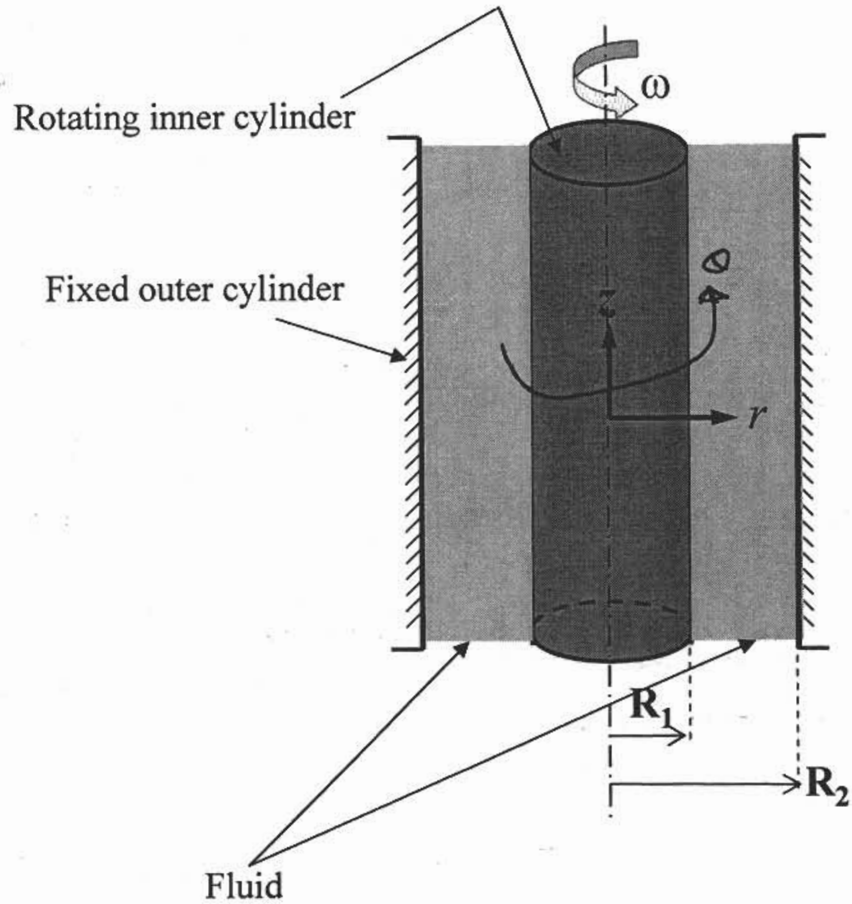
13. *Torque in a Couette viscometer—M.* Fig. P6.13 shows the horizontal cross section of a concentric cylinder or "Couette" viscometer, which is an apparatus for determining the viscosity μ of the fluid that is placed between the two vertical cylinders. The inner and outer cylinders have radii of r_1 and r_2, respectively. If the inner cylinder is rotated with a steady angular velocity ω, and the outer cylinder is stationary, derive an expression for v_θ (the θ velocity component) as a function of radial location r.



gravity ↓

Assumptions:

1. steady state
2. Newtonian fluid
3. There is one component of the velocity V_θ
   $V_r = V_z = 0$
4. Axisymmetric problem $\frac{\partial}{\partial \theta}(\text{any thing}) = 0$
5. $V_\theta = f(r)$ only.

Solution:

Continuity equation: $$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \cancelto{0}{\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r}} + \cancelto{0}{\frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta}} + \cancelto{0}{\frac{\partial(\rho v_z)}{\partial z}} = 0.$$

$\Rightarrow \quad 0 = 0$ no useful information

Navier-Stokes equations:

