1}{2}} x^{-\frac{1}{2}}\right] dx$$

$$+ \int_{1.152}^{5} \left[\frac{1}{2}\rho V_{x\infty}^2 (0.0576) \left(\frac{\rho V_{x\infty}}{\mu}\right)^{-\frac{1}{5}} x^{-\frac{1}{5}}\right) dx$$

$$= \frac{1}{2}\rho V_{x\infty}^2 \left\{ 0.656 \left(\frac{\rho V_{x\infty}}{\mu}\right)^{-\frac{1}{2}} 2\left(x^{\frac{1}{2}}\right)\Big|_0^{1.152} + 0.0576 \left(\frac{\rho V_{x\infty}}{\mu}\right)^{-\frac{1}{5}} \frac{5}{4} x^{4/5}\Big|_{1.152}^{5} \right\}$$

(14)

$$F_D = \frac{1}{2}\rho V_{x\infty}^2 \left\{ 1.511\left(\frac{\rho V_{x\infty}}{\mu}\right)^{-\frac{1}{2}} + 0.180\left(\frac{\rho V_{x\infty}}{\mu}\right)^{-\frac{1}{5}} \right\}$$

$$= \frac{1}{2}(0.075)(50)^2 \left\{ 1.511\left(\frac{50}{1.8\times10^{-4}}\right)^{-\frac{1}{2}} + 0.180\left(\frac{50}{1.8\times10^{-4}}\right)^{-\frac{1}{5}} \right\}.$$

$$= 1.644 \left(\frac{lb_m \cdot ft}{s^2}\right)\left(\frac{1}{ft}\right)$$

per unit width.

$$= 0.051 \ \frac{lbf}{ft}$$