Example 6.5—Flow Through an Annular Die

Following the discussion of polymer processing in the previous section, now consider flow through a die that could be located at the exit of the screw extruder of Example 6.4. Consider a die that forms a *tube* of polymer (other shapes being sheets and filaments). In the die of length L shown in Fig. E6.5, a pressure difference $p_2 - p_3$ causes a liquid of viscosity μ to flow steadily from left to right in the annular area between two fixed concentric cylinders. Note that p_2 is chosen for the inlet pressure in order to correspond to the extruder exit pressure from Example 6.4. The inner cylinder is solid, whereas the outer one is hollow; their radii are r_1 and r_2, respectively. The problem, which could occur in the extrusion of plastic tubes, is to find the velocity profile in the annular space and the total volumetric flow rate Q. Note that *cylindrical* coordinates are now involved.

Assumptions and continuity equation. The following assumptions are realistic:

1. There is only one nonzero velocity component, namely that in the direction of flow, v_z. Thus, $v_r = v_\theta = 0$.
2. Gravity acts vertically downwards, so that $g_z = 0$.
3. The axial velocity is independent of the angular location; that is, $\partial v_z / \partial \theta = 0$.

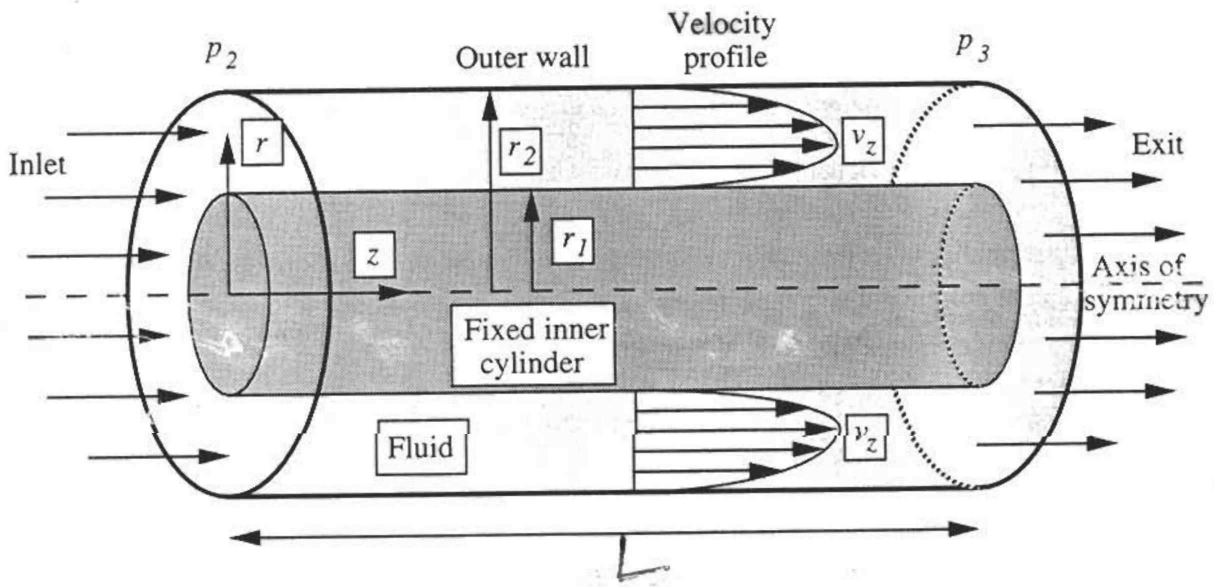

Fig. E6.5 Geometry for flow through an annular die.

Solution:

Continuity equation: $$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{0} + \underbrace{\frac{1}{r}\frac{\partial(\rho r \cancel{v_r})}{\partial r}}_{0} + \underbrace{\frac{1}{r}\frac{\partial(\rho \cancel{v_\theta})}{\partial \theta}}_{0} + \frac{\partial(\rho v_z)}{\partial z} = 0,$$

$\Rightarrow \dfrac{\partial V_z}{\partial z} = 0$, $V_z \neq f(z)$ also for axisymmetric flow
$V_z \neq f(\theta) \Rightarrow V_z = f(r)$ only.

Navier-Stokes Equations:

$v_r = v_\theta = 0$, $\frac{\partial}{\partial \theta}(\text{any thing}) = 0$ (axisymmetric problem)

ss, and $v_z = f(r)$ only.