$v_r = v_z = 0$, $v_\theta = f(r)$ only and $\frac{\partial}{\partial \theta}(\ ) = 0$

$$\rho\left(\cancelto{0}{\frac{\partial v_r}{\partial t}} + \cancelto{0}{v_r\frac{\partial v_r}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta^2}{r} + \cancelto{0}{v_z\frac{\partial v_r}{\partial z}}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left[\cancelto{0}{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_r)}{\partial r}\right)} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} - \cancelto{0}{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0}{\frac{\partial^2 v_r}{\partial z^2}}\right] + \cancelto{0}{\rho g_r},$$

$$\rho\left(\cancelto{0}{\frac{\partial v_\theta}{\partial t}} + \cancelto{0}{v_r\frac{\partial v_\theta}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0}{\frac{v_r v_\theta}{r}} + \cancelto{0}{v_z\frac{\partial v_\theta}{\partial z}}\right)$$
$$= \cancelto{0}{-\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right) + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancelto{0}{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancelto{0}{\frac{\partial^2 v_\theta}{\partial z^2}}\right] + \cancelto{0}{\rho g_\theta},$$

$$\rho\left(\cancelto{0}{\frac{\partial v_z}{\partial t}} + \cancelto{0}{v_r\frac{\partial v_z}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancelto{0}{v_z\frac{\partial v_z}{\partial z}}\right)$$
$$= -\frac{\partial p}{\partial z} + \mu\left[\cancelto{0}{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)} + \cancelto{0}{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z$$

$$-\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} \quad \text{----} \quad (1)$$

$$0 = \mu \frac{d}{dr}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) \quad \text{--} \quad (2)$$

$$0 = -\frac{\partial p}{\partial z} + \rho g_z \quad \text{---} \quad (3)$$

(2) $\Rightarrow \quad \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r v_\theta)\right) = 0$

integrate once $\Rightarrow \quad \frac{1}{r}\frac{d}{dr}(r v_\theta) = c_1$

, again $\Rightarrow \quad r v_\theta = \frac{c_1}{2} r^2 + c_2$

$\Rightarrow \quad v_\theta = c_1' r + \frac{c_2}{r} \qquad \left(c_1' = \frac{c_1}{2}\right)$

boundary conditions: $\quad r = R_1 \qquad v_\theta = \omega R_1$

$\qquad r = R_2 \qquad v_\theta = 0$

BC1 : $\quad \omega R_1 = c_1' R_1 + \frac{c_2}{R_1}$

BC2 : $\quad 0 = c_1' R_2 + \frac{c_2}{R_2}$

solve $\Rightarrow c_1' = \dfrac{-R_1 \omega}{\frac{R_2^2}{R_1} - R_1}$ & $c_2 = \dfrac{R_1 \omega}{\frac{1}{R_1} - \frac{R_1}{R_2^2}}$

$$\Rightarrow \boxed{v_\theta = \frac{R_1 \omega}{R_1 - \frac{R_2^2}{R_1}}\, r + \frac{R_1 \omega}{\frac{1}{R_1} - \frac{R_1}{R_2^2}}\, \frac{1}{r}}$$

<u>Stress:</u> the is one component of the stress tensor

$$\tau_{r\theta} = \tau_{\theta r} = \mu\, r \frac{d}{dr}\left(\frac{v_\theta}{r}\right)$$

$$\Rightarrow \tau_{r\theta} = \frac{-2\mu R_1 \omega}{\frac{1}{R_1} - \frac{R_1}{R_2}} \frac{1}{r^2}$$

Force required to rotate inner cylinder:

$$F = \tau_{r\theta}\Big|_{r=R_1} * \underbrace{2\pi R_1 L}_{\text{Area of inner cylinder}}$$

$$= \frac{-4\pi\mu \cancel{R_1^2} \omega L}{\left(\frac{1}{R_1} - \frac{R_1}{R_2}\right)} \frac{1}{\cancel{R_1^2}}$$

Torque $= F$ * unit arm

$= F * R_1$

$$\Rightarrow T = \frac{-4\pi\mu\omega L R_1}{\frac{1}{R_1} - \frac{R_1}{R_2}}$$

$$\Rightarrow \mu = \frac{T\left(\frac{1}{R_1} - \frac{R_1}{R_2}\right)}{-4\pi\omega L R_1}$$