Example 6.3—Film Flow on a Moving Substrate

Fig. E6.3.1 shows a coating experiment involving a flat photographic film that is being pulled up from a processing bath by rollers with a steady velocity U at an angle θ to the horizontal. As the film leaves the bath, it entrains some liquid, and in this particular experiment it has reached the stage where: (a) the velocity of the liquid in contact with the film is $v_x = U$ at $y = 0$, (b) the thickness of the liquid is constant at a value δ, and (c) there is no net flow of liquid (as much is being pulled up by the film as is falling back by gravity). (Clearly, if the film were to retain a permanent coating, a net upwards flow of liquid would be needed.)

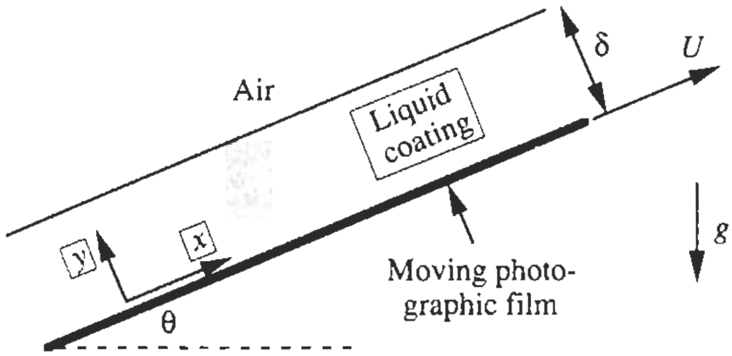


Fig. E6.3.1 Liquid coating on a photographic film.

Perform the following tasks:

1. Write down the differential mass balance and simplify it.
2. Write down the differential momentum balances in the x and y directions. What are the values of g_x and g_y in terms of g and θ? Simplify the momentum balances as much as possible.
3. From the simplified y momentum balance, derive an expression for the pressure p as a function of y, ρ, δ, g, and θ, and hence demonstrate that $\partial p/\partial x = 0$. Assume that the pressure in the surrounding air is zero everywhere.
4. From the simplified x momentum balance, assuming that the air exerts a negligible shear stress τ_{yx} on the surface of the liquid at $y = \delta$, derive an expression for the liquid velocity v_x as a function of U, y, δ, and α, where $\alpha = \rho g \sin\theta/\mu$.
5. Also derive an expression for the net liquid flow rate Q (per unit width, normal to the plane of Fig. E6.3.1) in terms of U, δ, and α. Noting that $Q = 0$, obtain an expression for the film thickness δ in terms of U and α.
6. Sketch the velocity profile v_x, labeling all important features.

Assumptions and continuity. The following assumptions are reasonable:

1. The flow is steady and Newtonian, with constant density ρ and viscosity μ.
2. The z direction, normal to the plane of the diagram, may be disregarded entirely. Thus, not only is v_z zero, but all derivatives with respect to z, such as $\partial v_x/\partial z$, are also zero.
3. There is only one nonzero velocity component, namely, that in the direction of motion of the photographic film, v_x. Thus, $v_y = v_z = 0$.
4. Gravity acts vertically downwards.

Solution:

Continuity Equation:

$v_y = 0 \quad v_z = 0$

$$\frac{\partial v_x}{\partial x} + \cancelto{0}{\frac{\partial v_y}{\partial y}} + \cancelto{0}{\frac{\partial v_z}{\partial z}} = 0.$$

$$\Rightarrow \frac{\partial v_x}{\partial x} = 0 \quad \Rightarrow v_x \neq f(x)$$

also for wide z-direction $v_x \neq f(z)$

conclusion $\underline{\underline{v_x = f(y)}}$ only

Navier-Stokes Equations:

SS, $v_y = 0$, $v_z = 0$ & $v_x = f(y)$ only $\Rightarrow$

$$\rho\left(\cancelto{0}{\frac{\partial v_x}{\partial t}} + \cancelto{0}{v_x\frac{\partial v_x}{\partial x}} + \cancelto{0}{v_y\frac{\partial v_x}{\partial y}} + \cancelto{0}{v_z\frac{\partial v_x}{\partial z}}\right) = -\cancelto{0}{\frac{\partial p}{\partial x}} + \mu\left(\cancelto{0}{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancelto{0}{\frac{\partial^2 v_x}{\partial z^2}}\right) + \rho g_x,$$

$$\rho\left(\cancelto{0}{\frac{\partial v_y}{\partial t}} + \cancelto{0}{v_x\frac{\partial v_y}{\partial x}} + \cancelto{0}{v_y\frac{\partial v_y}{\partial y}} + \cancelto{0}{v_z\frac{\partial v_y}{\partial z}}\right) = -\frac{\partial p}{\partial y} + \mu\left(\cancelto{0}{\frac{\partial^2 v_y}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v_y}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v_y}{\partial z^2}}\right) + \rho g_y,$$

$$\rho\left(\cancelto{0}{\frac{\partial v_z}{\partial t}} + \cancelto{0}{v_x\frac{\partial v_z}{\partial x}} + \cancelto{0}{v_y\frac{\partial v_z}{\partial y}} + \cancelto{0}{v_z\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left(\cancelto{0}{\frac{\partial^2 v_z}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 v_z}{\partial z^2}}\right) + \cancelto{0}{\rho g_z}.$$