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r,$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta,$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z.$$

simplify: $$0 = -\frac{\partial P}{\partial z} + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right]$$

pressure variation in z-direction is linear

$$\Rightarrow \frac{\partial P}{\partial z} = \frac{\Delta P}{L}$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{\Delta P}{\mu L}$$

integrate twice:

$$v_z = \frac{\Delta P}{4\mu L} r^2 + C_1 \ln(r) + C_2$$

boundary conditions:

$r = r_1 \qquad v_z = 0 \qquad$ (no-slip BC)

$r = r_2 \qquad v_z = 0 \qquad$ (no-slip BC)

$$\Rightarrow 0 = \frac{\Delta P}{4\mu L} r_1^2 + c_1 \ln(r_1) + c_2 \quad --- \quad (1)$$

$$0 = \frac{\Delta P}{4\mu L} r_2^2 + c_1 \ln(r_2) + c_2 \quad --- \quad (2)$$

$$(1) - (2) \Rightarrow \frac{\Delta P}{4\mu L}\left(r_1^2 - r_2^2\right) + c_1 \ln\left(\frac{r_1}{r_2}\right) = 0$$

$$\Rightarrow c_1 = \frac{\frac{\Delta P}{4\mu L}\left(r_2^2 - r_1^2\right)}{\ln\left(\frac{r_1}{r_2}\right)}$$

substitute in (2):

$$c_2 = -\frac{\Delta P}{4\mu L} r_2^2 - \frac{\frac{\Delta P}{4\mu L}\left(r_2^2 - r_1^2\right)}{\ln\left(\frac{r_1}{r_2}\right)} \ln(r_2)$$

substitute c_1 & c_2 in velocity profile

$$\Rightarrow v_z = \frac{\Delta P}{4\mu L}\left[\left(r^2 - r_2^2\right) + \frac{r_2^2 - r_1^2}{\ln\left(\frac{r_1}{r_2}\right)} \ln\left(\frac{r}{r_2}\right)\right]$$

<u>Volumetric flow rate</u>

$$Q = \int_A v_z \, dA$$

$$Q = \int_0^{2\pi} \int_{r_1}^{r_2} v_z \, r\,dr\,d\theta$$

⋮ do it yourself

$$Q = \frac{\pi (r_2^2 - r_1^2)}{8\mu} \frac{\Delta P}{L} \left[\frac{r_2^2 - r_1^2}{\ln\left(\frac{r_2}{r_1}\right)} - (r_2^2 + r_1^2) \right]$$

Stress Tensor:

recall $v_r = v_\theta = 0$ $\quad \frac{\partial}{\partial \theta}(\ ) = 0$

$v_z = f(r)$ only

$$\tau_{r\theta} = \tau_{\theta r} = \mu \left(\cancel{r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)}^{0} + \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}^{0} \right),$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu \left(\cancel{\frac{\partial v_\theta}{\partial z}}^{0} + \cancel{\frac{1}{r}\frac{\partial v_z}{\partial \theta}}^{0} \right),$$

$$\tau_{zr} = \tau_{rz} = \mu \left(\frac{\partial v_z}{\partial r} + \cancel{\frac{\partial v_r}{\partial z}}^{0} \right).$$

$$\sigma_{rr} = -p + \cancel{2\mu \frac{\partial v_r}{\partial r}}^{0} - \cancel{\frac{2}{3}\mu \nabla \cdot \mathbf{v}}^{0},$$

$$\sigma_{\theta\theta} = -p + 2\mu \left(\cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}^{0} + \cancel{\frac{v_r}{r}}^{0} \right) - \cancel{\frac{2}{3}\mu \nabla \cdot \mathbf{v}}^{0},$$

$$\sigma_{zz} = -p + \cancel{2\mu \frac{\partial v_z}{\partial z}}^{0} - \cancel{\frac{2}{3}\mu \nabla \cdot \mathbf{v}}^{0}.$$

$$\tau_{rz} = \tau_{zr} = \mu \frac{dv_z}{dr}$$

$$= \frac{\Delta P}{4L} \left[2r + \frac{r_2^2 - r_1^2}{\ln\left(\frac{r_1}{r_2}\right)} \frac{1}{r} \right]$$