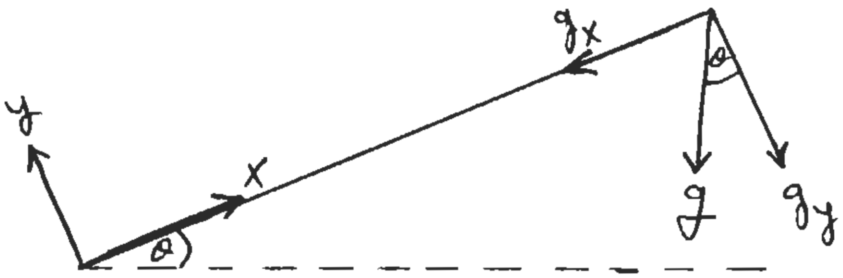
Note: In this problem flow is due to

$\boxed{\frac{\partial P}{\partial x} = 0} \rightarrow$ ① drag (friction between plate and fluid)
② gravity

simplify:
$$0 = \mu \frac{d^2 v_x}{dy^2} + \rho g_x \quad \text{-- ①}$$

$$0 = -\frac{\partial P}{\partial y} + \rho g_y \quad \text{--- ②}$$

$$\nearrow 0 \quad \text{--- ③}$$



$$g_x = -g \sin(\theta)$$

$$g_y = -g \cos(\theta)$$

Pressure Profile:

3 $\Rightarrow$ $P \neq f(z)$ also $P \neq f(x)$ (see note in previous page)

$\Rightarrow$ $P = f(y)$ only

integrate 2 $\int dP = \int \rho g_y \, dy$.

$$P = -\rho g \cos(\theta) y + C$$ ← const. of integration

pressure boundary condition: $y = \delta$ $P = P_{atm} = 0$ (gauge)

$\Rightarrow C = \rho g \cos(\theta) \delta$

integrate twice:

$$V_x = \frac{\alpha}{2} y^2 + c_1 y + c_2 \quad , \quad \frac{dV_x}{dy} = \alpha y + c_1$$

boundary conditions for velocity:

$y = 0 \qquad V_x = U \qquad$ BC1

$y = \delta \qquad \frac{dV_x}{dy} = 0 \qquad$ BC2

<u>physical interpretation of BC2:</u>

at $y = \delta$ friction between fluid and air is negligible $\Rightarrow$ at $y = \delta$ $\tau_{yx} = 0$

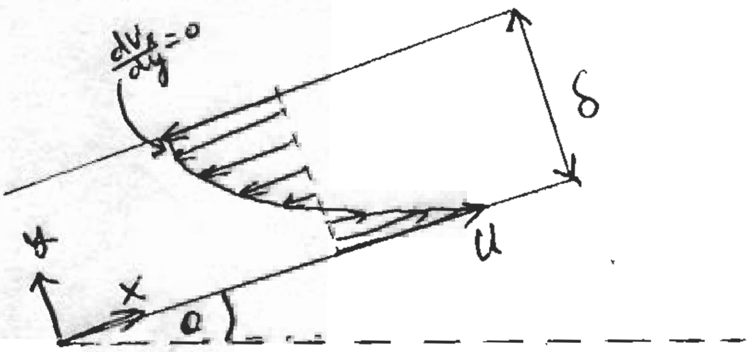
$$\tau_{yx} = \mu \frac{dV_x}{dy} = 0 \quad \Rightarrow \quad y = \delta \quad \frac{dV_x}{dy} = 0$$

apply BC1 $\Rightarrow$ $U = \frac{\alpha}{2}(0) + c_1(0) + c_2$

$$\Rightarrow \boxed{c_2 = U}$$

apply BC2 $\Rightarrow$ $0 = \alpha \delta + c_1 \Rightarrow \boxed{c_1 = -\alpha\delta}$

$$\Rightarrow \boxed{V_x = U - \alpha y \left[\delta - \frac{y}{2} \right]}$$



Shear Stress:

all components of stress tensor are zero except:

$$\tau_{xy} = \tau_{yx} = \mu \frac{dv_x}{dy} = \mu\alpha(y - \delta)$$

Volumetric Flow Rate:

$$Q = \int_A v_x \, dA$$

$$= \int_0^\delta \int_0^W v_x \, dz \, dy = W \int_0^\delta v_x \, dy$$

$$= \int_0^\delta v_x \, dy \quad \text{(per unit width)}$$

$$= \delta u - \frac{\alpha}{3}\delta^3 \quad \text{(do it yourself)}$$